136. *Periodic Linear Systems and a Class of Nonlinear Evolution Equations*

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1. **Periodic systems.** Consider a linear periodic system with t regarded as a parameter:

$$\frac{d\varphi}{dx}=A(x,t)\varphi, \qquad -\infty<x,t<+\infty, \tag{1.1}$$

where $\varphi=\varphi(x,t)$ is a complex n-column vector and $A(x,t)$ is a complex $n\times n$ matrix function. We assume that $A(x,t)$ is infinitely differentiable and periodic in x with period ω. Below, we are always in the category of infinite differentiability. From the well-known Floquet's theorem, we see that every fundamental matrix solution $X(x;t)$ of (1.1) has the form:

$$X(x;t)=P(x,t)\exp(x\log B(t)/\omega), \tag{1.2}$$

where $P(x,t)$ is a complex nonsingular $n\times n$ matrix function which is periodic in x with period ω and $B(t)$ is a complex nonsingular $n\times n$ matrix function which does not depend on x. $B(t)$ is called a monodromy matrix of (1.1) for $X(x,t)$. The eigenvalues $\rho_j(t)$ of a monodromy matrix of (1.1) are called the characteristic multipliers of (1.1). For any fixed t, every monodromy matrix of (1.1) is similar to each other. Hence, so long as t is fixed, the characteristic multipliers and their algebraic and geometric multiplicities (which we shall call the internal structure of a monodromy matrix) do not depend on the particular fundamental solution used to define the monodromy matrix. As is well known, in order that all solutions of (1.1) are bounded in the whole axis $-\infty<x<+\infty$, it is necessary and sufficient that all characteristic multipliers of (1.1) have modulii$=1$ and have simple elementary divisors (see Hale [1]). As t varies, the internal structure of $B(t)$ may change, that is, the qualitative properties of solutions of (1.1) may change. We now propose the following question: To find $A(x,t)$ for which the equation (1.1) admits a monodromy matrix which does not depend on t, that is, the internal structure of every monodromy matrix of (1.1) does not depend on t. For this question we have

Theorem 1. *There exists a monodromy matrix of* (1.1) *which does not depend on t if and only if there exists a matrix function* $\Gamma(x,t)$ *which is defined on* $-\infty<x,t<+\infty$, *periodic in* x *with period* ω *and satisfies*

(1.3) $$\frac{\partial}{\partial t}A(x,t)-\frac{\partial}{\partial x}\Gamma(x,t)+[A(x,t),\Gamma(x,t)]=0,$$

where the square bracket indicates the commutator.

A fundamental matrix solution of (1.1) such that $X(y;t)=I$, the identity matrix, for some fixed $y\in(-\infty,+\infty)$ and all $t\in(-\infty,+\infty)$ is called a principal matrix solution at initial point y and denoted by $X(x,y;t)$. We denote by $B(t;y)$ the monodromy matrix for $X(x,y;t)$. We have

Theorem 2. *Suppose that $B(t;y)$ is similar to a monodromy matrix which is independent of t. Then, $B(t,y)$ satisfies*

(1.4) $$\frac{\partial}{\partial t}B(t;y)=[\Gamma(y;t),B(t;y)].$$

2. Nonlinear evolution equations. We seek a class of nonlinear evolution equations having the following property: If the coefficient matrix $A(x,t)$ varies according to a nonlinear evolution equation of this class, the equation (1.1) admits a monodromy matrix which does not depend on t.

Theorem 3. *Suppose that the coefficient matrix $A(x,t)$ depends on a complex $n\times n$ matrix function $U=U(x,t)$ in the following fashion:*

(2.1) $$A(x,t)=F(U(x,t))+A_0(x),$$

where $A_0(x)$ is a $n\times n$ matrix function defined on $-\infty<x<+\infty$ which is periodic with period ω and F depends linearly on U. Suppose that there exists a $n\times n$ matrix function $\Gamma(x,t)$ which is periodic in x with period ω and satisfies

(2.2) $$\frac{\partial}{\partial x}\Gamma-[A,\Gamma]=F(S(U)),$$

where $S(\cdot)$ is some nonlinear partial differential (or integral or integro-differential) operator. Then, if U varies according to the equation

(2.3) $$\frac{\partial}{\partial t}U=S(U)$$

with periodic boundary condition

(2.4) $$U(x+\omega,t)=U(x,t)\qquad\text{for all } x \text{ and } t,$$

the internal structure of every monodromy matrix of (1.1) *does not depend on t.*

Corollary 1. *Under the same assumptions of Theorem* 3 *the characteristic multipliers are invariant integrals of* (2.3) *with condition* (2.4).

There are various examples of nonlinear equations of physical significance having the above remarkable property. We present a method of deriving them explicitly. For this we now consider the case when the coefficient matrix $A(x,t)$ depends on a complex parameter λ

in such a way:

$$A(x,t)=U(x,t)+\lambda J \tag{2.5}$$

where U is a $n\times n$ matrix function which is periodic in x with period ω and J is a constant matrix. Then, any fundamental matrix solution $X(x;t)$ of (1.1) is an entire function of λ. Hence, the matrices $P(x,t)$, $B(t)$ defined by (1.2) are also entire in λ. Therefore the matrix $\Gamma(x,t)$ mentioned in Theorem 1 must be an entire function of λ. In these cases, the equation (1.3) is

$$\frac{\partial}{\partial t}U-\frac{\partial}{\partial x}\Gamma+[U,\Gamma]+\lambda[J,\Gamma]=0. \tag{2.6}$$

We wish to choose Γ such that the derived equation

$$\frac{\partial}{\partial t}U=S(U)$$

does not depend on λ.

At first suppose that $\Gamma(x,t)$ is a polynomial of degree N in λ, N being an arbitrary positive integer. Denoting it by $\Gamma_N(x,t;\lambda)$, we may write

$$\Gamma_N(x,t;\lambda)=\sum_{i=0}^{N}\Gamma^{(i)}(x,t)\lambda^{N-i}, \tag{2.7}$$

where $\Gamma^{(i)}$ are $n\times n$ matrix functions to be determined so that the left hand side of (2.7) is independent of λ. Then, we get a recursion formula for the $\Gamma^{(i)}, i=0,1,\cdots,N,$:

$$[J,\Gamma^{(0)}]=0,$$

$$-\frac{\partial}{\partial x}\Gamma^{(i)}+[U,\Gamma^{(i)}]+[J,\Gamma^{(i+1)}]=0 \qquad i=0,1,\cdots,N-1, \tag{2.8}$$

and an evolution equation

$$\frac{\partial}{\partial t}U=\frac{\partial}{\partial x}\Gamma^{(N)}-[U,\Gamma^{(N)}]. \tag{2.9}$$

Adopting the notation $\mathrm{ad}_A\cdot B=[A,B]$, we can write the equation (2.9) formally in the form:

$$\begin{aligned}\frac{\partial}{\partial t}U&=\left(\frac{\partial}{\partial x}-\mathrm{ad}_U\right)\Gamma^{(N)}\\&=\left(\frac{\partial}{\partial x}-\mathrm{ad}_U\right)\mathcal{L}\left(\frac{\partial}{\partial x}-\mathrm{ad}_U\right)\Gamma^{(N-1)},\ \mathrm{mod}\ N(\mathrm{ad}_J)\\&=\cdots\\&=\mathcal{L}^{N-1}\frac{\partial^N}{\partial x^N}U+R^{(N)}\left(U,\frac{\partial}{\partial x}U,\cdots,\frac{\partial^{N-1}}{\partial x^{N-1}}U\right),\end{aligned} \tag{2.10}$$

where $\mathcal{L}=\alpha(\mathrm{ad}_{J*}\,\mathrm{ad}_J)\,\mathrm{ad}_{J*}$ (J^* being the complex conjugate transpose and $\alpha(\ \cdot\ )$ being a polynomial of degree two less than that of the minimal polynomial of $\mathrm{ad}_{J*}\,\mathrm{ad}_J$ (see [2]), $N(\mathrm{ad}_J)$ is the null space of ad_J and $R^{(N)}$ is a $n\times n$ matrix whose elements are polynomiald of the elements of the matrices $U, \partial U/\partial x, \cdots$ and $\partial^{N-1}U/\partial x^{N-1}$.

Theorem 4. *As $U(x,t)$ varies according to the nonlinear evolution equation* (2.10) *with periodic boundary condition*

$$U(x+\omega,t)=U(x,t), \tag{2.11}$$

every monodromy matrix of (1.1) *with* (2.5) *has the same internal structure.*

Corollary 2. *The eigenvalues of the system*

$$\frac{d\varphi}{dx}=(U(x,t)+\lambda J)\varphi \tag{2.12}$$

with boundary condition

$$\varphi(\omega,t)=e^{i\theta}\varphi(0,t), \tag{2.13}$$

where θ is an arbitrary real number, are invariant integrals of the nonlinear evolution equation (2.10) *with* (2.5).

Remark 1. The Korteweg-de Vries equation and its generalizations, the Modified Korteweg-de Vries equation, the nonlinear Schrödinger equation and so on, are the special case of the equation (2.10). They are naturally derived when one considers the case in which $A(x,t)=U(x,t)+\lambda J$ lies in the Lie algebra of unimodular group: $\mathfrak{sl}(2;C)$ (see [3]).

Remark 2. For the Korteweg-de Vries equation, the assertion of Corollary 2 was discovered by Gardner, Kruskal and Miura [4] (see also Lax [5], Menikoff [6], Tsutsumi [7]). This was a motivation of the present work.

Next suppose that $\Gamma(x,t)$ is a polynomial of degree M in λ^{-1}, M being an arbitrary positive integer. We denote it by $\tilde{\Gamma}_M(x,t;\lambda)$. Then we may write

$$\tilde{\Gamma}_M(x,t;\lambda)=\sum_{i=0}^{M}\tilde{\Gamma}^{(i)}(x,t)\lambda^{i-M-1}, \tag{2.14}$$

where $\tilde{\Gamma}^{(i)}(x,t)$ are $n\times n$ matrix functions to be determined so that the left hand side of (2.6) is independent of λ. We have a recursion formula

$$\begin{cases} -\dfrac{\partial}{\partial x}\tilde{\Gamma}^{(0)}+[U,\tilde{\Gamma}^{(0)}]=0, \\ -\dfrac{\partial}{\partial x}\tilde{\Gamma}^{(i)}+[U,\tilde{\Gamma}^{(i)}]+[J,\tilde{\Gamma}^{(i+1)}]=0, \qquad i=1,2,\cdots,M-1, \end{cases} \tag{2.15}$$

and an evolution equation

$$\frac{\partial}{\partial t}U+[J,\tilde{\Gamma}^{(M)}]=0, \tag{2.16}$$

which is equivalent to the equation

$$\frac{\partial}{\partial t}U=\mathrm{ad}_J\tilde{\mathcal{L}}\Big(\frac{\partial}{\partial x}-\mathrm{ad}_U\Big)\tilde{\Gamma}^{(M-1)}, \ \mathrm{mod}\ N(\mathrm{ad}_J), \tag{2.17}$$

where $\tilde{\mathcal{L}}=\beta(\mathrm{ad}_{J^*}\mathrm{ad}_J)\mathrm{ad}_{J^*}$ (J^* being the complex conjugate transpose of J and $\beta(\cdot)$ being a polynomial of degree two less than that of the minimal polynomial of $\mathrm{ad}_{J^*}\,\mathrm{ad}_J$) and $N(\mathrm{ad}_J)$ is the null space of ad_J.

Theorem 5. *As $U(x,t)$ varies according to the nonlinear evolution equation* (2.16) *with periodic boundary condition* (2.11), *every monodromy matrix of* (1.1) *with* (2.5) *has the same internal structure.*

Corollary 3. *The eigenvalue λ of the equation*

$$\frac{d\varphi}{dx} = (U(x,t) + \lambda J)\varphi \tag{2.18}$$

with boundary condition

$$\varphi(\omega, t) = e^{i\theta}\varphi(0,t), \tag{2.19}$$

where θ is an arbitrary real number, are invariant integrals of the nonlinear evolution equation (2.16) *with* (2.11).

Remark 3. The sine-Gordon equation and its generalizations are contained in this case. They are derived when one considers the case in which $A(x,t) = U(x,t) + \lambda J$ lies in $\mathfrak{sl}(2; C)$.

Definition. The equation (2.10) is called the conservative system of type I and the equation (2.16) the conservative system of type II.

Detailed proofs and further investigations will appear elsewhere.

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