32. Notes on Infinite Product Measure Spaces, I.

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The purpose of this note is to give a simple and new proof to the existence of an independent product measure on a Cartesian infinite product space.

Let $\{(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma) \mid \gamma \in \Gamma\}$ be a family of measure spaces satisfying $m^\gamma(\Omega^\gamma)=1$ for each $\gamma \in \Gamma$, where we mean by a *measure space* $(\Omega, \mathfrak{B}, m)$ a triple of a space Ω (without topology), a Borel field $\mathfrak{B}$ of subsets B of Ω, and a countably additive measure $m(B)$ defined on $\mathfrak{B}$ (with $0 < m(\Omega) < \infty$). We shall first define a measure space $(\Omega^*, \mathfrak{B}^*, m^*)$ which we call the *independent product measure space* of the family $\{(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma) \mid \gamma \in \Gamma\}$.

The space Ω^*, which is symbolically denoted as

(1) $$\Omega^* = \mathbf{P}_{\gamma \in \Gamma} \Omega^\gamma$$

is the set of all Γ-sequences (or functions defined on Γ)

(2) $$\omega^* = \{\omega^\gamma \mid \gamma \in \Gamma\}$$

such that $\omega^\gamma \in \Omega^\gamma$ for each $\gamma \in \Gamma$.

A subset R^* of Ω^* is called *rectangular* if it is of the form:

(3) $$R^* = B^{\gamma_1} \times \cdots \times B^{\gamma_n} \times \mathbf{P}_{\gamma \in \Gamma - \{\gamma_1, \dots, \gamma_n\}} \Omega^\gamma$$

where $B^{\gamma_i} \in \mathfrak{B}^{\gamma_i}$, $i=1, \dots, n$, and $\{\gamma_1, \dots, \gamma_n\}$ is an arbitrary finite system of elements from Γ. R^* is, by definition, the set of all $\omega^* = \{\omega^\gamma \mid \gamma \in \Gamma\} \in \Omega^*$ such that $\omega^{\gamma_i} \in B^{\gamma_i}$ for $i=1, \dots, n$. The family of all rectangular sets R^* of Ω^* is denoted by $\mathfrak{R}^*$.

Further, a subset E^* of Ω^* is called *elementary* if it is of the form:

(4) $$E^* = \mathbf{U}_{i=1}^n R_i^*$$

where $R_i^* \in \mathfrak{R}^*$ for $i=1, \dots, n$. We may assume that the R_i^* in (4) are mutually disjoint. This follows from the fact that the intersection of two rectangular set of Ω^* is again rectangular, and that the complementary of a rectangular set of Ω^* is expressible as the union of a finite number of mutually disjoint rectangular sets of Ω^*. The family of all elementary sets E^* of Ω^* is denoted by $\mathfrak{E}^*$. It is clear that $\mathfrak{E}^*$ is a field.

We shall next define a set function $m^*(R^*)$ on $\mathfrak{R}^*$ by

(5) $$m^*(R^*) = m^{\gamma_1}(B^{\gamma_1}) \times \cdots \times m^{\gamma_n}(B^{\gamma_n})$$

if R^* is of the form (3), and then $m^*(E^*)$ on $\mathfrak{E}^*$ by

(6) $$m^*(E^*) = \sum_{i=1}^n m^*(R_i^*)$$

if E^* is of the form (4) and if the R_i^* are mutually disjoint. It is easy to see that, although the expressions (3) and (4) are not unique for a given set R^* or E^*, the values $m^*(R^*)$ and $m^*(E^*)$ defined by (5) and (6) are uniquely determined. Further, it is clear that, if an elementary set E^* happens to be a rectangular set R^*, then the two definitions (5) and (6) give the same value. It is also clear that $m^*(E^*)$ is finitely additive on the field $\mathfrak{E}^*$.

A measure space $(\Omega^*, \mathfrak{B}^*, m^*)$ defined on the Cartesian product space Ω^* is called the *independent product measure space* of the family $\{(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma) \mid \gamma \in \Gamma\}$, if $\mathfrak{B}^*$ is the Borel field generated by $\mathfrak{E}^*$, and if $m^*(B^*)$ coincides with $m^*(E^*)$ on $\mathfrak{E}^*$. This fact is expressed symbolically as

$$(7) \qquad (\Omega^*, \mathfrak{B}^*, m^*) = \mathbf{P}_{\gamma \in \Gamma} \otimes (\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma) .$$

Then the main purpose of this paper is to give a proof tc the following

Theorem. Let $\{(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma) \mid \gamma \in \Gamma\}$ *be a family of measure spaces satisfying* $m^\gamma(\Omega^\gamma)=1$ *for each* $\gamma \in \Gamma$. *Then there exists an independent product measure space* $(\Omega^*, \mathfrak{B}^*, m^*) = \mathbf{P}_{\gamma \in \Gamma} \otimes (\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma)$.

This theorem was proved by A. Kolmogoroff[1] in case when each measure space $(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma)$, $\gamma \in \Gamma$, is the Lebesgue measure space defined on the closed interval (0,1) (i. e., when Ω^γ is the closed interval (0,1), $\mathfrak{B}^\gamma$ is the Borel field of all Lebesgue measurable subsets B^γ of (0,1), and $m^\gamma(B^\gamma)$ is the ordinary Lebesgue measure on $\mathfrak{B}^\gamma$). More general cases were discussed by J. L. Doob[2] by reducing them to the case of A. Kolmogoroff. The proof of A. Kolmogoroff, however, is based on the fact that the Cartesian product space Ω^* is compact (=bicompact) with respect to the ordinary weak topology of the product space whenever each factor space Ω^γ, $\gamma \in \Gamma$, is compact. In the following lines we shall give a simple proof to our theorem which is completely free from the notion of topology[3].

Proof. It is sufficient to show that the finitely additive measure $m^*(E^*)$ defined on the field $\mathfrak{E}^*$ can be extended to a countably additive measure $m^*(B^*)$ defined on the Borel field B^* generated by $\mathfrak{E}^*$. In order to show this, it suffices to verify that[4]

$$(8) \qquad E_k^* \in \mathfrak{E}^*, \quad m^*(E_k^*) \geqq \delta > 0, \quad E_k^* \geqq E_{k+1}^*, \quad k=1, 2, \dots$$

implies $\bigcap_{k=1}^{\infty} E_k^* \neq \Theta$, where Θ denotes the empty set. Since every rectangular, and hence every elementary, set is determined by a finite

1) A. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin, 1933.

2) J. L. Doob, Stochastic processes depending on an integral valued parameter, Trans. Amer. Math. Soc. **44** (1938).

3) Z. Lomnicki and S. Ulam, Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités I. Variables indépendantes, Fund. Math. **23** (1934), 237–278.

In this paper it is attempted to prove our theorem without appealing to the notions of topology, but unfortunately the proof given here contains a mistake.

4) See, for example, E. Hopf, Ergodentheorie, Berlin, 1937, p. 2.

number of coordinates[1], there exists a sequence of indices $\{\gamma_n \mid n=1, 2, \dots\}$ from Γ and an increasing sequence of positive integers $\{n_k \mid k=1, 2, \dots\}$ such that the set E_k^* is determined by the coordinates $\{\gamma_1, \gamma_2, \dots, \gamma_{n_k}\}$ for $k=1, 2, \dots$

Let us now put

$$(9) \qquad \Omega^{*(n)} = \mathbf{P}_{\gamma \in \Gamma - \{\gamma_1, \dots, \gamma_n\}} \Omega^\gamma$$

$n=1, 2, \dots$, and decompose the whole space Ω^* into factors:

$$(10) \qquad \Omega^* = \Omega^{\gamma_1} \times \dots \times \Omega^{\gamma_n} \times \Omega^{*(n)}.$$

We have also

$$(11) \qquad \Omega^{*(n)} = \Omega^{\gamma_{n+1}} \times \Omega^{*(n+1)}$$

$n=0, 1, 2, \dots$, where we put $\Omega^{*(0)} = \Omega^*$. In each $\Omega^{*(n)}$, we may define rectangular sets $R^{*(n)}$, elementary sets $E^{*(n)}$, and also the measures $m^{*(n)}(R^{*(n)})$ and $m^{*(n)}(E^{*(n)})$ defined on the family $\mathfrak{R}^{*(n)}$ and $\mathfrak{E}^{*(n)}$ of all these sets $R^{*(n)}$ and $E^{*(n)}$ respectively. This can be carried out in exactly the same way as in the case of the space $\Omega^{*(0)} = \Omega^*$.

Now, for each $\omega^{\gamma_1} \in \Omega^{\gamma_1}$, consider the sets

$$(12) \qquad (E_k^*)_{\omega^{\gamma_1}} = \{\omega^{*(1)} \mid (\omega^{\gamma_1}, \omega^{*(1)}) \in E_k^*\}$$

$k=1, 2, \dots$ $(E_k^*)_{\omega^{\gamma_1}}$ is, by definition, the set of all $\omega^{*(1)} \in \Omega^{*(1)}$ such that $(\omega^{\gamma_1}, \omega^{*(1)}) \in E_k^*$. It is clear that $(E_k^*)_{\omega^{\gamma_1}} \in \mathfrak{E}^{*(1)}$ and $(E_k^*)_{\omega^{\gamma_1}} \geqq (E_{k+1}^*)_{\omega^{\gamma_1}}$ for each $\omega^{\gamma_1} \in \Omega^{\gamma_1}$ and $k=1, 2, \dots$, and that

$$(13) \qquad \int_{\Omega^{\gamma_1}} m^{*(1)}\big((E_k^*)_{\omega^{\gamma_1}}\big) m^{\gamma_1}(d\omega^{\gamma_1}) = m^*(E_k^*) \geqq \delta > 0$$

for $k=1, 2, \dots$ Since the sequence of functions $\big\{m^{*(1)}\big((E_k^*)_{\omega^{\gamma_1}}\big) \mid k=1, 2, \dots\big\}$, defined and measurable on the measure space $(\Omega^{\gamma_1}, \mathfrak{B}^{\gamma_1}, m^{\gamma_1})$, is uniformly bounded between 0 and 1, and is monotone non-increasing, there exists an $\omega_0^{\gamma_1} \in \Omega^{\gamma_1}$ and a positive number $\delta_1 > 0$ such that

$$(14) \qquad m^{*(1)}\big((E_k^*)_{\omega_0^{\gamma_1}}\big) \geqq \delta_1 > 0$$

for $k=1, 2, \dots$

Thus we see that the same condition as (8) is satisfied in $\Omega^{*(1)}$ by the sequence of elementary sets $\{(E_k^*)_{\omega_0^{\gamma_1}} \mid k=1, 2, \dots\}$. Consequently, by proceeding in this way, we shall be able to obtain a sequence of points $\{\omega_0^{\gamma_n} \mid n=1, 2, \dots\}$ such that $\omega_0^{\gamma_n} \in \Omega^{\gamma_n}$ for $n=1, 2, \dots$, and a sequence of positive numbers $\{\delta_n \mid n=1, 2, \dots\}$ satisfying the condition:

$$(15) \qquad m^{*(n)}\big((E_k^*)_{\omega_0^{\gamma_1}, \dots, \omega_0^{\gamma_n}}\big) \geqq \delta_n > 0$$

for $k, n=1, 2, \dots$, where we put

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1) *A* subset A^* of Ω^* is *determined* by a finite number of coordinates if there exists a finite system of elements $\{\gamma_1, \dots, \gamma_n\}$ from Γ such that a point $\omega_0^* = \{\omega_\gamma^* \mid \gamma \in \Gamma\} \in \Omega^*$ belongs to A^* whenever there exists an $\omega^* = \{\omega^\gamma \mid \gamma \in \Gamma\} \in \Omega^*$ belonging to A^* with $\omega^{\gamma_i} = \omega_0^{\gamma_i}$ for $i=1, \dots, n$.

$$(16)\qquad (E_k^*)_{\omega_0^{\gamma_1},\dots,\omega_0^{\gamma_n}}=\{\omega^{*(n)}\mid(\omega_0^{\gamma_1},\dots,\omega_0^{\gamma_n},\omega^{*(n)})\in E_k^*\}$$
$$=\{\omega^{*(n)}\mid(\omega_0^{\gamma_n},\omega^{*(n)})\in(E_k^*)_{\omega_0^{\gamma_1},\dots,\omega_0^{\gamma_{n-1}}}\}$$

i. e., $(E_k^*)_{\omega_0^{\gamma_1}\dots\omega_0^{\gamma_n}}$ is the set of all points $\omega^{*(n)}\in\Omega^{*(n)}$ such that $(\omega_0^{\gamma_1},\dots,\omega_0^{\gamma_n},\omega^{*(n)})\in E_k^*$, or equivalently the set of all points $\omega^{*(n)}\in\Omega^{*(n)}$ such that $(\omega_0^{\gamma_n},\omega^{*(n)})\in(E_k^*)_{\omega_0^{\gamma_1},\dots,\omega_0^{\gamma_{n-1}}}$.

We claim that a point $\omega^*=\{\omega^\gamma\mid\gamma\in\Gamma\}\in\Omega^*$ satisfying $\omega^{\gamma_n}=\omega_0^{\gamma_n}$ for $n=1,2,\dots$ belongs to E_k^* for $k=1,2.\dots$ In fact, for each E_k^*, there exists an integer n_k such that E_k^* is determined by the coordinates $\{\gamma_1,\gamma_2,\dots,\gamma_{n_k}\}$. The relation (15) for $n=n_k$ then implies that there exists an $\omega^{*(n_k)}\in\Omega^{*(n_k)}$ such that $(\omega_0^{\gamma_1},\omega_0^{\gamma_2},\dots,\omega_0^{\gamma_{n_k}},\omega^{*(n_k)})\in E_k^*$ and, since E_k^* is determined by the coordinates $\{\gamma_1,\gamma_2,\dots,\gamma_{n_k}\}$, we must have $\omega^*\in E_k^*$, $k=1,2,\dots$, which immediately implies that $\bigcap_{k=1}^\infty E_k^*\neq\ominus$, as we wanted to prove.