35. *On the Extension of Klein's Geometrical Interpretation of Continued Fraction.*

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Klein gave in his work "Ausgewählte Kapitel aus der Zahlentheorie" a geometrical interpretation of continued fraction, and I have made use of it to prove Hurwitz's theorem and its extensions on continued fraction.[1] I wish in this note to extend the idea of Klein to give some precise account of the order of approximation of $|\alpha x-y+\beta|$ to zero.

Let L be the straight line $\alpha x-y+\beta=0$, where α denotes an arbitrary positive irrational number and β any real number between 0 and 1, and suppose that it passes through no lattice point, (that is the point whose coordinates are integers). Let $(Z_1) = A_0A_1A_2\cdots$, $(Z_2) = B_0B_1B_2\cdots$, where $A_0 = (0,1)$, $B_0 = (0,0)$, be two polygonal lines, convex towards L, whose vertices are all lattice points, and such that there is no lattice point between (Z_1), (Z_2). Next let (Z_3), (Z_4) be the analogous polygonal lines in the left half plane. We call the vertices of (Z_1), (Z_2), (Z_3), (Z_4) the principal approximate points, while the lattice points on the sides the intermediate approximate points.

To construct (Z_1), (Z_2) we proceed as follows. Since the lattice points in the upper half plane nearest to A_0B_0 lie on a parallel line to A_0B_0, we take two consecutive lattice points A', B' on this line, which intercept L, the sense $A'B'$ being the same as A_0B_0. We take also a fixed lattice point H on the same line in the opposite side of A' with respect to B'. Then we can determine a positive integer b' such that $HA' = b_1\cdot B'A'$. Next, if the prolonged portion of A_0A' cut L, then determine two consecutive lattice points A_1, A'' on this line which intercept L, and let $A_0A_1 = a_1\cdot A_0A'$, $A_0A'' = (a_1+1)\ A_0A'$. On the other hand, if the prolonged portion of B_0B' cut L, then determine two consecutive lattice points B_1, B'' on this line which intercept L, and let

1) FUKASAWA, Über Kleinsche geometrische Darstellung des Kettenbuchs, Japanese Journ. of Math., 2 (1925), 101-114.

$B_0B_1 = a_1 \cdot B_0B'$, $B_0B'' = (a_1+1)\, B_0B'$. To distinguish these two cases, we introduce a number τ, which is equal to 1 or 0 according as the first or the second case occurs. Thus we determine as the first step a triple system of integers (a_1, b_1, τ_1).

If the first case occurs, then we proceed similarly, taking B_0, A_1, A'' instead of A_0, B_0, H, and determine the second system (a_2, b_2, τ_2). If the second case occurs, then we take A_0, B_1, B'' instead of A_0, B_0, H, and determine (a_2, b_2, τ_2). In this way we can determine a system of characteristic numbers (a_i, b_i, τ_i), $i = 1, 2, 3, \ldots$

By means of the affin-transformation, which does not change the lattice system as a whole, and the area, we can prove that

$$\alpha = b_1 - \frac{\nu_1}{a_1} + \frac{1}{b_2} - \frac{\nu_2}{a_2} + \frac{1}{b_3} - \frac{\nu_3}{a_3} + \cdots\cdots,$$

$$\beta = (1-\tau_1) - \frac{(1-\tau_2)\nu_1}{1+\alpha_1 a_1} + \frac{(1-\tau_3)\nu_1\nu_2}{(1+\alpha_1 a_1)(1+\alpha_2 a_2)} - \frac{(1-\tau_4)\nu_1\nu_2\nu_3}{(1+\alpha_1 a_1)(1+\alpha_2 a_2)(1+\alpha_3 a_3)} + \cdots\cdots,$$

where $\nu_k = 1$ or -1 according as $\tau_k = 0$ or 1, and

$$\alpha_n = b_{n+1} - \frac{\nu_{n+1}}{a_{n+1}} + \frac{1}{b_{n+2}} - \frac{\nu_{n+2}}{a_{n+2}} + \cdots\cdots.$$

From these geometrical considerations we can prove the following facts.

Let $P = (x, y)$ be a lattice point and put $\lambda(P) = |x(\alpha x - y + \beta)|$, which represents the area of the parallelogram formed by L, the y-axis and two parallel lines to them passing through P. Then:

(1) If P be any intermediate approximate point on the side $P_n P_{n+1}$ of the polygonal lines $(Z_1), \cdots, (Z_4)$, then $\lambda(P) > \lambda(P_n)$, $\lambda(P) > \lambda(P_{n+1})$.

(2) For any principal approximate point P, $\lambda(P) < 1$.

(3) Let P_n be a principal approximate point on (Z_2), and P_m be the principal approximate point (Z_1), which comes just before P_n in the way of construction of (Z_1), (Z_2), and P_l be the lattice point on the side of (Z_2), passing through P_n such that P_lP_n contains no lattice point. Further let P'_n be the lattice point on (Z_1) such that $P_mP'_n$ contains no lattice point, and Q_n be vertex of the parallelogram $P_lP_mP_nQ_n$. Then

$$\text{Mini.}\ (\lambda(P_n), \lambda(P_m), \lambda(P_l), \lambda(Q_n))$$

or

$$\text{Mini.}\ (\lambda(P_n), \lambda(P_m), \lambda(P'_n), \lambda(Q_n)) < \frac{1}{4}.$$

Since Q_n does not remain always at finite for $n \to \infty$, this inequality represents nothing but Minkowski's theorem: There are infinitely many pairs of integers (x, y) which satisfy

$$|x(\alpha x-y+\beta)|<\frac{1}{4}.$$

(4) The necessary and sufficient condition that there exists only a finite number of integers satisfying

$$|x(\alpha x-y+\beta)|<\frac{1}{\mu}, \ (\mu>4)$$

is that there exists an integer n_0 such that for $n>n_0$

$$(a_{2n}, b_{2n}, \tau_{2n})=(1, 1, 0),\ a_{2n+1},\ b_{2n+1}\to\infty,\ b_{2n+1}/a_{2n-1}\to 1,$$

or

$$(a_{2n+1}, b_{2n+1}, \tau_{2n+1})=(1, 1, 0),\ a_{2n},\ b_{2n}\to\infty,\ b_{2n+2}/a_{2n}\to 1.$$

A special case $\beta=1/2$ was first treated by Grace in his paper, Note on a Diophantine Approximation, Proc. London Math. Society, Ser. II, **17** (1918).