PAPERS COMMUNICATED

49. *A Non-Homogeneous Ergodic Theorem.*

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There are two ergodic theorems, one of which is the mean ergodic theorem due to von Neumann and the other is the Birkhoff's ergodic theorem.[1] These theorems treat of the convergence of the arithmetic mean

$$(1) \qquad M_n(f, T) = \frac{f(TP)+f(T^2P)\cdots+f(T^nP)}{n}$$

in the limit in the mean or in the limit in the ordinary sense, where T and its inverse T^{-1} are the measure preserving transformation and P is a point in the space Ω, in which the Lebesgue measure is defined and $|\Omega| < \infty$.[2]

This paper is to study the almost everywhere convergence of the series[3]

$$(2) \qquad \frac{f(TP)}{1}+\frac{f(T^2P)}{2}+\cdots+\frac{f(T^nP)}{n}+\cdots,$$

where

$$(3) \qquad \int_\Omega f(P)\,dP=0.$$

The almost everywhere convergence of the series (2) implies that of (1). The condition (3) is necessary for the almost everywhere convergence of (2), which is seen by the Abel's lemma and the mean ergodic theorem.

If T is of the mixture type, then

$$\lim_{n\to\infty} |T^nA\cdot B| = |A|\cdot|B|/|\Omega|$$

for any measurable set A and B in Ω. We will suppose that

$$|T^nA\cdot B| = \frac{|A|\cdot|B|}{|\Omega|} + o\left(\frac{1}{\log^3|n|}\right)$$

uniformly for all measurable set A and B such as $A \supseteqq B$. In this case we will call that T is of the uniform mixture type.

The object of this paper is to prove the following theorem.

Theorem 1. If $f(P)\,\varepsilon L^2$ in Ω, (3) *holds and T is of the measure preserving transformation of the uniform mixture type, then the series* (3) *converges almost everywhere.*

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1) For the detail, see E. Hopf, Ergodentheorie and the Wiener's papers in the American Journal of Math., **5** (1938) and the Duke Mathematical Journal, **5** (1939).

2) $|A|$ means the measure of A.

3) The author owes this problem to Mr. Fukamiya.

Proof. Let N be any positive integer and let us put

$$J_N(P)=\underset{1\leq m\leq N}{\text{l. u. b.}}\sum_{n=1}^{m}\frac{f(T^nP)}{n}=\sum_{n=1}^{n(P)}\frac{f(T^nP)}{n},$$

where $n(P)$ is an integral-valued measurable function and $1\leqq n(P)\leqq N$. We can write

$$J_N=\int_\Omega J_N(P)\,dP=\sum_{n=1}^{N}\int_{\Omega_n}\frac{f(T^nP)}{n}\,dP,$$

where $\Omega_n\geqq\Omega_{n+1}\ (n=1,2,\dots,N-1)$.

If we denote by $\varphi_n(P)$ the characteristic function of Ω_n, then

$$\begin{aligned}J_N&=\sum_{n=1}^{N}\frac{1}{n}\int_\Omega\varphi_n(P)f(T^nP)\,dP\\&=\sum_{n=1}^{N}\frac{1}{n}\int_\Omega\varphi_n(T^{-n}P)\,f(P)\,dP=\int_\Omega\Big(\sum_{n=1}^{N}\frac{1}{n}\varphi_n(T^{-n}P)\Big)f(P)\,dP.\end{aligned}$$

By (3) we have

$$J_N=\int_\Omega\Big(\sum_{n=1}^{N}\frac{1}{n}\varphi_n(T^{-n}P)-c_N\Big)f(P)\,dP,$$

where c_N is a constant and is determined later. By the Schwarz's inequality we get

$$J_N\leqq\Big[\int_\Omega\Big(\sum_{n=1}^{N}\frac{1}{n}\varphi_n(T^{-n}P)-c_N\Big)^2dP\Big]^{\frac12}\Big[\int_\Omega|f(P)|^2dP\Big]^{\frac12}.$$

If we denote by $K^{\frac12}$ the first factor of the right hand side, then we have

$$\begin{aligned}K&=\int_\Omega\Big(\sum_{n=1}^{N}\frac{1}{n}\varphi_n(T^{-n}P)-c_N\Big)^2dP\\&=\int_\Omega\Big[\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{1}{mn}\varphi_m(T^{-m}P)\varphi_n(T^{-n}P)\\&\qquad\qquad-2c_N\sum_{n=1}^{N}\varphi_n(T^{-n}P)+c_N^2\Big]dP\\&=\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{1}{mn}\int_\Omega\varphi_m(T^{-m}P)\varphi_n(T^{-n}P)\,dP\\&\qquad\qquad-2c_N\sum_{n=1}^{N}\frac{1}{n}\int_\Omega\varphi_n(T^{-n}P)\,dP+c_N^2\,|\Omega|\\&=\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{1}{mn}\int_\Omega\varphi_m(T^{n-m}P)\varphi_n(P)\,dP\\&\qquad\qquad-2c_N\sum_{n=1}^{N}\frac{1}{n}\int_\Omega\varphi_n(P)\,dP+c_N^2\,|\Omega|\\&=\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{1}{mn}\,|T^{n-m}\Omega_m\cdot\Omega_n|-2c_N\sum_{n=1}^{N}\frac{|\Omega_n|}{n}+c_N^2\,|\Omega|,\end{aligned}$$

where $T^{n-m}\Omega_m$ denotes the set of points $T^{n-m}P$ for $P \varepsilon \Omega_m$. Since T is of the uniform mixture type, we have

$$|T^{n-m}\Omega_m \cdot \Omega_n| = \frac{|\Omega_m|\cdot|\Omega_n|}{|\Omega|} + \frac{A_{m,n}}{\log^3(|m-n|+2)},$$

where $A_{m,n}$ is bounded. Therefore

$$\begin{aligned} K &\leqq \sum_{m=1}^{N}\sum_{n=1}^{N} \frac{1}{mn}\frac{|\Omega_m|\cdot|\Omega_n|}{|\Omega|} - 2c_N \sum_{n=1}^{N}\frac{|\Omega_n|}{n} \\ &\qquad + c_N^2|\Omega| + \sum_{m=1}^{N}\sum_{n=1}^{N}\frac{A_{m,n}}{\log^3(|m-n|+2)} \\ &= \Big(\sum_{n=1}^{N}\frac{1}{n}\frac{|\Omega_n|}{|\Omega|} - c_N\Big)^2|\Omega| + \sum_{m=1}^{N}\sum_{n=1}^{N}\frac{A_{m,n}}{mn\log^3(|m-n|+2)}. \end{aligned}$$

If we take

$$c_N = \frac{1}{|\Omega|}\sum_{n=1}^{N}\frac{|\Omega_n|}{n},$$

then

$$\begin{aligned} K &\leqq \sum_{m=1}^{N}\sum_{n=1}^{N}\frac{A_{m,n}}{mn\log^3(|m-n|+2)} \leqq A_1\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{1}{mn\log^3(|m-n|+2)} \\ &\leqq A_2\sum_{\lambda=1}^{\infty}\frac{1}{\log^3(\lambda+2)}\sum_{|m-n|=\lambda}\frac{1}{mn} \leqq A_3\sum_{\lambda=1}^{\infty}\frac{1}{\lambda\log^2(\lambda+2)}. \end{aligned}$$

Thus K is bounded. Therefore $J_N(P)$ is almost everywhere upper bounded. Similarly $J_N(P)$ is almost everywhere lower bounded. The almost everywhere convergence of (3) is proved considering

$$\underset{N_0\leq m\leq N}{\text{l. u. b.}}\sum_{n=N_0}^{m}\frac{f(T^nP)}{n}, \quad \underset{N_0\leq m\leq N}{\text{g. l. b.}}\sum_{n=N_0}^{m}\frac{f(T^nP)}{n}.$$

Theorem 2. In Theorem 1, if we take off the condition (3), *then the series*

$$\sum_{n=1}^{\infty}\frac{f(T^nP)-M_n(T,P)}{n},$$

converges almost everywhere, $M_n(T,P)$ being defined by (1).

Proof is done similarly as Theorem 1. Let us put

$$J'_N(P) = \underset{1\leq m\leq n}{\text{l. u. b.}}\sum_{n=1}^{m}\frac{f(T^nP)-M_n(T,P)}{n}.$$

We have

$$\begin{aligned} \int_\Omega J'_N(P)\,dP &= \int_\Omega \sum_{n=1}^{n(P)}\frac{f(T^nP)-M_n(T,P)}{n}\,dP \\ &= \sum_{n=1}^{N}\int_\Omega \frac{1}{n}\{f(T^nP)-M_n(T,P)\}\,\varphi'_n(P)\,dP, \end{aligned}$$

where $n'(P)$ and $\varphi'_n(P)$ is defined as $n(P)$ and $\varphi_n(P)$ in the proof of Theorem 1.

$$\int_\Omega J'_N(P)dP=\sum_{n=1}^N \frac{1}{n}\int_\Omega \varphi'_n(P)f(T^nP)-\sum_{n=1}^N \frac{1}{n}\int_\Omega \varphi'_n(P)M_n(T,P)\,dP$$

$$=\sum_{n=1}^N \frac{1}{n}\int_\Omega \varphi'_n(T^{-n}P)f(P)\,dP-\sum_{n=1}^N \frac{1}{n^2}\sum_{m=1}^N\int_\Omega \varphi'_n(T^{-m}P)f(P)\,dP$$

$$=\int_\Omega\Big[\sum_{n=1}^N \frac{1}{n}\varphi'_n(T^{-n}P)-\sum_{n=1}^N \frac{1}{n^2}\sum_{m=1}^N \varphi'_n(T^{-m}P)\Big]f(P)\,dP$$

$$\leqq\Big\{\int_\Omega\Big[\sum_{n=1}^N \frac{1}{n}\varphi'_n(T^{-n}P)-\sum_{n=1}^N \frac{1}{n^2}\sum_{m=1}^N \varphi'_n(T^{-m}P)\Big]^2dP\Big\}^{\frac{1}{2}}$$

$$\cdot\Big\{\int_\Omega |f(P)|^2dP\Big\}^{\frac{1}{2}}.$$

If we define Ω'_n as Ω_n in the proof of Theorem 1, then we have

$$\int_\Omega\Big[\sum_{n=1}^N \frac{1}{n^2}\sum_{m=1}^N\{\varphi'_n(T^{-n}P)-\varphi'_n(T^{-m}P)\}\Big]^2dP$$

$$=\sum_{m=1}^N\sum_{\nu=1}^N \frac{1}{n^2\nu^2}\sum_{m=1}^N\sum_{\mu=1}^N\{|T^{-n}\Omega'_n\cdot T^{-\nu}\Omega'_n|-|T^{-m}\Omega'_n\cdot T^{-\nu}\Omega'_\nu|$$

$$-|T^{-n}\Omega'_n\cdot T^{-\mu}\Omega'_\nu|+|T^{-m}\Omega'_n\cdot T^{-\mu}\Omega'_\nu|\},$$

which is bounded. Thus the theorem is proved.