117. *On Linear Functions of Abelian Groups.*

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(Comm. by M. FUJIWARA, M.I.A., Dec. 12, 1940.)

1. Let a set G of elements $a_i, b_i, c_i \dots\dots$, $(i=1, 2, \dots, n)$, satisfy the following axioms:

(1) There exists an operation in G which associates with each class of n elements $a_1, a_2, \dots, a_n$ of G an $(n+1)$-th element a_0 of G, i.e.,

$$(a_1, a_2, \dots, a_n) = a_0 .$$

(2) The operation satisfies the associative law

$$\big((a_1, a_2, \dots, a_n), \quad (b_1, b_2, \dots, b_n), \quad \dots\dots, \quad (d_1, d_2, \dots, d_n)\big)$$
$$= \big((a_1, b_1, \dots, d_1), \quad (a_2, b_2, \dots, d_2), \quad \dots\dots, \quad (a_n, b_n, \dots, d_n)\big) .$$

(3) There exists at least one unit element 0 such that

$$(0, 0, \dots, 0) = 0 .$$

(4) For any given elements a, b, each of the equations

$$(x, a, 0, \dots, 0) = b \quad \text{and} \quad (a, y, 0, \dots, 0) = b$$

has a unique solution with respect to the unknown x and y respectively.

We know[1] that the mean value of n real numbers $x_1, x_2, \dots, x_n$, say,

$$(x_1, x_2, \dots, x_n) = \frac{x_1 + x_2 + \dots + x_n}{n}$$

satisfies the above axioms (1), (2), (4), and, in place of (3), the axiom: "every element is unit element," and the symmetrical condition. We shall consider the converse problem which is answered as follows:

Theorem[2]. *The set G forms an abelian group with respect to the new operation which is defined by the equation*

$$x + y = (a, b, 0, \dots, 0) ,$$

assuming that $x = (a, 0, 0, \dots, 0)$ *and* $y = (0, b, 0, \dots, 0)$.

Moreover, the operation $(x_1, x_2, \dots, x_n)$ *of G is expressed as a linear function of* $x_1, x_2, \dots, x_n$ *such that*

$$(x_1, x_2, \dots, x_n) = A_1 x_1 + A_2 x_2 + \dots + A_n x_n ,$$
$$A_i A_k = A_k A_i , \qquad (i, k = 1, 2, \dots, n) ,$$

1) This result is due to the remark of Mr. M. Takasaki.

2) K. Toyoda, On Axioms of Mean Transformations and Automorphic Transformations of Abelian Groups, Tôhoku Math. Journal, **47** (1940), pp., 239–251.

K. Toyoda, On Affine Geometry of Abelian Groups, Proc. **16** (1940), 161–164.

where A_1, A_2 denote automorphisms of G and $A_3, A_4, \dots, A_n$ denote homomorphisms of G.

2. In the following lines, we shall give the proof for the above theorem.

Lemma 1. If we suppose

$$(p, 0, 0, \dots, 0) = (b, r, 0, \dots, 0),$$
$$(q, 0, 0, \dots, 0) = (b, s, 0, \dots, 0),$$

then it follows that

$$(p, s, 0, \dots, 0) = (q, r, 0, \dots, 0).$$

Proof.

$$\big((p, s, 0, \dots, 0),\ \ 0, 0, \dots, 0\big)$$
$$= \big((p, s, 0, \dots, 0),\ \ (0, 0, 0, \dots, 0),\ \ \dots\dots,\ \ (0, 0, 0, \dots, 0)\big)$$
$$= \big((p, 0, 0, \dots, 0),\ \ (s, 0, 0, \dots, 0),\ \ 0, 0, \dots, 0\big)$$
$$= \big((b, r, 0, \dots, 0),\ \ (s, 0, 0, \dots, 0),\ \ 0, 0, \dots, 0\big)$$
$$= \big((b, s, 0, \dots, 0),\ \ (r, 0, 0, \dots, 0),\ \ 0, 0, \dots, 0\big)$$
$$= \big((q, 0, 0, \dots, 0),\ \ (r, 0, 0, \dots, 0),\ \ 0, 0, \dots, 0\big)$$
$$= \big((q, r, 0, \dots, 0),\ \ 0, 0, \dots, 0\big).$$

Hence, we get

Theorem 1. The set G forms an abelian group with respect to the new operation

$$(a, 0, 0, \dots, 0) + (0, b, 0, \dots, 0) = (a, b, 0, \dots, 0).$$

Proof. If we put

$$x = (a, 0, 0, \dots, 0) = (0, a', 0, \dots, 0),$$
$$y = (b, 0, 0, \dots, 0) = (0, b', 0, \dots, 0),$$
$$z = (c, 0, 0, \dots, 0) = (0, c', 0, \dots, 0),$$

then, by means of Lemma 1,

$$x + y = (a, b', 0, \dots, 0) = (a', b, 0, \dots, 0)$$
$$= y + x.$$

Also, putting

$$x + y = (p, 0, 0, \dots, 0) = (0, p', 0, \dots, 0),$$
$$y + z = (q, 0, 0, \dots, 0) = (0, q', 0, \dots, 0),$$

we have

$$(p, 0, 0, \dots, 0) = (b, a', 0, \dots, 0),$$
$$(q, 0, 0, \dots, 0) = (b, c', 0, \dots, 0),$$

whence, by means of Lemma 1,

$$(p, c', 0, \dots, 0) = (q, a', 0, \dots, 0).$$

Therefore, we obtain

$$(x+y)+z=(p, c', 0, \dots, 0)=(a, q', 0, \dots, 0)$$
$$=x+(y+z).$$

Consequently, we can prove that G forms an abelian group.

3. Next, we shall show that the operation of G becomes a linear function in the space G.

Lemma 2.

$$(x_1, x_2, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)+(0, 0, \dots, 0, x_i, 0, \dots, 0)$$
$$=(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n).$$

Proof. If we put

$$x_k=(a_k, 0, 0, \dots, 0), \quad \text{for} \quad k=1, 2, .., n \quad \text{and} \quad k \neq i,$$

and $\quad x_k=(0, b_k, 0, \dots, 0), \quad \text{for} \quad k=i,$

then it follows that

$$(x_1, x_2, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$$
$$=\big((a_1, 0, 0, \dots, 0), \quad (a_2, 0, 0, \dots, 0), \quad \dots\dots, \quad (a_{i-1}, 0, 0, \dots, 0),$$
$$(0, 0, 0, \dots, 0), \quad (a_{i+1}, 0, 0, \dots, 0), \quad \dots\dots, \quad (a_n, 0, 0, \dots, 0)\big)$$
$$=\big((a_1, a_2, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n), \quad 0, 0, \dots, 0\big),$$

and

$$(0, 0, \dots, 0, x_i, 0, \dots, 0)$$
$$=\big(0, 0, \dots, 0, \quad (0, b_i, 0, \dots, 0), \quad 0, \dots, 0\big)$$
$$=\big(0, \quad (0, 0, \dots, 0, b_i, 0, \dots, 0), \quad 0, 0, \dots, 0\big).$$

Therefore, we get

$$(x_1, x_2, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)+(0, 0, \dots, 0, x_i, 0, \dots, 0)$$
$$=\big((a_1, a_2, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n), \quad (0, 0, \dots, 0, b_i, 0, \dots, 0), \quad 0, 0, \dots, 0\big)$$
$$=\big((a_1, 0, 0, \dots, 0), \quad (a_2, 0, 0, \dots, 0), \quad \dots\dots, \quad (a_{i-1}, 0, 0, \dots, 0),$$
$$(0, b_i, 0, \dots, 0), \quad (a_{i+1}, 0, 0, \dots, 0), \quad \dots\dots, \quad (a_n, 0, 0, \dots, 0)\big)$$
$$=(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n).$$

Lemma 3.

$$(0, 0, \dots, 0, x_i, 0, \dots, 0)+(0, 0, \dots, 0, y_i, 0, \dots, 0)$$
$$=(0, 0, \dots, 0, x_i+y_i, 0, \dots, 0).$$

Proof. If we put

$$x_i=(a_i, 0, 0, \dots, 0),$$
$$y_i=(0, b_i, 0, \dots, 0),$$

then it follow that

$$(0, 0, \dots, 0, x_i, 0, \dots, 0)+(0, 0, \dots, 0, y_i, 0, \dots, 0)$$
$$=\big(0, 0, \dots, 0, \quad (a_i, 0, 0, \dots, 0), \quad 0, \dots, 0\big)$$

$$+\Big(0, 0, \ldots, 0, \quad (0, b_i, 0, \ldots, 0), \quad 0, \ldots, 0\Big)$$

$$=\Big((0, 0, \ldots, 0, a_i, 0, \ldots, 0), \quad 0, 0, \ldots, 0\Big)$$

$$+\Big(0, \quad (0, 0, \ldots, 0, b_i, 0, \ldots, 0), \quad 0, 0, \ldots, 0\Big)$$

$$=\Big((0, 0, \ldots, 0, a_i, 0, \ldots, 0), \quad (0, 0, \ldots, b_i, 0, \ldots, 0), \quad 0, 0, \ldots, 0\Big)$$

$$=\Big(0, 0, \ldots, 0, \quad (a_i, b_i, 0, \ldots, 0), \quad 0, \ldots, 0\Big)$$

$$=(0, 0, \ldots, 0, x_i+y_i, 0, \ldots, 0) .$$

Consequently, we obtain

Theorem 2. The operation of G is expressed as a linear function in the space G such that

$$(x_1, x_2, \ldots, x_n)=\sum_{k=1}^{n} A_k x_k ,$$

$$A_i A_k = A_k A_i , \qquad (i, k=1, 2, \ldots, n) ,$$

where A_1, A_2 denote automorphisms of G and $A_2, A_4, \ldots, A_n$ denote homorphisms of G.

Theorem 3[1]. *Let a function $(x_1, x_2, \ldots, x_n)$ of n real numbers $x_1, x_2, \ldots, x_n$ satisfy the above four axioms* (1), (2), (3), (4) *and be continuous*[2] *with respect to every component x_i in a domain $|x| \leqq a$, $(i=1, 2, \ldots, n)$. Then, the function is given by the expression such that*

$$(x_1, x_2, \ldots, x_n)=f^{-1}\big(\lambda_1 f(x_1)+\lambda_2 f(x_2)+\cdots+\lambda_n f(x_n)\big) ,$$

assuming that λ_i, $(i=1, 2, \ldots, n)$, denote some real numbers under the additional conditions $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, and that $y=f(x)$ denotes a one-valued continuous function of a real number x and possesses a one-valued inverse continuous function $x=f^{-1}(y)$.

4. Let a set G of elements $a, b, c, \ldots,$ satisfy the following axioms.

(1) For every pair of elements a, b, the product (a, b) of G determines uniquely a third element c in G, i. e., $(a, b)=c$.

(2) For two given elements a (or b) and c, the equation $(a, b)=c$ can be solved by b (or a) in G.

(3) Let a, b, p and q be arbitrary four elements in G. If the simultaneous equations

$$(p, b)=(a, r) ,$$

$$(q, b)=(a, s) ,$$

hold, then it follows that

$$(p, s)=(q, r) .$$

Then, we have

1) K. Toyoda, loc. cit.

2) Van der Waerden, Vorlesungen über kontinuierliche Gruppen, (1929).
L. Pontrjagin, Topological groups, (1939).

Theorem 4[1]. *The set G forms an abelian group with respect to the new operations*

$$(x, b)+(a, y)=(x, y),$$

where a, b denote any elements in G.

Proof. We can prove in the same way as Theorem 1.

Corollary. Let a continuous function (x, y) *of two real variables* x, y *satisfy the above three axioms* (1), (2), (3). *Then, the function* (x, y) *is expressed as follows:*

$$(x, y)=f\big(\varphi(x)+\psi(y)\big),$$

where f, φ, ψ *denote topological transformations.*

Theorem 5[2]. *If we introduce two new operations*

$$(x, b)+(a, y)=(x, y) \quad and \quad (x, b') * (a', y)=(x, y)$$

into G, then we have

$$x * y=x+y-(a', b').$$

Proof. By the definitions, we have

$$(x, b')=(x, b)+(a, b'),$$
$$(a', y)=(a', b)+(a, y),$$

whence

$$\begin{aligned}(x, b') * (a', y)&=(x, y)=(x, b)+(a, y)\\ &=\{(x, b')-(a, b')\}+\{(a', y)-(a', b)\}\\ &=(x, b')+(a', y)-\{(a', b)+(a, b')\}\\ &=(x, b')+(a', y)-(a', b').\end{aligned}$$

(4) Let a, b, p and q be arbitrary four elements in G. If the simultaneous equations

$$(p, a)=(b, r),$$
$$(q, a)=(b, s),$$

hold, then it follows that

$$(p, s)=(q, r).$$

Then, we have

Theorem 6. The set G forms an abelian group with respect to the new operation

$$(a, 0)+(0, b)=(a, b).$$

Moreover, the operation (x, y) *of G becomes a linear function of* x, y *in the space G such that*

$$(x, y)=Ax+By,$$

where A and B denotes automorphism of G. But, A and B are not necessarily commutative.

Proof. We can proceed in the same way as Theorems 1, 2, 3.

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1), 2) K. Toyoda, loc. cit.