14. *A Theorem concerning the Dynamical Systems with Slow Variation.*

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1. In certain dynamical systems such as often appear in celestial mechanics the solution is not always convergent[1] if it is expressed in series of a form for practical use. Although its convergence in series of a certain form is actually proved in the way of the theory of analytic functions,[2] the solution in convergent series can not reveal its nature from the dynamical point of view. Thus we have to employ *semi-convergent* series in order to get the information as to the behaviour of the motion. It is proposed to estimate the duration of time in which the solution can be approximated by an adopted representation to a previously given degree. A first step toward attacking this problem was trodden by Birkhoff[3] for a motion near an equilibrium point in which the characteristic numbers are all distinct, different from zero and not connected by any linear homogeneous relation with rational coefficients. The problem of this note[4] is to extend the research to the case in which a certain number of characteristic numbers are zero at least in the first approximation.

2. Consider a function H of $2m+2n$ variables x_i, y_i, ξ_j, η_j and t $(i=1, 2, \ldots\ldots, m;\ j=1, 2, \ldots\ldots, n)$, which, together with its partial derivatives of the first order with respect to any of the variables, satisfy the Lipschitz condition for all the variables ξ_j, η_j and t and for all values of ξ_j, η_j and t in a domain: $|x_i|, |y_i|<D$, $(i=1, 2, \ldots\ldots, m)$ with a finite positive constant D. With this function H associate the following system of differential equations:

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1) H. Poincaré: Acta Math. **13** (1889), 1; *Méthodes Nouvelles de la Mécanique Céleste.* T. **2** (1893).

2) K. Sundman: Acta Math. **36** (1912), 105; T. Levi-Civita: Acta Math. **42** (1918), 99.

3) G. D. Birkhoff: Amer. Jour. Math. **49** (1927), 1; *Dynamical Systems.* (1927) Chap. IV.

4) It is a great pleasure to express my indebtedness to Professor G. D. Birkhoff for his valuable criticisms and suggestions and to the Rockfellor Foundation for enabling me to work at the Harvard University.

$$
(1) \begin{cases} \dfrac{dx_i}{dt}=\dfrac{\partial H}{\partial y_i}, & \dfrac{dy_i}{dt}=-\dfrac{\partial H}{\partial x_i}, \qquad (i=1, 2, \ldots\ldots, m) \\ \dfrac{d\xi_j}{dt}=\dfrac{\partial H}{\partial \eta_j}, & \dfrac{d\eta_j}{dt}=-\dfrac{\partial H}{\partial \xi_j}. \qquad (j=1, 2, \ldots\ldots, n) \end{cases}
$$

Assume that we have a solution $x_i=y_i=0$, $\xi_j=A_j$, $\eta_j=B_j$ $(i=1, 2, \ldots\ldots, m;$ $j=1, 2, \ldots\ldots, n)$ for all values of t, where A_j and B_j are arbitrary constants.

From this assumption we infer that H can be expressed in the domain D of x_i and y_i and for all values of ξ_j, η_j and t in a convergent form:

$$
H=\sum_{k,\,l}^{1,2,\ldots\ldots,m}(a_{kl}x_kx_l+b_{kl}x_ky_l+c_{kl}y_ky_l)+H_1(x_i, y_i, \xi_j, \eta_j, t),
$$

where $a_{kl}=a_{lk}$, $c_{kl}=c_{lk}$ and H_1 is a power series of x_i and y_i beginning with the terms of the third degree in x_i and y_i, the coefficients of which are Lipschitzian functions of all the variables ξ_j, η_j and t for all values of ξ_j, η_j and t. This is the case in which more than two characteristic numbers are zero in the first approximation. We propose to study the nature of the solution of (1) in D.

Assume further that a_{kl}, b_{kl}, c_{kl} are all constants, that the characteristic equation of the quadratic terms of H has only linear invariant factors, that the characteristic numbers are all purely imaginary, and finally that there is no linear homogeneous relation with rational coefficients among the characteristic numbers.

By a linear canonical transformation the system (1) can be transformed into a system for pairs of conjugate imaginary variables x_i' and y_i' with an associated system for real pairs ξ_j and η_j:

$$
(2) \begin{cases} \dfrac{dx_i'}{dt}=\dfrac{\partial F}{\partial y_i'}, & \dfrac{dy_i'}{dt}=-\dfrac{\partial F}{\partial x_i'}, \qquad (i=1, 2, \ldots\ldots, m) \\ \dfrac{d\xi_j}{dt}=\dfrac{\partial F}{\partial \eta_j}, & \dfrac{d\eta_j}{dt}=-\dfrac{\partial F}{\partial \xi_j}, \qquad (j=1, 2, \ldots\ldots, n) \end{cases}
$$

$$
F=\sum_{k}^{1,2,\ldots\ldots,m}\lambda_k x_k'y_k'+F_3+F_4+\cdots\cdots,
$$

where F_3, F_4, ..., are respectively the terms of the third, fourth, ..., degree in x_i' and y_i', of which the coefficients are functions of ξ_j, η_j and t. F is also convergent. λ_k's are the characteristic numbers.

3. Next apply a transformation;

$$
\bar{x}_i=\frac{\partial G}{\partial \bar{y}_i}, \qquad y_i'=\frac{\partial G}{\partial x_i'}, \qquad (i=1, 2, \ldots\ldots, m)
$$

with

$$G=\sum_k^{1,2,\ldots\ldots,m} x_k'\bar{y}_k+G_3+G_4+\cdots\cdots+G_u+G^{(u)}.$$

G_3, G_4, ..., G_u are homogeneous functions of $\bar{y}_i$ and x_i' of degree indicated by the suffixes and $G^{(u)}$ is the remainder term as yet undetermined but does not contain terms of the degree 1, 2, ..., u in $\bar{y}_i$ and x_i'. By the $s-2$ times' repetition of this transformation for $u=3, 4, \ldots\ldots, s$ we can bring F in the form:

$$K=K^{(s)}+R^{(s)}, \qquad R^{(s)}=F_{s+1}+F_{s+2}+\cdots\cdots,$$

$$K^{(s)}=\sum_k^{1,2,\ldots\ldots,m}\lambda_k\bar{x}_k\bar{y}_k+\sum f_{a_1a_2\ldots\ldots a_m}(\bar{x}_1\bar{y}_1)^{a_1}(\bar{x}_2\bar{y}_2)^{a_2}\ldots\ldots(\bar{x}_m\bar{y}_m)^{a_m},$$

where the last sum extends to all integral values, positive or zero, of $a_1, a_2, \ldots\ldots, a_m$ satisfying

$$2<2(a_1+a_2+\cdots\cdots+a_m)\leqq s, \qquad \text{if } s \text{ is even},$$

$$2<2(a_1+a_2+\cdots\cdots+a_m)\leqq s-1, \quad \text{if } s \text{ is odd},$$

and $f_{a_1a_2\ldots\ldots a_m}$'s are functions of ξ_j, η_j and t. And the system (2) is transformed into:

$$(3)\quad\begin{cases}\dfrac{d\bar{x}_i}{dt}=\dfrac{\partial K^{(s)}}{\partial\bar{y}_i}+\dfrac{\partial R^{(s)}}{\partial\bar{y}_i}, & \dfrac{d\bar{y}_i}{dt}=-\dfrac{\partial K^{(s)}}{\partial\bar{x}_i}-\dfrac{\partial R^{(s)}}{\partial\bar{x}_i},\\ & (i=1,2,\ldots\ldots,m)\\ \dfrac{d\bar{\xi}_j}{dt}=\dfrac{\partial K^{(s)}}{\partial\bar{\eta}_j}+\dfrac{\partial R^{(s)}}{\partial\bar{\eta}_j}, & \dfrac{d\bar{\eta}_j}{dt}=-\dfrac{\partial K^{(s)}}{\partial\bar{\xi}_j}-\dfrac{\partial R^{(s)}}{\partial\bar{\xi}_j},\\ & (j=1,2,\ldots\ldots,n)\end{cases}$$

where we have replaced ξ_j and η_j by $\bar{\xi}_j$ and $\bar{\eta}_j$ in order to make the comparison with (1) more comprehensible.

Denote the result of substituting a system of arbitrary constants c_i for $(\bar{x}_i\bar{y}_i)$ $(i=1, 2, \ldots\ldots, m)$ in $K^{(s)}$ by a parenthesis. Consider a curtate system:

$$(4)\quad\begin{cases}\dfrac{d\bar{x}_i}{dt}=\left(\lambda_i+\dfrac{\partial K'}{\partial c_i}\right)\bar{x}_i, & \dfrac{d\bar{y}_i}{dt}=-\left(\lambda_i+\dfrac{\partial K'}{\partial c_i}\right)\bar{y}_i, \quad (i=1,2,\ldots\ldots,m)\\ \dfrac{d\bar{\xi}_j}{dt}=\left(\dfrac{\partial K'}{\partial\bar{\eta}_j}\right), & \dfrac{d\bar{\eta}_j}{dt}=-\left(\dfrac{\partial K'}{\partial\bar{\xi}_j}\right), \qquad (j=1,2,\ldots\ldots,n)\end{cases}$$

in which K' is a finite power series arranged in ascending powers of the constants $c_1, c_2, \ldots\ldots, c_m$ beginning with the terms of the second degree, the coefficients of the various powers of c_i's being Lipschitzian functions of $\bar{\xi}_j$, $\bar{\eta}_j$ and t for all these variables. The system of the form (4) may be said to be *normalised.*

Suppose that we can find an exact solution of the normalised associated curtate system:

$$(5)\quad \frac{d\bar{\xi}_i}{dt}=\left(\frac{\partial K'}{\partial\bar{\eta}_j}\right),\qquad \frac{d\bar{\eta}_j}{dt}=-\left(\frac{\partial K'}{\partial\bar{\xi}_j}\right).\qquad (j=1,\,2,\,\ldots\ldots,\,n)$$

Denote by $x_i(t)$, $y_i(t)$, $\xi_j(t)$, $\eta_j(t)$ the solution of the system of differential equations (1) with a certain initial condition with the restriction that $x_iy_i=c_i$ for $t=t_0$. Let $\bar{x}_i(t)$, $\bar{y}_i(t)$, $\bar{\xi}_j(t)$, $\bar{\eta}_j(t)$ be the corresponding solution of the system (4) with the same initial condition and the same restriction. Then the relations of the form:

$$|x_i(t)-\bar{x}_i(t)|,\ |y_i(t)-\bar{y}_i(t)|,\ |\xi_j(t)-\bar{\xi}_j(t)|,\ |\eta_j(t)-\bar{\eta}_j(t)|<\delta,$$
$$(i=1,\,2,\,\ldots\ldots,\,m;\quad j=1,\,2,\,\ldots\ldots,\,n)$$

hold with a previously assigned positive number δ in the time interval:

$$|t-t_0|<\text{Min.}\begin{pmatrix}\dfrac{1}{2m(s-5)N\varepsilon_0^{s-5}}, & \sqrt{\dfrac{\delta}{2^{s-4}mNL\varepsilon_0^{s-4}}}, & \\[2ex] \dfrac{\delta}{L_1\varepsilon_0^{2s+2}}, & \dfrac{\delta}{L_2\varepsilon_0^{2s+3}}, & \dfrac{\delta}{L_3\varepsilon_0^{2s+1}},\end{pmatrix},$$

where ε_0 is the value of $\varepsilon=(\sum_k^{1,2,\ldots\ldots,m}\bar{x}_k\bar{y}_k)^{\frac{1}{2}}$ for $t=t_0$ and L, L_1, L_2, L_3 and N are finite positive constants depending on ε_0, c_i, s and on the functional form of F but neither vanishing nor becoming infinite when ε_0 tends to zero. Or, if $\delta=O(\varepsilon_0^{\varkappa})$ in Landau's notation, then

$$|x_i(t)-\bar{x}_i(t)|,\ |y_i(t)-\bar{y}_i(t)|,\ |\xi_j(t)-\bar{\xi}_j(t)|,\ |\eta_j(t)-\bar{\eta}_j(t)|=O(\varepsilon_0^{\varkappa}),$$

in the time interval contained between the limits extending on both sides of t_0:

$$\pm|t-t_0|=O(\varepsilon_0^{\frac{\varkappa-s+4}{2}})\qquad\text{for}\qquad \varkappa>-s+6,$$

or

$$\pm|t-t_0|=O(\varepsilon_0^{-s+5})\qquad\text{for}\qquad \varkappa<-s+6.$$

4. Especially if the system (5) is satisfied by *quasi-periodic* functions of Bohl and Esclangon,[1] then the solution of (1) can be approximated by a stable motion expressed by quasi-periodic functions in the domain just mentioned to the degree of approximation δ.

If (5) admits the point $\xi_j=A_j$, $\eta_j=B_j$, $(j=1,\,2,\,\ldots\ldots,\,n)$ as a *centre*[2] in the sense of Poincaré, then (1) is also approximated by a stable solution. Vranceanu's case[3] is a further specification of this corollary.

1) P. Bohl: Dorpat. Dissertation (1893); Crelle J. **131** (1906), 268; E. Esclangon: Paris. Thèse (1904); Ann. Obs. Bordeaux (1904); C. R. **135** (1902), 891; **137** (1903), 305; H. Bohr: Acta Math. **46** (1925), 101.

2) H. Poincaré: Jour. de Math. [iv] **1** (1885), 167; Oeuvres. T. **1**.

3) C. Vranceanu: Atti Accademia Lincei. Rendiconti. [vi] **7** (1928), 630.