150. On the Jordan-Hölder Theorem

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Let $\{A_n, f_n\}$ be a family of groups A_n and homomorphisms $f_n: A_n \to A_{n-1}$, defined for all $n \in Z$ ($Z=\{0, \pm 1, \pm 2, \cdots\}$). If a sequence

$$\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots$$

is exact, then we denote it by $(A_n : f_n)$ and we say $(A_n : f_n)$ to be *well defined*. Generalizations of Isomorphism Theorem and the Jordan-Hölder Theorem in group theory have been given in some papers (for example, [2] and [3]). The purpose of this note is also to give those theorems for a sequence $(A_n : f_n)$.

1. Isomorphism Theorem. In this section, let $(A_n : f_n)$ and $(B_n : g_n)$ be well defined. A *translation* $\{\alpha_n\}$ of $(A_n : f_n)$ into $(B_n : g_n)$ is the set of homomorphisms $\alpha_n : A_n \to B_n$ such that $\alpha_{n-1} f_n = g_n \alpha_n$ for all $n \in Z$. Moreover, if each α_n is an isomorphism, we say that $(A_n : f_n)$ is *isomorphic* to $(B_n : g_n)$. If for each $n \in Z$, B_n is a subgroup of A_n, i.e., $A_n \geqslant B_n$, and $f_n = g_n$ on B_n, then we denote $(B_n : g_n)$ by $(B_n : f_n)$. In this case, we call $(B_n : f_n)$ a *subsequence* of $(A_n : f_n)$ and write it in the notation: $(A_n : f_n) \geqslant (B_n : f_n)$. Moreover, if $A_n \triangleright B_n$ for all $n \in Z$, we call $(B_n : f_n)$ a *normal subsequence* of $(A_n : f_n)$ and write it in the notation: $(A_n : f_n) \triangleright (B_n : f_n)$.

It is easy to prove the following

Lemma 1. *Let $(A_n : f_n)$ be well defined. For each $n \in Z$, let M_n be a subgroup of A_n. Then $(M_n : f_n)$ is well defined iff $f_n(M_n) = f_n(A_n) \cap M_{n-1}$ for all $n \in Z$.*

By Lemma 1 and the same way as in proofs of [1, Lemma 2] and [1, Lemma 3], we can prove the following

Lemma 2. *Let $(A_n : f_n) \geqslant (P_n : f_n)$. For each $n \in Z$, let $A_n \geqslant M_n \triangleright P_n$. Then $(M_n : f_n)$ is well defined iff $(M_n/P_n : \bar{f}_n)$ is well defined where each $\bar{f}_n$ is a mapping which is naturally induced by f_n.*

Theorem 1. *Let $\{\alpha_n\} : (A_n : f_n) \to (B_n : g_n)$ be a translation. Then $(\alpha_n(A_n) : g_n)$ is well defined iff* $(\mathrm{Ker}\,(\alpha_n) : f_n)$ *is well defined. In this case, $(A_n/\mathrm{Ker}\,(\alpha_n) : \bar{f}_n)$ is also well defined and isomorphic to $(\alpha_n(A_n) : g_n)$, where for each $n \in Z$, $\bar{f}_n$ is a mapping which is naturally induced by f_n.*

Proof. The first assertion follows from routine arguments and the remainder follows from Lemma 2.

Theorem 2. *Let $(A_n : f_n) \triangleright (M_n : f_n)$ and $(A_n : f_n) \geqslant (H_n : f_n)$. Then $(M_n H_n : f_n)$ is well defined iff $(M_n \cap H_n : f_n)$ is well defined. In this case, $(M_n H_n / M_n : \bar{f}_n)$ and $(H_n / M_n \cap H_n : \hat{f}_n)$ are well defined and mutually isomorphic, where for each $n \in Z$, $\bar{f}_n$ and $\hat{f}_n$ are mappings which are naturally induced by f_n.*

Proof. By Lemma 2, $(A_n / M_n : \bar{f}_n)$ is well defined. We consider the translation $\{\alpha_n\} : (H_n : f_n) \to (A_n / M_n : \bar{f}_n)$ where each α_n is a natural homomorphism. By Theorem 1, $(M_n H_n / M_n : \bar{f}_n)$ is well defined iff $(M_n \cap H_n : f_n)$ is well defined. Hence the first assertion follows from Lemma 2. A proof of the remainder is obvious.

2. Jordan-Hölder Theorem. Now we simplify our notation, that is, we write G^* instead of $(G_n : f_n)$. Let $G^* \geqslant A^*$, B^* and $G^* \triangleright M^*$. If $(A_n \cap B_n : f_n)$, $(A_n B_n : f_n)$ and $(G_n / M_n : \bar{f}_n)$ are well defined where for each $n \in Z$, $\bar{f}_n$ is a mapping which is naturally induced by f_n, then we write $A^* \cap B^*$, $A^* B^*$ and G^*/M^* instead of those and say that $A^* \cap B^*$, $A^* B^*$ and G^*/M^* are well defined, respectively. If there is a family $\{K_i^*\}$ such that $G^* = K_0^* \triangleright K_1^* \triangleright \cdots \triangleright K_r^* = A^*$, A^* is said to be *subnormal* in G^*, $G^* \triangleright \triangleright A^*$. Let $G^* \geqslant A^*$. We say that A^* has the *I-property* in G^* if for every subnormal subsequence B^* of G^*, $A^* \cap B^*$ is well defined. Let $G^* = K_0^* \triangleright K_1^* \triangleright \cdots \triangleright K_r^* = A^*$. This series is called an *I-normal series* if each K_i^* has the I-property in G^*.

From the definition, we have easily the following

Proposition 1. *Let $G^* = K_0^* \triangleright K_1^* \triangleright \cdots \triangleright K_r^* = A^*$. Then this is an I-normal series iff each K_{i+1}^* has the I-property in K_i^*.*

Let $G^* \geqslant A^*$. If there is $n \in Z$ such that A_n is a proper subgroup of G_n, then A^* is said to be a *proper subsequence* of G^*. We say that G^* is *I-simple* if no proper normal subsequence of G^* has the I-property in G^*. Furthermore, an I-normal series $G^* = K_0^* \triangleright K_1^* \triangleright \cdots \triangleright K_r^* = A^*$ is called an *I-composition series* from G^* to A^* if each K_{i+1}^* is a proper subsequence of K_i^* such that K_i^* / K_{i+1}^* is I-simple.

Proposition 2. *Let $G^* \triangleright M^*$ and suppose M^* has the I-property in G^*. Then G^*/M^* is I-simple iff for every H^* having the I-property in G^*, $G^* \triangleright H^* \geqslant M^*$ implies $H^* = G^*$ or $H^* = M^*$.*

Proof. If part: Let $G^*/M^* \triangleright X^*$ and suppose X^* has the I-property in G^*/M^*. Then, by Lemma 2, there is a subsequence H^* of G^* such that $G^* \triangleright H^* \triangleright M^*$ and $X^* = H^*/M^*$. Now let $G^* \triangleright \triangleright L^*$. Then $L^* \cap M^*$ is well defined and so is $L^* M^*$ by Theorem 2. Hence $L^* M^* / M^*$ is well defined by Lemma 2 and $G^*/M^* \triangleright \triangleright L^* M^* / M^*$. Thus $H^*/M^* \cap L^* M^* / M^*$ is well defined and so is $H^*(L^* M^*)/M^*$. Hence $H^*(L^* M^*) = H^* L^*$ is well defined by Lemma 2 and so is $H^* \cap L^*$ by Theorem 2. This shows that H^* has the I-property in G^*. Hence $H^* = M^*$ or $H^* = G^*$. Therefore G^*/M^* is I-simple. Only if part: By the same way as in the stated above, the application of Lemma 2 and Theorem 2 gives its proof and so we omit it.

Lemma 3. *Let $G^*\triangleright A^*$ and $G^*\geqslant B^*$. Suppose A^* and B^* have the I-property in G^*. Then A^*B^* is well defined. Furthermore if $G^*\triangleright A^*B^*$, then A^*B^* has the I-property in G^*.*

Proof. Let $G^*\triangleright\triangleright H^*$. Then $A^*\cap H^*$ is well defined and so is A^*H^* by Theorem 2. Furthermore $G^*\triangleright\triangleright A^*H^*$ and so $B^*\cap A^*H^*$ is well defined. On the other hand, $A^*\cap B^*$ is well defined and $B^*\cap A^*H^*\geqslant A^*\cap B^*$. Hence $(A^*\cap B^*)\cap(B^*\cap A^*H^*)$ is well defined and so is $A^*\cap(B^*\cap A^*H^*)$. Thus $A^*(B^*\cap A^*H^*)$ and A^*B^* are well defined by Theorem 2. Hence, simultaneously with $A^*(B^*\cap A^*H^*)=A^*B^*\cap A^*H^*$, we obtain that $A^*B^*\cap A^*H^*$ is well defined. Let $G^*\triangleright A^*B^*$. Then $(A^*B^*)(A^*H^*)$ is well defined and so is $(A^*B^*)H^*$. Thus, by Theorem 2, $A^*B^*\cap H^*$ is well defined. Hence A^*B^* has the I-property in G^*.

Lemma 4. *Let $G^*\triangleright\triangleright A^*\triangleright B^*$ and let $G^*\triangleright\triangleright H^*\triangleright C^*$. Suppose A^*,B^* and C^* have the I-property in G^*. Then $B^*(A^*\cap C^*)$ and $B^*(A^*\cap H^*)$ are well defined. Furthermore $B^*(A^*\cap C^*)$ has the I-property in $B^*(A^*\cap H^*)$.*

Proof. It is easy to see that $B^*(A^*\cap C^*)$ and $B^*(A^*\cap H^*)$ are well defined. Furthermore $G^*\triangleright\triangleright B^*(A^*\cap H^*)$. Since B^* and $A^*\cap C^*$ have the I-property in G^*, those have the I-property in $B^*(A^*\cap H^*)$. Moreover $B^*(A^*\cap H^*)\triangleright B^*(A^*\cap C^*)$ and $B^*(A^*\cap H^*)\triangleright B^*$. Hence, by Lemma 3, $B^*(A^*\cap C^*)$ has the I-property in $B^*(A^*\cap H^*)$.

From Proposition 1, Lemma 4 and the well known results, we have following

Lemma 5. *Let*

(i) $$G^*=K_0^*\triangleright K_1^*\triangleright\cdots\triangleright K_r^*=A^*,$$

(ii) $$G^*=L_0^*\triangleright L_1^*\triangleright\cdots\triangleright L_s^*=A^*$$

be two I-normal series from G^* to A^*. Then $K_i^*(K_{i-1}^*\cap L_j^*)$ $(=K_{i,j}^*;\ r\geqslant i\geqslant 1;\ s\geqslant j\geqslant 0)$ and $L_j^*(L_{j-1}^*\cap K_i^*)$ $(=L_{j,i}^*;\ s\geqslant j\geqslant 1;\ r\geqslant i\geqslant 0)$ are well defined. Furthermore, for each i, j $(r\geqslant i\geqslant 1;\ s\geqslant j\geqslant 0)$, $K_{i,j}^*$ has the I-property in G^* and

(1) $$K_{i-1}^*=K_{i,0}^*\triangleright K_{i,1}^*\triangleright\cdots\triangleright K_{i,s}^*=K_i^*.$$

Moreover, for each i, j $(r\geqslant i\geqslant 0;\ s\geqslant j\geqslant 1)$, $L_{j,i}^*$ has the I-property in G^* and

(2) $$L_{j-1}^*=L_{j,0}^*\triangleright L_{j,1}^*\triangleright\cdots\triangleright L_{j,r}^*=L_j^*.$$

Joining the I-normal series (1), respectively (2), together, we obtain refinements of the I-normal series (i) and (ii) for which $K_{i,j-1}^*/K_{i,j}^*\leftrightarrow L_{j,i-1}^*/L_{j,i}^*$ is a one to one correspondence of their factors such that corresponding factors are isomorphic.

By Lemma 5 and the well known procedure, we have the following

Theorem 3 (Jordan-Hölder Theorem). *If*

$$G^*=K_0^*\geqslant K_1^*\geqslant\cdots\geqslant K_r^*=A^* \quad \textit{and} \quad G^*=L_0^*\geqslant L_1^*\geqslant\cdots\geqslant L_s^*=A^*$$

are two I-composition series from G^* to A^*, then $r=s$. Furthermore there is a permutation π of $\{1,\cdots,r\}$ such that K_{i-1}^*/K_i^* is isomorphic to $L_{\pi(i)-1}^*/L_{\pi(i)}^*$ for each $i=1,\cdots,r$.

References

[1] Z. Goseki: On Sylow subgroups and an extension of groups. Proc. Japan Acad., **50**, 576–579 (1974).
[2] O. Tamaschke: A generalization of subnormal subgroup. Arch. Math., **19**, 337–347 (1968).
[3] O. Wyler: Ein Isomorphiesatz. Arch. Math., **14**, 13–15 (1963).