127. On Compact Topological Rings.

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In this paper we shall study compact or locally compact topological rings, where, by a topological ring, we mean a ring with topology with respect to which the operations $x-y$ and xy are continuous as a function of two variables. We do not assume that the multiplication is commutative. When a topological ring R is observed as an abelian group with respect to addition, R is denoted by G. In § 1, we shall discuss the case when R is compact, by representing R as the ring of endomorphisms of the character group G^* of G. In § 2, we shall give some remarks on locally compact rings by making use of the results obtained in § 1.

§ 1. Let R be a compact topological ring, and G^* the character group of $R(=G)$. The mapping $x\to(\varphi, xa)$, where $x\in G$, $\varphi\in G^*$, and $a\in R$ is fixed, gives rise to a new character $\theta_a\varphi\in G^*$ which is defined by $(\theta_a\varphi, x)=(\varphi, xa)$. It is easy to see that $\varphi\to\theta_a\varphi$ is an endomorphism of G^* into itself. The set of all endomorphisms θ_a of G^*, where a runs through R, is denoted by R^*. Clearly $\theta_{a+b}=\theta_a+\theta_b$ and $\theta_{ab}=\theta_a\theta_b$. Thus $a\to\theta_a$ determines a homomorphism Γ from R onto R^*.

Let us introduce a topology into the ring Θ of all endomorphisms θ of G^*[1]. To this end it suffices to give a system of neighborhoods of the zero endomorphism. We define a neighborhood of zero as follows:

$$V^*_{\varphi_1,\dots,\varphi_n;F,\varepsilon}(0)=\Big\{\theta\Big|\,|(\theta\varphi_i, x)|<\varepsilon \text{ for all } x\in F, i=1,\dots,n\Big\},$$

where $\varphi_i\in G^*, i=1,\dots,n$, F is an arbitrary compact set in G, and $\varepsilon>0$ is an arbitrary positive number. With respect to this topology, Θ is obviously a topological ring. As a subset of Θ, R^* is also topologized. We shall now prove that Γ is continuous as a mapping of R onto R^*. For this purpose let us consider the set

$$A=\Big\{a\Big|a\in R,\ |(\theta_a\varphi_i, x)|<\varepsilon \text{ for all } x\in F,\ i=1,\dots,n\Big\}$$
$$=\Big\{a\Big|a\in R,\ |(\varphi_i, xa)|<\varepsilon \text{ for all } x\in F,\ i=1,\dots,n\Big\},$$

where $\varphi_i\in G^*$, $i=1,\dots,n$, F is an arbitrary compact set in G, and $\varepsilon>0$ is an arbitrary positive number. We first note that, for any $\varphi\in G^*$, $\Big\{x\Big|\,|(\varphi, x)|<\varepsilon\Big\}$ is an open set in G. Then, by appealing to the following lemma, it is easy to see that A is an open set in G, which implies that Γ is continuous.

Lemma. Let F be a compact set in R, and let U be an open set

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1) S. Kakutani informed the author of the fact that the topological ring Θ had been discussed by M. Abe in his note: Über die Automorphismen der lokalbikompakten abelschen Gruppen, Proc. **15** (1940), 59.

in R. *If* $Fa_0 \subset U$ *for some* $a_0 \in R$, *then there exists a neighborhood* $V(a_0)$ *of* a_0 *such that* $FV(a_0) \subseteqq U$.

This lemma follows easily from the continuity of multiplication and the compactness of F.

Now, for any fixed $\varphi \in G^*$, the mapping $\theta \rightarrow \theta\varphi$ from Θ into G^* is continuous. (This proposition is an immediate consequence of the definition of topology in Θ). Consequently the mapping $a \rightarrow \theta_a \rightarrow \theta_a\varphi$ from R into G^* is also continuous. As a continuous image of a compact set R, the set $C_\varphi = \{\theta_a\varphi \mid a \in R\}$ must also be compact. Since G^* is a discrete space as the character group of a compact abelian group G, so we see that C_φ is a finite set.

Theorem 1. There exists no compact and connected topological ring except the trivial one in which the product of any two elements is always the zero element.

Proof. Let R be a compact and connected ring. Then the set C_φ defined above consists only of one element; for it is a finite and connected set in G^*. This element must be the zero character; for the set C_φ contains the image of the zero element of R. From this follows that each element of R is a left total zero divisor[1], and consequently the product of any two elements of R is zero.

From now on we assume that no element of R is a left total zero divisor. This assumption is fulfilled if, for example. R contains a unit element or if R has no nilpotent ideal. As is easily verified, under this assumption, the homomorphism Γ defined above is an isomorphism.

Theorem 2. A compact ring without left (or right)[2] *total zero divisor is totally disconnected.*

Proof. Let R be a compact ring without left total zero divisor. We construct the product space $\Omega = P_{\varphi \in G^*} C_\varphi$ of all C_φ, φ running through G^*, and introduce the usual weak topology of Tychonoff into Ω.

Let us consider the correspondence between $a \in R$ and $\omega_a \in \Omega$, where $\omega_a = \{\theta_a\varphi \mid \varphi \in G^*\}$. This is a one-to-one correspondence, since R has no left total zero divisor. Further we may conclude, from the compactness of R and the continuity of the mapping $a \rightarrow \theta_a\varphi$, that the mapping $a \rightarrow \omega_a$ is a homeomorphism. This last statement follows from the continuity of the mapping $a \rightarrow \theta_a\varphi$ for fixed $\varphi \in G^*$, and from the fact that Ω is topologized by the Tychonoff topology. Thus R is homeomorphic with a subset of Ω which is totally disconnected as a product space of finite sets C_φ. This completes the proof of Theorem 2 in case R has no left total zero divisor. The case of right total zero divisor may be discussed in a similar way.

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1) An element $a \in R$ $(a \neq 0)$ is called a *left total zero divisor* is $xa = 0$ for all $x \in R$. A *right total zero divisor* is defined similarly. Further, an element $a \in R$ $(a \neq 0)$ is called a *two-sided total zero divisor* or simply a *total zero divisor* if it is a left and a right total zero divisor at the same time, i. e. if $ax = xa = 0$ for all $x \in R$.

2) A ring without right total zero divisor may have a left total zero divisor. In fact, if we consider the ring of all matirices of the form: $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$, where a and b are elements of a prime field of characteristic p, then this ring has a left total zero divisor $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$, $c \neq 0$, while it is obvious that it has no right total zero divisor.

Theorem 3. A compact ring without left (or right) total zero divisor is a limit ring[1] *of finite rings. (Conjecture of S. Kakutani)*

Proof. By Theorem 2 $R(=G)$ is a totally disconnected compact abelian group with respect to addition. Hence, in an arbitrary neighborhood U of zero, we may choose an open subgroup V of G. By the previous lemma, we can find a neighborhood V_1 of zero such that $RV_1 \leqq V$. Similarly there exists a neighborhood V_2 of zero such that $V_2R \leqq V_1$. Let now W be an open subgroup of G satisfying $W \leqq V \frown V_1 \frown V_2$. Then it is easy to see that the two-sided ideal $I = W + RW + WR + RWR$ generated by W is contained in U. Thus, for any neighborhood U of zero, there exists an open two-sided ideal I contained in U. Theorem 3 follows from this easily if we observe that the factor ring R/I is a finite ring[2].

§ 2. Let R be a locally compact ring, which has no compact nilpotent ideal. If we consider R as a locally compact abelian group with respect to addition, we get the following decomposition of R: $R = V + B$, where V is a vector group of finite dimension, and B is a closed subgroup of R whose component C of zero is compact. For any element $a \in R$, the set aC is also a compact connected subgroup of R. Since the projection of aC on V must also be a compact subgroup of V, it consists only of the zero element of V, i. e. aC must be contained in B. Further, since C is the component of zero of B, we must have $aC \leqq C$. Similarly, $Ca \leqq C$. Therefore, C is a compact connected two-sided ideal in R. If we observe C itself as a compact connected ring, then Theorem 1 implies that $C^2 = 0$. Because of our assumption C must then be the zero ideal, which shows that B is totally disconnected.

In case R is connected, R must coincide with V. Since V is a vector group of finite dimension, it is easy to conclude that R is a hyper-complex number system over the field of real numbers[3].

Returning to the general case, for any element $a \in R$, the set aV is connected. Hence the projection of aV on B must be zero, i. e. $aV \leqq V$. Similarly, $Va \leqq V$. V is thus a two-sided ideal of R. Therefore we may consider B as a family of R-operator endomorphisms of V. In a special case, when all elements of B, considered as an operator on V, are zero operators, R is clearly a direct product of a hypercomplex number system V over the field of real numbers, and a locally compact totally disconnected ideal B. In general, the aggregate of all zero endomorphisms of V forms an open subgroup in B. This follows from the fact that B, being a locally compact totally disconnected group, contains a compact open subgroup C. Exactly in the same way as in above, it may be easily shown that $VC = CV = 0$.

1) A limit ring can be defined exactly in the same way as a limit group. It is to be noted that the number of groups or rings used in the definition of a limit group or a limit ring is not necessarily countable. Cf. A. Weil, L'intégration dans les groupes et leurs applications, Actualités, 1939.

2) We owe this proof to T. Nakayama.

3) This result was already obtained by N. Jacobson and O. Taussky, Proc. N.A.S. USA, **21** (1935), 107, under a more general assumption that R has no total zero divisor.