105. A General Local Ergodic Theorem

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1. Introduction and the theorem. The purpose of this note is to prove a local ergodic theorem for a one-parameter semi-group of positive bounded linear operators on $L_1(X)$. A local ergodic theorem for a one-parameter semi-group of positive linear contractions was proved by Krengel [5], Ornstein [6], Akcoglu-Chacon [1] and Terrell [7] under little different conditions. Fong-Sucheston gave a proof of a local ergodic theorem for a special class of one-parameter semi-groups of positive uniformly bounded linear operators [4].

Let $(X, \mathfrak{B}, m)$ be a σ-finite measure space and $L_1(X)=L_1(X, \mathfrak{B}, m)$ be the Banach space of real integrable functions on X. Let $(T_t)(t \geqslant 0)$ be **a strongly continuous one-parameter semi-group of positive bounded linear operators on $L_1(X)$.** This means that ① T_t is a positive bounded linear operator on $L_1(X)$ for every $t \geqslant 0$ and $T_0=I$ (identity) (The positivity of T means that $Tf \geqslant 0$, if $f \geqslant 0$.), ② $T_{t+s}f=T_t \circ T_s f$ for any t, $s \geqslant 0$ and $f \in L_1(X)$, ③ $\lim_{t\to 0}\|T_t f-f\|=0$ for any $f \in L_1(X)$ (strong continuity). Then there exist constants M, β such that $\|T_t\| \leqslant Me^{\beta t}$ [9]. (If we can take $M=1$, $\beta=0$, then (T_t) is said to be a strongly continuous one-parameter semi-group of positive linear contractions.) By the strong continuity of (T_t), there exists a function $g(t, x)$ such that for a fixed $t \geqslant 0$, $g(t, x)=(T_t f)(x)$ for a.e. x and $g(t, x)$ is $\mathfrak{L}^+ \times \mathfrak{B}$-measurable, where $\mathfrak{L}^+$ is the σ-algebra of Lebesgue measurable sets on the half real line $[0, \infty)$. The function with this property is uniquely determined in the class of $\mathfrak{L}^+ \times \mathfrak{B}$-measurable functions [3, 8]. We define the integral $\int_a^b (T_t f)(x)dt$ $(0 \leqslant a < b < \infty)$ by $\int_a^b g(t, x)dt$.

We shall prove the following

Theorem. *Let (T_t) be a strongly continuous one-parameter semi-group of positive bounded linear operators on $L_1(X)$. Then we have*

$$\lim_{\alpha\to 0} \frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt = f(x) \qquad \text{a.e. for any } f \in L_1(X).$$

Corollary. *If $g \geqslant 0$ and $g \in L_1(X)$, then we have*

$$\lim_{\alpha\to 0} \frac{\int_0^\alpha (T_t f)(x)\,dt}{\int_0^\alpha (T_t g)(x)\,dt} = \frac{f(x)}{g(x)} \qquad \text{a.e. for any } f \in L_1(X)$$

on $\{x: g(x)>0\}$.

2. The proof of the theorem.

Lemma 1. *Let* $f\in L_1(X)$. *For a.e.* s (*with respect to the Lebesgue measure on the half real line*) *we have*

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_{t+s}f)(x)dt=(T_sf)(x) \qquad \textit{for a.e. } x.$$

The proof is based upon the Lebesgue theorem that for any real integrable function $f(t)$ on the real line, we have

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha f(t+s)dt=f(s) \qquad \text{for a.e. } s.$$

The proof of Lemma 2 in U. Krengel [5] is valid for that of Lemma 1.

Lemma 2 (**a maximal ergodic lemma**). *Let* $f\in L_1(X)$. *If*

$$\limsup_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt>0$$

on E, *then we have*

$$\int_E f^-(x)dm\leqslant\int_X f^+(x)dm.$$

Proof. Let $\varepsilon(0<\varepsilon<1)$ be an abitrary positive number. By the strong continuity of (T_t), there exists a positive number δ such that

$$\|T_tf^-\chi_E\|\geqslant(I-\varepsilon)\|f^-\chi_E\| \quad\text{and}\quad \|T_tf^+\|\leqslant(I+\varepsilon)\|f^+\| \tag{1}$$
for any $t(0\leqslant t\leqslant\delta)$,

$$\sup_{0\leqslant t\leqslant\delta}\|T_t\|=K<\infty. \tag{2}$$

(χ_G denotes the characteristic function of a set G)

Let us choose a positive number $\eta(0<\eta<\delta)$ such that

$$\frac{2\eta K}{\delta-\eta}\|f^-\|<\varepsilon. \tag{3}$$

There exists a positive integer l such that there exists a subset F of E with properties,

$$\sup_{0\leqslant j\leqslant[l\eta]}\sum_{i=0}^{j}(T_{1/l}^if)(x)>0 \qquad \text{on } F, \tag{4}$$

$$K\int_{E-F}f^-(x)dm<\varepsilon, \qquad \text{where } [a] \text{ is the integral part of } a. \tag{5}$$

This may be proved as follows. It follows from the assumption that

$$\sup_{0<\alpha<\eta}\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt>0 \qquad \text{on } E. \tag{6}$$

Since the integral $1/\alpha\int_0^\alpha (T_tf)(x)dt$ is a continuous function of the variable $\alpha>0$ for a.e. x,

$$\lim_{p\to\infty}\sup_{\substack{0<\alpha<\eta\\ \alpha\in Q_p}}\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt=\sup_{0<\alpha<\eta}\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt \tag{7}$$
for a.e. x,

where Q_p is the set of fractions with the denominator p (p is a positive integer.). We can choose a positive number ε' by (6) such that

$$(8)\qquad \text{if}\quad m(A)<\varepsilon',\quad \text{then}\quad \mu(A)<\frac{\varepsilon}{3}\quad \text{and}\quad \mu(E-E(\varepsilon'))<\frac{\varepsilon}{3},$$

where $\mu(A)=K\int_A f^-(x)dm$ and $E(\varepsilon')=\left\{x:\sup_{0<\alpha<\eta} 1/\alpha\int_0^\alpha (T_t f)(x)>2\varepsilon'\right\}\cap E$. It follows from (7) by the Egorov's theorem, there exists an integer q such that, if $p\geqslant q$,

$$(9)\qquad \sup_{\substack{0<\alpha<\eta\\ \alpha\in Q_p}} \frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt>\varepsilon' \qquad \text{for any } x \text{ in a set } F_1 \text{ with}$$

$$F_1\subset E(\varepsilon') \quad\text{and}\quad \mu(E(\varepsilon')-F_1)<\varepsilon/3.$$

Since the integral $1/\alpha\int_0^\alpha (T_t f)(x)dt$ is equal to the strong limit of $1/[n\alpha]\sum_{i=0}^{[n\alpha]}(T_{i/n}f)(x)$ $(n\to\infty)$ [3, 8], there exists a positive integer l such that

$$(10)\qquad \left\|\frac{1}{[l(j/q)]}\sum_{i=0}^{[l(j/q)]}(T_{1/l}^i f)(x)-\frac{q}{j}\int_0^{j/q}(T_t f)(x)\,dt\right\|<\frac{\varepsilon'^2}{[q\eta]}$$

$$(j=1,2,\cdots,[q\eta]).$$

And it follows from this that

$$(11)\qquad \left|\frac{1}{[l(j/q)]}\sum_{i=0}^{[l(j/q)]}(T_{1/l}^i f)(x)-\frac{q}{j}\int_0^{j/q}(T_t f)(x)dt\right|<\varepsilon'$$

$$(j=1,2,\cdots,[q\eta]),$$

except on a set F_2 with $m(F_2)<\varepsilon'$. By (8), $\mu(F_2)<\varepsilon/3$. Letting $F=F_1\cap F_2^c$ we have (4) and (5) by (8), (9) and (11).

We denote $T_{1/l}$ by T so that (4) and (5) are written by (12) and (13), respectively.

$$(12)\qquad \sup_{0\leqslant j\leqslant[l\eta]}\sum_{i=0}^{j}(T^i f)(x)>0 \qquad \text{on } F,$$

$$(13)\qquad K\int_{E-F} f^-(x)dm<\varepsilon.$$

We use the Chacon-Ornstein lemma:

Lemma (Chacon-Ornstein) [2]. *If* $\sup_{0\leqslant j\leqslant N}\sum_{i=0}^{j}(T^i f)(x)>0$ *on* F, *then there exist sequences of non-negative functions* $\{d_k\}$ *and* $\{f_k\}$ $(0\leqslant k\leqslant N)$ *such that*

$$(14)\qquad T^n f^+=\sum_{k=0}^{n}T^{n-k}d_k+f_n \qquad (0\leqslant n\leqslant N),$$

$$(15)\qquad \sum_{k=0}^{N}d_k\leqslant f^- \quad \textit{and}\quad \sum_{k=0}^{N}d_k=f^- \qquad \textit{on } F.$$

Remark. Though the lemma was proved by them under the assumption that $\|T\|\leqslant 1$ and $N=\infty$, conditions (14) and (15) hold good without the assumption.

Let us apply the lemma with $N=[l\eta]$. Put $n=[l(\delta-\eta)]$ and $S_n f=\sum_{k=0}^{n-1}T^k f$. We have by (1), 14) and $f_N\geqslant 0$,

$$(16)\quad \begin{aligned}(1+\varepsilon)\int f^+dm &\geqslant \int \frac{S_n}{n}T^N f^+dm\\ &\geqslant \int \frac{S_n}{n}\sum_{k=0}^{N} d_k dm + \sum_{k=0}^{N}\int \frac{S_n}{n}(T^{N-k}d_k - d_k)dm.\end{aligned}$$

Since $\left|\int S_n/n(T^j d-d)dm\right| \leqslant (2j/n)K\|d\|$, it follows from (16) and (15) that

$$(17)\quad \begin{aligned}\int \frac{S_n}{n}\sum_{k=0}^{N} d_k\chi_F dm &\leqslant (1+\varepsilon)\int f^+dm + \frac{2KN}{n}\int \sum_{k=0}^{N} d_k dm\\ &\leqslant (1+\varepsilon)\int f^+dm + \frac{2KN}{n}\int f^- dm.\end{aligned}$$

By (1) and (15),

$$(18)\quad \begin{aligned}(1-\varepsilon)\int_E f^- dm &\leqslant \int \frac{S_n}{n} f^-\chi_E dm\\ &= \int \frac{S_n}{n}\sum_{k=0}^{N} d_k\chi_F dm + \int \frac{S_n}{n} f^-\chi_{E-F} dm.\end{aligned}$$

And so by (17) and (2)

$$(19)\quad (1-\varepsilon)\int_E f^- dm \leqslant (1+\varepsilon)\int f^+dm + \frac{2KN}{n}\int f^- dm + K\int f^-\chi_{E-F}dm.$$

If l tends to infinity, $N/n=[l\eta]/[l(\delta-\eta)]$ tends to $\eta/(\delta-\eta)$, and by (19), (3) and (13) we have,

$$(1-\varepsilon)\int_E f^- dm \leqslant (1+\varepsilon)\int f^+dm + 2\varepsilon.$$

Arbitrariness of ε implies the Lemma 2.

The proof of the theorem. If the theorem does not hold, then there exists a positive number $\delta (0<\delta<1)$, a function f and a set E such that

$$(20)\quad \limsup_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt - f(x) > \delta$$

on E and $0<m(E)<\infty$.

Let ε' be an arbitrary positive number with $0<\varepsilon'<1/10$. Put $\varepsilon=\varepsilon'\delta$. By Lemma 1 we can choose a function g such that

(21) $|f-g|<\varepsilon$, except on a set with a measure less than $\varepsilon \min(m(E), 1)$,

$$(22)\quad \|f-g\|<\varepsilon$$

and

$$(23)\quad \lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_t g)(x)dt = g(x) \qquad \text{a.e.}$$

Then we have by (20), (21) and (23)

$$(24)\quad \begin{aligned}&\limsup_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha T_t(f-g)(x)dt\\ &=\limsup_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt - f(x) + f(x) - g(x) > \frac{\delta}{2} \quad \text{on } F,\end{aligned}$$

where $F=E\cap\{x:|f-g|<\varepsilon\}$ and therefore by (21)

$$m(E-F)<\varepsilon\min(m(E),1).$$

Again by Lemma 1 we can choose a non-negative function h (Put $h(x)=(T_s(1-\varepsilon/2)\chi_F)(x)$ for a suitable s.) such that

(25) $1-\varepsilon\leqslant h(x)\leqslant 1$ on G with $G\subset F$ and $m(F-G)<\varepsilon\min(m(E),1)$,

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_t h)(x)dt=h(x) \qquad \text{a.e.} \tag{26}$$

Then we have by (24), (25) and (26),

$$\limsup_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha T_t\left(f-g-\frac{\delta}{2}h\right)(x)dt>0 \qquad \text{on } G. \tag{27}$$

By Lemma 2, $h\geqslant 0$ and (22), we have

$$\begin{aligned}\int_G\left(f-g-\frac{\delta}{2}h\right)^-(x)dm&\leqslant\int_X\left(f-g-\frac{\delta}{2}h\right)^+(x)dm\\&<\int(f-g)^+(x)dm<\varepsilon.\end{aligned} \tag{28}$$

Since $(f-g-(\delta/2)h^-(x)>\delta/3$ on G by (21), (24) and (25) we have, remembering $\varepsilon=\varepsilon'\delta(0<\varepsilon'<1/10,\ 0<\delta<1)$,

$$m(E)\leqslant m(G)+2\varepsilon<3\varepsilon'+2\varepsilon<5\varepsilon'. \tag{29}$$

Arbitrariness of ε' implies that $m(E)=0$. This contradicts the assumption (20) and the proof is complete.

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