115. *On almost Primes in Arithmetic Progressions. II*

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§ 1. Let P_r denote as usual a number which has at most r prime factors counting multiplicities. In our previous paper [2] we have proved that there are numbers such that

$$P_2 \ll k^{11/10}, \qquad P_2 \equiv l \pmod{k},$$
$$P_3 \ll k(\log k)^{70}, \qquad P_3 \equiv l \pmod{k},$$

for almost all reduced residue classes $l \pmod{k}$. The purpose of the present note is to study briefly the dual problem in which the reduced residue class l is fixed and the modulus k runs over certain interval. We prove

Theorem. *Let l be a fixed non-zero integer. Then there is a P_3 such that*

$$P_3 \ll k(\log k)^{70}, \qquad P_3 \equiv l \pmod{k},$$

for almost all k, $(k, l)=1$.

Our proof depends on two recent results: one from [2] which concerns to a compact presentation of the sieve procedure of Jurkat and Richert, and the other from [1] which is a simple variant of the dispersion method of Linnik. These are embodied in lemmas of the next paragraph.

Notations. In what follows we always have $(k, l)=1$, and we may assume that l is a positive integer. x is a positive and sufficiently large parameter. $\varphi(n)$ denotes the Euler function, and $d(n)$, $d_5(n)$ are divisor functions. (n, m) and $[n, m]$ denote the greatest common divisor and the least common multiple between n and m, respectively.

§ 2. Let $z \geqq 2$ be arbitrary, and let

$$P_k(z) = \prod_{\substack{p \leqq z \\ p \nmid k}} p, \qquad \Gamma_k(z) = \prod_{\substack{p \leqq z \\ p \nmid k}} \left(1 - \frac{1}{p}\right),$$

p being generally a prime number. We introduce another parameter w such that $z \leqq w$, and we put, for any non-negative constant ζ,

$$V_\zeta(x; k, l; z, w) = \sum_{\substack{n \equiv l \ (\mathrm{mod}\ k) \\ n \leqq x \\ (n,l)=1 \\ (n, P_k(z))=1}} \left\{1 - \zeta \sum_{\substack{p \mid n \\ p \nmid kl \\ z \leqq p < w}} \left(1 - \frac{\log p}{\log w}\right)\right\},$$

$$S(x; k, l; z, w) = \sum_{\substack{n \equiv l \ (\mathrm{mod}\ k) \\ n \leqq x \\ (n,l)=1}} \sum_{\substack{p^2 \mid n \\ p \nmid kl \\ z \leqq p < w}} 1.$$

Also we write

(1) $\qquad W_\zeta(x;k,l;z,w)=V_\zeta(x;k,l;z,w)-S(x;k,l;z,w).$

We define the functions f and F by the difference-differential equation

$$\frac{d}{du}(uf(u))=F(u-1),\quad \frac{d}{du}(uF(u))=f(u-1),\quad u\geqq 2,$$

$$f(u)=0,\quad F(u)=2e^\gamma/u,\quad 0<u\leqq 2,$$

where γ is the Euler constant, and we put, for any $1<u<v$,

$$\Psi_\zeta(u,v)=f(v)-\zeta\int_u^v F\Big(v\Big(1-\frac{1}{t}\Big)\Big)\Big(1-\frac{u}{t}\Big)\frac{dt}{t}.$$

Then we have, by an easy modification of the argument of [2],

Lemma 1. *Let* R *be a parameter with* $R\leqq x^{3/2}$, *and let*

$$2\leqq z\leqq w\leqq x/\sqrt{R}.$$

Then there are numbers $\Phi(m)=\Phi_\zeta(m;x,z,w,R)$ *such that*

(2) $$V_\zeta(x;k,l;z,w)\geqq \sum_{\substack{n\equiv l \ (\mathrm{mod}\ k)\\ (n,l)=1\\ n\leqq x}}\Big(\sum_{m\mid n}\Phi(m)\Big).$$

And these numbers have the properties:

$$\Phi(m)=O(d_5(m))$$
$$\Phi(m)=0,\qquad \text{for } m>x/\sqrt{R}.$$

Moreover we have

(3) $$\sum_{(m,kl)=1}\frac{\Phi(m)}{m}\geqq\frac{\varphi(l)}{l}\Gamma_{kl}(z)\Psi_\zeta\Big(\frac{\log x/\sqrt{R}}{\log w},\frac{\log x/\sqrt{R}}{\log z}\Big)-O((\log x)^{-1/15}).$$

The next lemma is a slightly generalized presentation of a recent result of Hooley [1].

Lemma 2. *Let* $c(m)$ *be any complex numbers, and let* l *be a positive integer. Then we have, uniformly for all involved parameters,*

$$\sum_{\substack{K\leqq k<2K\\ (k,l)=1}}\Bigg|\sum_{\substack{n\equiv l\ (\mathrm{mod}\ k)\\ (n,l)=1\\ n\leqq x}}\Big(\sum_{\substack{m\mid n\\ m\leqq M}}c(m)\Big)-\frac{\varphi(l)(x-l)}{lk}\sum_{\substack{(m,kl)=1\\ m\leqq M}}\frac{c(m)}{m}\Bigg|^2$$
$$\ll\Big(\frac{x}{K}\Big)^2\Big(\sum_{m\leqq M}|c(m)|\Big)^2+x\log x\sum_{m_1,m_2\leqq M}\frac{|c(m_1)c(m_2)|}{[m_1,m_2]}(d(m_1)+d(m_2)).$$

§ 3. We now show a brief proof of our theorem. We set in Lemma 1

$$z=(x/\sqrt{R})^{1/4},\qquad w=(x/\sqrt{R})^{9/10},$$
$$\frac{1}{\zeta}=4-\frac{10}{9}(\Lambda_3-\delta),$$

where

$$\Lambda_3=4-\frac{\log 27/7}{\log 3}$$

and δ is an arbitrary small positive constant. Then we have, by [3],

(4) $$\Psi_\zeta\Big(\frac{\log x/\sqrt{R}}{\log w},\frac{\log x/\sqrt{R}}{\log z}\Big)\geqq\frac{\delta}{2}e^\gamma\frac{\log 9}{\log 8}.$$

On the other hand, writing $V_\zeta^{(1)}(x;k)$ for the right side of (2), we get, from Lemma 2,

$$\sum_{\substack{K\leqq k<2K\\(k,l)=1}}\left|V_\zeta^{(1)}(x;k)-\frac{\varphi(l)(x-l)}{lk}\sum_{(m,kl)=1}\frac{\Phi(m)}{m}\right|^2$$
$$\ll\frac{x^4}{K^2R}(\log x)^8+x(\log x)^{66}.$$

Hence, noticing (3) and (4), we have

$$V_\zeta^{(1)}(x;k)=(1+O((\log x)^{-1/2}))\frac{\varphi(l)(x-l)}{lk}\sum_{(m,kl)=1}\frac{\Phi(m)}{m},\tag{5}$$

save for at most

$$\ll\frac{x^2}{R}(\log x)^{11}+\frac{K^2}{x}(\log x)^{69}$$

modulus k from the interval $[K,2K)$.

Also, again appealing to Lemma 2, we see easily

$$\sum_{\substack{K\leqq k<2K\\(k,l)=1}}\{S(x;k,l;z,w)\}^2\ll\left(\frac{x}{K}\right)^2w^2+\frac{x^2}{zK}.$$

Thus we have

$$S(x;k,l;z,w)\ll\frac{x}{K}(\log x)^{-2}\tag{6}$$

save for at most

$$\left(\left(\frac{x^2}{R}+\frac{K}{z}\right)\log^4 x\right)$$

modulus k from the interval $[K,2K)$.

Finally we set

$$R=x(\log x)^{82},$$
$$K=x(\log x)^{-70}.$$

Then we have, from (1) to (6), that, for sufficiently large x,

$$W_\zeta(x;k,l;z,w)\geqq\frac{\delta x}{3k}\Gamma_k(z)e^{\gamma}$$

save for at most $K(\log x)^{-1}$ modulus k from the interval $[K,2K)$. According to [3] this means that there is a P_3 such that

$$P_3\leqq x\ll K(\log K)^{70}$$
$$P_3\equiv l\ (\mathrm{mod}\ k),$$

for almost all k.

References

[1] C. Hooley: On the Brun-Titchmarch theorem. II. Proc. London Math. Soc., **30**, 114–128 (1975).

[2] Y. Motohashi: On almost primes in arithmetic progressions (to appear in J. Math. Soc. Japan.).

[3] H. E. Richert: Selberg's sieve with weight. Mathematatika, **16**, 1–22 (1969).