34. *A New Elementary Proof of a Theorem of Minkowski.*

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In the following Note, No. 35, Fukasawa extends Klein's geometrical interpretation of continued fraction and proves Minkowski's theorem[1] in a more precise form. I will here add another simple proof based on the same standpoint as in my previous paper on the approximation of an irrational number by rational numbers.[2]

Let a be any positive irrational number, β any real number between 0 and 1, for which there is no pair of integers (x, y) which satisfies $ax-y+\beta=0$. Further let L be the straight line $ax-y+\beta=0$ and $A_0=(0,0)$, $B_0=(-1,0)$. Then construct two polygonal lines $(A)=A_0A_1A_2\cdots$, $(B)=B_0B_1B_2\cdots$, convex towards L, such that their vertices are all lattice points, that is points whose coordinates are integers, and that there is no lattice point in the domain D enclosed by the x-axis, (A) and (B).

Let C_{n+1} be any lattice point on (A) or (B), say (B), and C_n, C_{n+2} two consecutive lattice points on (A), such that the abscissa of C_{n+1} lies between those of C_n and C_{n+2}. If we construct the parallelogram $C_{n-1}C_n C_{n+1} C_{n+1}$, then C_{n-1} must lie below the x-axis; for, the straight line passing through C_{n+1}, parallel to C_nC_{n+2} cuts the x-axis at a point M between A_0, B_0, and if $C_{n+1}M < C_nC_{n+1}$, then there will be a lattice point on the segment $C_{n+1}M$, which lies in the domain D, contrary to the assumption.

Let the coordinates of C_k be (Q_k, P_k), and M_k be the intersection of L with the line passing through C_k parallel to the y-axis, then $aQ_k-P_k+\beta$ is equal to C_kM_k with the sign + or − according as C_k lies above or below L. Therefore from the assumption we have

1) Minkowski, Diophantische Approximationen. See also Remak, Neuer Beweis eines Minkowskischen Satzes, Journal f. Math., 142 (1913); Scherrer, Zur Geometrie der Zahlen, Math. Annalen, 89 (1923).

2) These Proceedings, 2, 1 − 3.

$$\begin{aligned} S_{n+2} &= Q_{n+2}(\alpha Q_{n+2} - P_{n+2} + \beta) \\ - S_{n+1} &= Q_{n+1}(\alpha Q_{n+1} - P_{n+1} + \beta) \\ S_n &= Q_n \quad (\alpha Q_n - P_n + \beta), \end{aligned} \tag{1}$$

where $\quad S_k = |Q_k(\alpha Q_k - P_k + \beta)|.$

Eliminating α, β and observing that the area of the triangle $C_n C_{n+1} C_{n+2}$ is equal to $1/2$, we get

$$\frac{Q_{n+2}-Q_{n+1}}{Q_n} S_n + \frac{Q_{n+2}-Q_n}{Q_{n+1}} S_{n+1} + \frac{Q_{n+1}-Q_n}{Q_{n+2}} S_{n+2} = 1. \tag{2}$$

On the other hand we have the relation

$$\frac{S_{n+2}}{Q_{n+2}} + \frac{S_{n+1}}{Q_{n+1}} = \frac{S_n}{Q_n} + \frac{S_{n-1}}{Q_{n-1}}, \tag{3}$$

when C_{n-1} lies below the line L, for, $C_n M_n - C_{n+2} M_{n+2} = C_{n+1} M_{n+1} + C_{n-1} M_{n-1}$, $Q_{n-1} < 0$. This relation is also deducible from (1) and $-S_{n-1} = Q_{n-1}(\alpha Q_{n-1} - P_{n-1} + \beta)$.

From (2), (3) it follows immediately, taking $Q_{n+2} - Q_{n+1} = Q_n - Q_{n-1}$, $Q_{n-1} < 0$ into account

$$\begin{aligned} 1 &= \frac{Q_{n+2}}{Q_{n+1}} S_{n+1} + \frac{Q_{n+1}}{Q_{n+2}} S_{n+2} + \frac{Q_n}{|Q_{n-1}|} S_{n-1} + \frac{|Q_{n-1}|}{Q_n} S_n \\ &\geqq \left(\frac{Q_{n+2}}{Q_{n+1}} + \frac{Q_{n+1}}{Q_{n+2}} + \frac{Q_n}{|Q_{n-1}|} + \frac{|Q_{n-1}|}{Q_n} \right) \text{Mini.}\,(S_{n-1}, S_n, S_{n+1}, S_{n+2}) \\ &\geqq 4 \text{ Mini.}\,(S_{n-1}, S_n, S_{n+1}, S_{n+2}), \end{aligned} \tag{4}$$

for $\mu + \dfrac{1}{\mu} \geqq 2$, if $\mu > 0$.

In the case where C_{n-1}, C_{n+1}, C_{n+2} lie below L and C_n above L, we have

$$\frac{Q_{n+2}-Q_{n+1}}{Q_n} S_n + \frac{Q_{n+2}-Q_n}{Q_{n+2}} S_{n+1} - \frac{Q_{n-1}-Q_n}{Q_{n+2}} S_{n+2} = 1, \tag{2'}$$

$$\frac{S_{n+2}}{Q_{n+2}} + \frac{S_n}{Q_n} = \frac{S_{n+1}}{Q_{n+1}} + \frac{S_{n-1}}{Q_{n-1}}, \tag{3'}$$

whence follows

$$\begin{aligned} 1 &= \frac{Q_{n+2}}{Q_n} S_n + \frac{Q_n}{Q_{n+2}} S_{n+2} + \frac{Q_{n+1}}{|Q_{n-1}|} S_{n-1} + \frac{|Q_{n-1}|}{Q_{n+1}} S_{n+1}, \\ &\geqq 4 \text{ Mini.}\,(S_{n-1}, S_n, S_{n+1}, S_{n+2}). \end{aligned} \tag{4'}$$

The equality sign in (4), (4′) holds good when and only when C_{n-1}, C_n, C_{n+1}, C_{n+2} coincide all together. Thus we have

$$\text{Mini.}\,(S_{n-1}, S_n, S_{n+1}, S_{n+2}) > \frac{1}{4},$$

which is nothing but Minkowski's theorem in Fukasawa's form.

If C_{n-1} lie above L, then we take instead of C_n, C_{n+2} two consecutive lattice points C'_n, C'_{n+2} on the prolonged portion of $C_n\, C_{n+2}$, which intercept L. In this case $C'_n C'_{n+2} C_{n+1} C_{n-1}$ is also a parallelogram whose area is equal to 1, and

$$\frac{S'_{n+2}}{Q'_{n+2}} + \frac{S'_n}{Q'_n} = \frac{S_{n+1}}{Q_{n+1}} + \frac{S_{n-1}}{Q_{n+1}},$$

whence we get

$$\text{Mini. } (S_{n-1}, S'_n, S_{n+1}, S'_{n+2}) < \frac{1}{4}.$$

If we make use of the fact $Q_{n+2} - Q_{n+1} > Q_n$ only, we have from (2)

$$1 > \frac{Q_{n+2} - Q_{n+1}}{Q_n} S_n + \frac{Q_{n+2} - Q_n}{Q_{n+1}} S_{n+1} > S_n + S_{n+1},$$

whence
$$\text{Mini. } (S_n, S_{n+1}) < \frac{1}{2};$$

this is a precise form of Tchebycheff's theorem, which corresponds to Vahlen's theorem in the theory of continued fraction.

In the second case above treated, where C_{n-1}, C_{n+1}, C_{n+2} lie below L, while C_n above L, we can obtain Mini. $(S_{n-1}, S_n, S_{n+1}, S_{n+2}) < \frac{1}{4\frac{1}{2}}$, if we make use of the fact $Q_{n+2} = Q_{n+1} + Q_n + |\,Q_{n-1}\,| > 2Q_n$.