18. Notes on Divergent Series and Integrals.

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§ 1. The purpose of this paper is to prove the following two theorems:

Theorem 1. Let $x(\omega)$ and $y(\omega)$ be two real-valued non-negative measurable functions defined on the interval $\Omega=\{\omega \mid 0 \leqq \omega \leqq 1\}$ of real numbers which are not necessarily integrable on Ω. If

$$(1) \qquad \int_E y(\omega)d\omega < \infty \qquad \textit{implies} \qquad \int_E x(\omega)d\omega < \infty$$

for any measurable subset E of Ω, then there exist a constant K and a real-valued non-negative measurable function $z(\omega)$ defined and integrable on Ω such that

$$(2) \qquad x(\omega) \leqq Ky(\omega)+z(\omega) \qquad \textit{for any} \qquad \omega \in \Omega .$$

Theorem 2. Let $\{a_n \mid n=1, 2, \dots\}$ and $\{b_n \mid n=1, 2, \dots\}$ be two sequences of real non-negative numbers not greater than 1, for which the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are not necessarily convergent. If

$$(3) \qquad \sum_{k=1}^{\infty} b_{n_k} < \infty \qquad \textit{implies} \qquad \sum_{k=1}^{\infty} a_{n_k} < \infty$$

for any subsequence $\{n_k \mid k=1, 2, \dots\}$ of the sequence $\{n \mid n=1, 2, \dots\}$ of all integers, then there exist a constant K and a sequence $\{c_n \mid n=1, 2, \dots\}$ of real non-negative numbers, for which the series $\sum_{n=1}^{\infty} c_n$ is convergent, such that

$$(4) \qquad a_n \leqq Kb_n+c_n \qquad \textit{for} \qquad n=1, 2, \dots$$

The proof of these theorems will be given in § 3.

§ 2. Let Ω be an arbitrary set and let $\mathfrak{B}=\{E\}$ be a Borel field of subsets E of Ω. Let further $\varphi(E)$ be a countably additive measure defined on $\mathfrak{B}$. We admit the value $+\infty$ for $\varphi(E)$; but in case $\varphi(\Omega)=\infty$, it is assumed that there exists a sequence $\{E_n \mid n=1, 2, \dots\}$ of sets $E_n \in \mathfrak{B}$ such that $\varphi(E_n) < \infty$, $n=1, 2, \dots$ and $\Omega=\bigcup_{n=1}^{\infty} E_n$.

A countably additive measure $\varphi(E)$ defined on $\mathfrak{B}$ is *regular* if, for any $E \in \mathfrak{B}$ with $1 \leqq \varphi(E) \leqq \infty$, there exists an $E' \in \mathfrak{B}$ with $E' \subseteqq E$ and $0 < \varphi(E') \leqq 1$. It is easy to see that, if $\varphi(E)$ is a regular countably additive measure defined on $\mathfrak{B}$, then for any positive number M and for any $E \in \mathfrak{B}$ with $M \leqq \varphi(E) \leqq \infty$, there exists an $E' \in \mathfrak{B}$ with $E' \subseteqq E$ and $M \leqq \varphi(E') \leqq M+1$.

Theorem 3. Let $\varphi(E)$ and $\psi(E)$ be two regular countably additive measures defined on a Borel field $\mathfrak{B}=\{E\}$ of subsets E of a set Ω. If

$$(5) \qquad \psi(E) < \infty \qquad \text{implies} \qquad \varphi(E) < \infty ,$$

then there exists a constant K such that

(6) $$\varphi(E) \leqq K\psi(E) + K \quad \textit{for any } E \in \mathfrak{B}.$$

Remark. It is not difficult to see that from (6) follows the existence of a regular countably additive measure $\varkappa(E)$ defined on $\mathfrak{B}$ with $\varkappa(\Omega) < \infty$ such that

(7) $$\varphi(E) \leqq K\psi(E) + \varkappa(E) \quad \textit{for any } E \in \mathfrak{B}.$$

Proof of Theorem 3. We shall first show that there exists a constant K such that

(8) $$\psi(E) \leqq 2 \quad \textit{implies} \quad \varphi(E) \leqq K \quad \textit{for any} \quad E \in \mathfrak{B}.$$

In fact, otherwise there would exist a sequence $\{E_n \mid n=1, 2, \dots\}$ of subsets E_n of Ω such that $E_n \in \mathfrak{B}$, $\psi(E_n) \leqq 2$ and $\sum_{k=1}^{n-1} \varphi(E_k) + 2^{n+1} \leqq \varphi(E_n) < \infty$, $n=1, 2, \dots$ Let us put $E'_n = E_n - E_n \frown \bigcup_{k=1}^{n-1} E_k$, $n=1, 2, \dots$ Then it is clear that $\{E'_n \mid n=1, 2, \dots\}$ are mutually disjoint and $\psi(E'_n) \leqq 2$, $2^{n+1} \leqq \varphi(E'_n) < \infty$, $n=1, 2, \dots$ Let us decompose each E'_n into 2^n disjoint parts: $E'_n = \bigcup_{p=1}^{2^n} E''_{n,p}$ in such a way that $E''_{n,p} \in \mathfrak{B}$, $p=1, \dots, 2^n$, $1 \leqq \varphi(E''_{n,p}) \leqq 2$, $p=1, \dots, 2^n - 1$ and $1 \leqq \varphi(E''_{n,p_n})$. This is possible since $\varphi(E)$ is regular. Then it is clear that, for each n, there exists an integer p_n $(1 \leqq p_n \leqq 2^n)$ such that $\psi(E''_{n,p_n}) \leqq 2^{1-n}$. Let us now put $E^* = \bigcup_{n=1}^{\infty} E''_{n,p_n}$. Then it is easy to see that $\varphi(E^*) = \sum_{n=1}^{\infty} \varphi(E''_{n,p_n}) \geqq \sum_{n=1}^{\infty} 1 = \infty$ while $\psi(E^*) \leqq \sum_{n=1}^{\infty} \psi(E''_{n,p_n}) \leqq \sum_{n=1}^{\infty} 2^{1-n} = 2$ contrary to the assumption (5). Thus we see that there exists a constant K which satisfies (8).

Now, for any $E \in \mathfrak{B}$ with $\psi(E) < \infty$, let n be a positive integer such that $n-1 \leqq \psi(E) < n$. Let us then decompose E into m disjoint parts: $E = \bigcup_{p=1}^{m} E_p$, where m is an integer satisfying $1 \leqq m \leqq n$, in such a way that $E_p \in \mathfrak{B}$, $p=1, \dots, m$, $1 \leqq \psi(E_p) \leqq 2$, $p=1, \dots, m-1$, $0 \leqq \psi(E_m) \leqq 2$. This is possible since $\psi(E)$ is regular by assumption. From this follows easily, because of (7), that $\varphi(E) = \sum_{p=1}^{m} \varphi(E_p) \leqq Km \leqq Kn \leqq K\big(\psi(E)+1\big)$ as we wanted to prove.

§ 3. It is now easy to see that Theorem 3 implies Theorem 1. We have only to put $\varphi(E) = \int_E x(\omega)d\omega$ and $\psi(E) = \int_E y(\omega)d\omega$. Since $\varphi(E)$ and $\psi(E)$ are clearly regular, so we see that there exists a constant K which satisfies (6) or equivalently

(9) $$\int_E x(\omega)d\omega \leqq K\int_E y(\omega)d\omega + K \quad \textit{for any measurable set } E.$$

If we now put

(10) $$z(\omega) = \max\big(x(\omega) - Ky(\omega), 0\big),$$

then it is clear that (2) is satisfied. Further, by putting $E_0 = \{\omega \mid z(\omega) \geqq 0\}$ and $E_n = \{\omega \mid y(\omega) \leqq n\}$, $n=1, 2, \dots$, we see from (9) and (10) that $\int_{E_n} z(\omega)d\omega = \int_{E_n \frown E_0} z(\omega)d\omega = \int_{E_n \frown E_0} x(\omega)d\omega - K\int_{E_n \frown E_0} y(\omega)d\omega \leqq K < \infty$, $n = 1, 2, \dots$, from which follows immediately that $\int_\Omega z(\omega)d\omega \leqq K < \infty$.

In order to show that Theorem 3 implies Theorem 2, let us put $\varphi(E)=\sum_{n\in E}a_n$ and $\psi(E)=\sum_{n\in E}b_n$ where $E=\{n_k \mid k=1, 2, \dots\}$ is an arbitrary (finite or infinite) subsequence of the sequence $\Omega=\{n \mid n=1, 2, \dots\}$ of all positive integers. Since $\varphi(E)$ and $\psi(E)$ are clearly regular, so we see that there exists a constant K such that

$$\sum_{n\in E}a_n \leqq K\sum_{n\in E}b_n + K \tag{11}$$

for any (finite or infinite) subsequence E of Ω. If we now put

$$c_n = \max(a_n - Kb_n, 0), \qquad n=1, 2, \dots, \tag{12}$$

then it is clear that (4) is satisfied. Further, by putting $E_0=\{n \mid c_n \geqq 0\}$ we see from (11) and (12) that $\sum_{n=1}^{N} c_n = \sum_{n\in E_0, n\leq N} c_n = \sum_{n\in E_0, n\leq N} a_n - K\sum_{n\in E_0, n\leq N} b_n \leqq K < \infty$, $N=1, 2, \dots$, from which follows that $\sum_{n=1}^{\infty} c_n \leqq K < \infty$.