# 32. On the Uniqueness of Global Generalized Solutions for the Equation $F(x, u, \operatorname{grad} u)=0$

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**1. Introduction.** If we intend to treat the Cauchy problem for the Hamilton-Jacobi equation

$$u_t+f(\operatorname{grad} u)=0, \qquad x\in R^n, \quad t>0,$$
$$(\operatorname{grad} u=(u_{x_1}, \cdots, u_{x_n}))$$

from the point of view of the theory of semigroups of nonlinear transformations, it is necessary ([1]) to establish the existence and uniqueness of certain bounded (possibly generalized) solutions of the associated equation

$$\text{(AE)} \qquad u+f(\operatorname{grad} u)=h(x), \qquad x\in R^n,$$

for given $h$. In this note we shall consider a more general equation of the form

$$\text{(E)} \qquad F(x, u, \operatorname{grad} u)=0, \qquad x\in R^n,$$

and prove a uniqueness theorem for certain bounded generalized (Lipschitz-continuous) solutions of (E). A semigroup treatment of the Hamilton-Jacobi equation in several space variables will be taken up in a later paper.

**2. Definition of a generalized solution.** We shall assume that the function $F(x,u,p)$ in (E) is real-valued and of class $C^2$ with respect to all its arguments in $R^n_x\times R^1_u\times R^n_p$ and satisfies the following three conditions:

i) The matrix $(F_{ij}(x,u,p))$, where $F_{ij}=\partial^2F/\partial p_i\partial p_j$ $(i,j=1,\cdots,n)$, is nonnegative, i.e.,

$$\sum_{i,j=1}^{n} F_{ij}(x,u,p)\lambda_i\lambda_j\geq 0$$

for each $(x,u,p)\in R^n_x\times R^1_u\times R^n_p$ and each real $\lambda_1,\cdots,\lambda_n$;

ii) There exists a positive constant $c$ such that

$$F_u(x,u,p)\geq c$$

for all $(x,u,p)\in R^n_x\times R^1_u\times R^n_p$;

iii) The partial derivatives $F_{p_i}, F_{p_ix_i}, F_{p_iu}$ and $F_{p_ip_i}$ $(i=1,\cdots,n)$ are bounded in any subdomain

$$(1) \qquad \mathscr{D}=\{(x,u,p);\ x\in R^n,\ |u|\leq U,\ |p|\leq P\},$$

where $U$ and $P$ are arbitrary constants.

Under the assumption i), we shall give the following definition (cf. [3], [4]).

**Definition.** A bounded and uniformly Lipschitz-continuous function $u: R^n \to R^1$ that satisfies (E) at almost all points of $R^n$ is called a bounded generalized solution of (E) if it satisfies the following semi-concavity condition:

(SC) $$u(x+y)+u(x-y)-2u(x) \leq k|y|^2, \qquad x, y \in R^n,$$

for some constant $k$.

**3. Uniqueness.** Our aim is to prove the

**Theorem** (Uniqueness). *Under Assumptions* i)–iii), *there exists at most one bounded generalized solution of* (E).

As is easily seen, there are, in general, infinitely many bounded, Lipschitz-continuous functions that satisfy (E) at almost all points of $R^n$. In fact, this failure of uniqueness is shown by the following example. Consider the equation

(2) $$u+(1/2)(u_x^2+u_y^2)=0, \qquad (x, y) \in R^2.$$

Then, obviously, the functions $u_{\alpha\beta}$ defined by

$$u_{\alpha\beta}(x)=\begin{cases} 0, & x \leq \alpha \quad \text{or} \quad x \geq \beta, \\ -(1/2)(x-\alpha)^2, & \alpha \leq x \leq (1/2)(\alpha+\beta), \\ -(1/2)(x-\beta)^2, & (1/2)(\alpha+\beta) \leq x \leq \beta, \end{cases}$$

for all pairs of $(\alpha, \beta)$ with $\alpha \leq \beta$ are bounded, Lipschitz-continuous solutions of (2), $u \equiv 0$ being semi-concave.

**Proof of Theorem.** To prove the theorem by contradiction let $u$ and $v$ be two bounded generalized solutions of (E). For $u$ and $v$, let $U$ denote a common absolute bound in $R^n$, let $P$ be a common Lipschitz constant, and let $k$ be a common semiconcavity constant. Let $\mathscr{D}$ denote a domain defined by (1) and we set

$$K_1=\sup_{\mathscr{D}} \Big(\sum_i (F_{p_i}(x, u, p))^2\Big)^{1/2},$$
$$K_2=\sup_{\mathscr{D}} \Big|\sum_i F_{p_i x_i}\Big|+P \sum_i \sup_{\mathscr{D}} |F_{p_i u}|,$$
$$K_3=\sup_{\mathscr{D}} \sum_i F_{p_i p_i}.$$

For a part of our proof below the author owes to a technique suggested by Douglis [3].

Since

$$F(x, u, \operatorname{grad} u)=0, \qquad F(x, v, \operatorname{grad} v)=0,$$

*a.e.* in $R^n$, the difference $w=u-v$ satisfies the equation

$$Gw+\sum_{i=1}^n G_i w_{x_i}=0$$

*a.e.* in $R^n$, where

$$\begin{aligned} G&=G(x, u, v) \\ &=\int_0^1 F_u(x, v+\theta(u-v), \operatorname{grad} v+\theta\,(\operatorname{grad} u-\operatorname{grad} v))d\theta, \\ G_i&=G_i(x, u, v) \\ &=\int_0^1 F_{p_i}(x, v+\theta(u-v), \operatorname{grad} v+\theta\,(\operatorname{grad} u-\operatorname{grad} v))d\theta. \end{aligned}$$

If we set $W=w^q$, where $q$ is an even integer, we have

$$qGW+\sum_{i=1}^{n} G_i W_{x_i}=0 \tag{3}$$

*a.e.* in $R^n$.

By convolving $u$ and $v$ with mollifying kernels, we can find two approximating sequences $\{u^m\}$ and $\{v^m\}$ of infinitely differentiable functions, each having the same absolute bound $U$, Lipschitz constant $P$ and semiconcavity constant $k$ as $u$ and $v$, such that $\{\operatorname{grad} u^m\}$ and $\{\operatorname{grad} v^m\}$ converge *a.e.* in $R^n$ to $\operatorname{grad} u$ and $\operatorname{grad} v$ respectively. If we set

$$G_i^m=G_i(x, u^m, v^m), \qquad i=1,\cdots,n,$$

then equation (3) can be written as

$$qGW+\sum_{i=1}^{n}(G_i^m W)_{x_i}=\sum_{i=1}^{n}(G_i^m-G_i)W_{x_i}+W\sum_{i=1}^{n}(G_i^m)_{x_i}. \tag{4}$$

Let $r$ be an arbitrary positive number, and we integrate the both sides of (4) over the ball $|x|\le r$. We thus get

$$\begin{aligned} &q\int_{|x|\le r} GW dx+\int_{|x|=r} W\sum_{i=1}^{n} G_i^m \cos(n, x_i)dS \\ &\quad =\int_{|x|\le r}\sum_{i=1}^{n}(G_i^m-G_i)W_{x_i}dx+\int_{|x|\le r} W\sum_{i=1}^{n}(G_i^m)_{x_i}dx, \end{aligned} \tag{5}$$

where $n$ is the outer normal to the sphere $S: |x|=r$ and $dS$ is the surface element. On the other hand, we have

$$\int_{|x|\le r} GWdx\ge c\int_{|x|\le r} Wdx,$$

$$\int_{|x|=r} W\sum_{i=1}^{n} G_i^m\cos(n, x_i)dS\ge -K_1\int_{|x|=r} WdS,$$

and

$$\int_{|x|\le r} W\sum_{i=1}^{n}(G_i^m)_{x_i}dx\le (K_2+kK_3)\int_{|x|\le r} Wdx,$$

since

$$\begin{aligned} \sum_{i=1}^{n}(G_i^m)_{x_i}=&\int_0^1\sum_{i=1}^{n} F_{p_i x_i}(\cdots)d\theta \\ &+\sum_{i=1}^{n}\left(u_{x_i}^m\int_0^1 \theta F_{p_i u}(\cdots)d\theta+v_{x_i}^m\int_0^1(1-\theta)F_{p_i u}(\cdots)d\theta\right) \\ &+\sum_{i,j=1}^{n}\left(u_{x_i x_j}^m\int_0^1\theta F_{p_i p_j}(\cdots)d\theta+v_{x_i x_j}^m\int_0^1(1-\theta)F_{p_i p_j}(\cdots)d\theta\right) \end{aligned}$$

$((\cdots)=(x, v^m+\theta(u^m-v^m), \operatorname{grad} v^m+\theta(\operatorname{grad} u^m-\operatorname{grad} v^m)))$.

(Note that, by virtue of assumption i) and the semiconcavity condition (SC),

$$\begin{aligned} \sum_{i,j=1}^{n} u_{x_i x_j}^m F_{p_i p_j}(\cdots)&=\operatorname{tr}[(M-kI)F]+k\sum_{i=1}^{n} F_{p_i p_i}(\cdots) \\ &\le k\sum_{i=1}^{n} F_{p_i p_i}(\cdots), \end{aligned}$$

$M$ and $F$ denoting the matrices $(u_{x_i x_j}^m)$ and $(F_{p_i p_j}(\cdots))$ respectively.)

Substituting these inequalities into (5), we get

$$cq\int_{|x|\le r} W dx - K_1 \int_{|x|=r} W dS$$
$$\le \int_{|x|\le r} \sum_{i=1}^{n} (G_i^m - G_i) W_{x_i} dx + (K_2 + kK_3) \int_{|x|\le r} W dx$$

and, hence, by letting $m$ tend to infinity and using the bounded convergence theorem

$$cq\int_{|x|\le r} W dx - K_1 \int_{|x|=r} W dS \le (K_2 + kK_3) \int_{|x|\le r} W dx. \tag{6}$$

If we set

$$I(r) = \int_{|x|\le r} W dx \qquad \text{for } r > 0$$

and choose an even integer $q$ so large that $cq > K_2 + kK_3$, then inequality (6) can be written as a differential inequality for $I(r)$

$$aI(r) - dI(r)/dr \le 0, \qquad r > 0,$$

where $a = (cq - K_2 - kK_3)/K_1$ is a positive constant.

Now suppose that there is a positive number $r_0$ for which $I(r_0) > 0$. Then the differential inequality (7) gives a lower bound $I(r_0)\exp(a(r - r_0))$ for the growth order of $I(r)$ as $r$ tends to infinity. But this is a contradiction, since the integral $I(r)$ increases at most polynomially with $r$ because of the boundedness of $W = w^q$. Therefore, $I(r) = 0$ for $r \ge 0$ and, hence, the difference $w = u - v$ must vanish identically in $R^n$. This completes the proof.

**Corollary.** *If $f: R^n \to R^1$ is of class $C^2$ and satisfies condition* i) *with $F(x, u, p)$ replaced by $f(p)$, then there exists at most one bounded generalized solution of* (AE).

## References

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