An averaging formula for the coincidence Reidemeister trace

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Abstract

In the setting of continuous maps between compact orientable manifolds of the same dimension, there is a well known averaging formula for the coincidence Lefschetz number in terms of the Lefschetz numbers of lifts to some finite covering space. We state and prove an analogous averaging formula for the coincidence Reidemeister trace. This generalizes a recent formula in fixed point theory by Liu and Zhao.

We give two separate and independent proofs of our main result: one using methods developed by Kim and the first author for averaging Nielsen numbers, and one using an axiomatic approach for the local Reidemeister trace.

We also give some examples and state some open questions for the nonorientable case.

1 The Reidemeister trace in coincidence theory

Let M_1 and M_2 be closed orientable manifolds of the same dimension. Let $p_1 : \tilde{M}_1 \to M_1$ and $p_2 : \tilde{M}_2 \to M_2$ be the universal covering projections with the groups of covering transformations Π_1 and Π_2, respectively. Let $f, g : M_1 \to M_2$ be a pair of continuous maps. By fixing lifts $\tilde{f}$ and $\tilde{g}$ of f and g, the

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maps f and g induce homomorphisms $\phi : \Pi_1 \to \Pi_2$ and $\psi : \Pi_1 \to \Pi_2$ between the groups of covering transformations as follows.

$$\phi(\alpha)\tilde{f} = \tilde{f}\alpha, \ \psi(\alpha)\tilde{g} = \tilde{g}\alpha, \quad \forall \alpha \in \Pi_1. \tag{1–1}$$

The set of Reidemeister classes $\mathcal{R}[\phi,\psi]$ is defined as the quotient of Π_2 by the doubly-twisted conjugacy relation $\alpha \sim \beta$ if and only if there is some $\gamma \in \Pi_1$ with $\alpha = \psi(\gamma)^{-1}\beta\phi(\gamma)$. The Reidemeister class containing β will be written as $[\beta] \in \mathcal{R}[\phi,\psi]$, or when we wish to emphasize the homomorphisms, as $[\beta]_{\phi,\psi}$.

Then it is well known that the coincidence set $\mathrm{Coin}(f,g)$ splits into a disjoint union of coincidence classes. That is,

$$\mathrm{Coin}(f,g) = \bigsqcup_{[\beta]\in\mathcal{R}[\phi,\psi]} p_1\big(\mathrm{Coin}(\beta\tilde{f},\tilde{g})\big)\,. \tag{1–2}$$

Therefore any coincidence class $\mathbb{S} = p_1\big(\mathrm{Coin}(\beta\tilde{f},\tilde{g})\big)$ can be indexed by the Reidemeister class $[\beta]$; $\mathbb{S} = \mathbb{S}_{[\beta]}$. Here we allow some coincidence classes to be empty.

The Reidemeister number $R(f,g) = R(\phi,\psi)$ is the number of Reidemeister classes. The coincidence Lefschetz number $L(f,g)$ defined using the (co)homology groups is the sum of local indices. That is,

$$L(f,g) = \sum_{[\beta]\in\mathcal{R}[\phi,\psi]} \mathrm{ind}(f,g;\mathbb{S}_{[\beta]}) \in \mathbb{Z},$$

where ind denotes the coincidence index of a coincidence class.

Definition 1.1. The *coincidence Reidemeister trace* of (f,g) is defined as follows:

$$\begin{aligned} RT(f,\tilde{f},g,\tilde{g}) &= \sum_{[\beta]\in\mathcal{R}[\phi,\psi]} \mathrm{ind}(f,g;\mathbb{S}_{[\beta]})[\beta] \\ &= \sum_{[\beta]\in\mathcal{R}[\phi,\psi]} \mathrm{ind}(f,g;p_1\big(\mathrm{Coin}(\beta\tilde{f},\tilde{g})\big))[\beta] \in \mathbb{Z}\mathcal{R}[\phi,\psi]. \end{aligned} \tag{1–3}$$

We remark that the coincidence Reidemeister trace contains information of the coincidence Nielsen number and the coincidence Lefschetz number. Indeed, the number of non-zero terms in this sum is the coincidence Nielsen number $N(f,g)$, and the sum of the coefficients is the coincidence Lefschetz number $L(f,g)$.

Let Γ_1 and Γ_2 be finite index normal subgroups of Π_1 and Π_2 so that $\phi(\Gamma_1) \subset \Gamma_2$ and $\psi(\Gamma_1) \subset \Gamma_2$. Then $\phi : \Pi_1 \to \Pi_2$ restricts to a homomorphism $\phi' : \Gamma_1 \to \Gamma_2$, and then induces a homomorphism $\bar{\phi} : \Pi_1/\Gamma_1 \to \Pi_2/\Gamma_2$. We denote the canonical inclusion by $i_1 : \Gamma_1 \to \Pi_1$ and the canonical projection by $u_1 : \Pi_1 \to \Pi_1/\Gamma_1$. Similarly, we obtain homomorphisms ψ', $\bar{\psi}$, i_2 and u_2. We then have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_1 & \xrightarrow{i_1} & \Pi_1 & \xrightarrow{u_1} & \Pi_1/\Gamma_1 & \longrightarrow & 1 \\ & & {\scriptstyle\phi'}\big\downarrow{\scriptstyle\psi'} & & {\scriptstyle\phi}\big\downarrow{\scriptstyle\psi} & & {\scriptstyle\bar{\phi}}\big\downarrow{\scriptstyle\bar{\psi}} & & \\ 1 & \longrightarrow & \Gamma_2 & \xrightarrow{i_2} & \Pi_2 & \xrightarrow{u_2} & \Pi_2/\Gamma_2 & \longrightarrow & 1 \end{array} \tag{1–4}$$

Denoting $\bar{M}_1 = \Gamma_1\backslash\tilde{M}_1$ and $\bar{M}_2 = \Gamma_2\backslash\tilde{M}_2$, we see that $\bar{M}_1$ and $\bar{M}_2$ are regular coverings of M_1 and M_2 such that

$$p_1 : \tilde{M}_1 \xrightarrow{p_1'} \bar{M}_1 \xrightarrow{\bar{p}_1} M_1,$$
$$p_2 : \tilde{M}_2 \xrightarrow{p_2'} \bar{M}_2 \xrightarrow{\bar{p}_2} M_2$$

are covering projections.

Since $\phi(\Gamma_1) \subset \Gamma_2$ and $\psi(\Gamma_1) \subset \Gamma_2$, the pair of lifts $(\tilde{f}, \tilde{g})$ induces the pair of lifts $(\bar{f}, \bar{g})$ of (f, g). Indeed, $\bar{f} : \bar{M}_1 \to \bar{M}_2$ is defined by $\Gamma_1\tilde{x} \in \bar{M}_1 \mapsto \Gamma_2\tilde{f}(\tilde{x}) \in \bar{M}_2$. This map is well defined because if $\tilde{x}' = \gamma\tilde{x}$ for some $\gamma \in \Gamma_1$ then $\phi(\gamma) \in \Gamma_2$ and

$$\tilde{f}(\tilde{x}') = \tilde{f}(\gamma\tilde{x}) = (\tilde{f}\gamma)(\tilde{x}) = (\phi(\gamma)\tilde{f})(\tilde{x}) = \phi(\gamma)(\tilde{f}(\tilde{x}))$$

where the third identity follows from the definition of ϕ, (1–1). Similarly, $\tilde{g}$ restricts to $\bar{g} : \bar{M}_1 \to \bar{M}_2$. Furthermore, for any $\bar{\beta} \in \Pi_1/\Gamma_2$ and $\beta \in u_2^{-1}(\bar{\beta})$, the following diagram is commutative

$$\begin{array}{ccc} \tilde{M}_1 & \underset{\tilde{g}}{\overset{\beta\tilde{f}}{\longrightarrow}} & \tilde{M}_2 \\ \big\downarrow{\scriptstyle p_1'} & & \big\downarrow{\scriptstyle p_2'} \\ \bar{M}_1 & \underset{\bar{g}}{\overset{\bar{\beta}\bar{f}}{\longrightarrow}} & \bar{M}_2 \\ \big\downarrow{\scriptstyle \bar{p}_1} & & \big\downarrow{\scriptstyle \bar{p}_2} \\ M_1 & \underset{g}{\overset{f}{\longrightarrow}} & M_2 \end{array} \tag{1–5}$$

Note that $\{\beta\tilde{f} \mid \beta \in \Pi_2\}$ is the set of all lifts on the universal covering manifolds of f, and $\{\bar{\beta}\bar{f} \mid \beta \in \Pi_2\}$ is the set of all lifts on the regular covering manifolds of f.

There is an averaging formula for the coincidence Lefschetz number. The following formula is apparently due to McCord:

$$L(f, g) = \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar{\beta}\in\Pi_2/\Gamma_2} L(\bar{\beta}\bar{f}, \bar{g}).$$

The summation on the right side above is discussed in [10, page 360], and the full formula appears in [11, page 88]. The corresponding averaging formula for coincidence Nielsen numbers does not hold in general, this is studied in [8].

Recall from (1–1) that the endomorphism ϕ was defined using the lift $\tilde{f}$. Taking projections by p_1' and p_2' we have

$$\bar{\phi}(\bar{\alpha})\bar{f} = \bar{f}\bar{\alpha}, \quad \forall\alpha \in \Pi_1.$$

If we use other lifts $\beta\tilde{f}$ and $\bar{\beta}\bar{f}$ of f, then we obtain the corresponding endomorphisms $\tau_\beta\phi$ and $\tau_{\bar{\beta}}\bar{\phi}$ respectively. (Here, τ_β is the conjugation by

β, $\delta \mapsto \beta\delta\beta^{-1}$.) Similarly, the lift $\beta\tilde{f}$ of $\bar{\beta}\bar{f} : \bar{M}_1 \to \bar{M}_2$ induces an endomorphism $\tau_\beta\phi' : \Gamma_1 \to \Gamma_2$. We remark also that the pair of homomorphisms $(\tau_\beta\phi', \psi')$ are the homomorphisms induced by $(\bar{\beta}\bar{f}, \bar{g})$ by fixing the lifts $(\beta\tilde{f}, \tilde{g})$. Hence by definition

$$\begin{aligned} &RT(\bar{\beta}\bar{f}, \beta\tilde{f}, \bar{g}, \tilde{g}) \\ &= \sum_{[\gamma]\in\mathcal{R}[\tau_\beta\phi',\psi']} \operatorname{ind}(\bar{\beta}\bar{f}, \bar{g}; p_1'(\operatorname{Coin}(\gamma(\beta\tilde{f}), \tilde{g})))[\gamma] \in \mathbb{Z}\mathcal{R}[\tau_\beta\phi', \psi']. \end{aligned} \tag{1–6}$$

For each $\bar{\beta} \in \Pi_2/\Gamma_2$ and $\beta \in u_2^{-1}(\bar{\beta})$, we have the following commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \Gamma_1 \arrow[r, "i_1"] \arrow[d, "\tau_\beta\phi'"', shift right] \arrow[d, "\psi'", shift left] & \Pi_1 \arrow[r, "u_1"] \arrow[d, "\tau_\beta\phi"', shift right] \arrow[d, "\psi", shift left] & \Pi_1/\Gamma_1 \arrow[r] \arrow[d, "\tau_{\bar{\beta}}\bar{\phi}"', shift right] \arrow[d, "\bar{\psi}", shift left] & 1 \\ 1 \arrow[r] & \Gamma_2 \arrow[r, "i_2"] & \Pi_2 \arrow[r, "u_2"] & \Pi_2/\Gamma_2 \arrow[r] & 1 \end{tikzcd} \tag{1–7}$$

Moreover, the inclusion i_1 and the projection u_1 induce the following exact sequence of groups

$$1 \to \operatorname{coin}(\tau_\beta\phi', \psi') \xrightarrow{i_1^\beta} \operatorname{coin}(\tau_\beta\phi, \psi) \xrightarrow{u_1^\beta} \operatorname{coin}(\tau_{\bar{\beta}}\bar{\phi}, \bar{\psi}),$$

where coin denotes the coincidence subgroup of a pair of homomorphisms. Here, i_1^β and u_1^β are (co)restrictions of i_1 and u_1, respectively.

We remark also that $i_2 : \Gamma_2 \to \Pi_2$ and $u_2 : \Pi_2 \to \Pi_2/\Gamma_2$ induce maps between the Reidemeister sets

$$\hat{i}_2^\beta : \mathcal{R}[\tau_\beta\phi', \psi'] \to \mathcal{R}[\tau_\beta\phi, \psi],\ [\gamma]_{\tau_\beta\phi',\psi'} \mapsto [\gamma]_{\tau_\beta\phi,\psi}$$

and

$$\hat{u}_2^\beta : \mathcal{R}[\tau_\beta\phi, \psi] \to \mathcal{R}[\tau_{\bar{\beta}}\bar{\phi}, \bar{\psi}],\ \ [\delta]_{\tau_\beta\phi,\psi} \mapsto [u_2(\delta)]_{\tau_{\bar{\beta}}\bar{\phi},\bar{\psi}} = [\bar{\delta}]_{\tau_{\bar{\beta}}\bar{\phi},\bar{\psi}}$$

such that $\hat{u}_2^\beta$ is surjective and $(\hat{u}_2^\beta)^{-1}([\bar{1}]) = \operatorname{im}(\hat{i}_2^\beta)$. That is, the following sequence of sets is exact:

$$\mathcal{R}[\tau_\beta\phi', \psi'] \xrightarrow{\hat{i}_2^\beta} \mathcal{R}[\tau_\beta\phi, \psi] \xrightarrow{\hat{u}_2^\beta} \mathcal{R}[\tau_{\bar{\beta}}\bar{\phi}, \bar{\psi}] \longrightarrow 1. \tag{1–8}$$

This exactness is used in [8, 5] to give a convenient relabeling of the coincidence classes. Typically the coincidence classes of (f, g) are labeled by the Reidemeister classes $\mathcal{R}[\phi, \psi]$ of (ϕ, ψ). We may instead relabel them in terms of the Reidemeister classes $\mathcal{R}[\bar{\phi}, \bar{\psi}]$ and $\mathcal{R}[\tau_\beta\phi', \psi']$. This gives the following:

Lemma 1.2 ([8, Lemma 2.2], [5, Lemma 3.1]). *For each $[\bar{\beta}] \in \mathcal{R}[\bar{\phi}, \bar{\psi}]$, choose some $[\beta] \in \mathcal{R}[\phi, \psi]$ with $\hat{u}_2([\beta]) = [\bar{\beta}]$. Then we have the disjoint union:*

$$\operatorname{Coin}(f, g) = \bigsqcup_{[\bar{\beta}]\in\mathcal{R}[\bar{\phi},\bar{\psi}]} \left(\bigsqcup_{[\gamma]_{\tau_\beta\phi,\psi}\in\operatorname{im}(\hat{i}_2^\beta)} p_1(\operatorname{Coin}(\gamma\beta\tilde{f}, \tilde{g})) \right).$$

It will be important for us to relate the elements $RT(f,\tilde{f},g,\tilde{g}) \in \mathbb{Z}\mathcal{R}[\phi,\psi]$ and $RT(f,\beta\tilde{f},g,\tilde{g}) \in \mathbb{Z}\mathcal{R}[\tau_\beta\phi,\psi]$ for some $\beta \in \Pi_2$ of free Abelian groups with generators given by the Reidemeister classes. By (1–3), we have that

$$RT(f,\beta\tilde{f},g,\tilde{g}) = \sum_{[\delta]_{\tau_\beta\phi,\psi}\in\mathcal{R}[\tau_\beta\phi,\psi]} \mathrm{ind}(f,g;\mathbb{S}_{[\delta]_{\tau_\beta\phi,\psi}})[\delta]_{\tau_\beta\phi,\psi}.$$

$$RT(f,\alpha\tilde{f}) = \sum_{[\beta]_{\tau_\alpha\phi}\in\mathcal{R}[\tau_\alpha\phi]} \mathrm{ind}(f;\mathbb{F}_{[\beta]_{\tau_\alpha\phi}})[\beta]_{\tau_\alpha\phi}.$$

In this sum, we note that

$$\mathbb{S}_{[\delta]_{\tau_\beta\phi,\psi}} = p_1\left(\mathrm{Coin}(\delta(\beta\tilde{f}),\tilde{g})\right) = p_1\left(\mathrm{Coin}((\delta\beta)\tilde{f},\tilde{g})\right) = \mathbb{S}_{[\delta\beta]_{\phi,\psi}}$$

and that multiplication on the right by β gives a one-to-one correspondence $\rho_\beta : \mathcal{R}[\tau_\beta\phi,\psi] \to \mathcal{R}[\phi,\psi]$ such that

$$\rho_\beta([\delta]_{\tau_\alpha\phi,\psi}) = [\delta\beta]_{\phi,\psi}.$$

Linearizing (and again using the notation ρ_β) gives an isomorphism between free Abelian groups:

$$\rho_\beta : \mathbb{Z}\mathcal{R}[\tau_\beta\phi,\psi] \longrightarrow \mathbb{Z}\mathcal{R}[\phi,\psi].$$

Thus we have

$$\begin{aligned}
\rho_\beta(RT(f,\beta\tilde{f},g,\tilde{g})) &= \rho_\beta\left(\sum_{[\delta]_{\tau_\beta\phi,\psi}\in\mathcal{R}[\tau_\beta\phi,\psi]} \mathrm{ind}(f,g;\mathbb{S}_{[\delta]_{\tau_\beta\phi,\psi}})[\delta]_{\tau_\beta\phi,\psi}\right)\\
&= \sum_{[\delta\beta]_{\phi,\psi}\in\mathcal{R}[\phi,\psi]} \mathrm{ind}(f,g;\mathbb{S}_{[\delta\beta]_{\phi,\psi}})[\delta\beta]_{\phi,\psi}\\
&= \sum_{[\alpha]_{\phi,\psi}\in\mathcal{R}[\phi,\psi]} \mathrm{ind}(f,g;\mathbb{S}_{[\alpha]_{\phi,\psi}})[\alpha]_{\phi,\psi}
\end{aligned}$$

and so

$$RT(f,\tilde{f},g,\tilde{g}) = \rho_\beta(RT(f,\beta\tilde{f},g,\tilde{g})). \tag{1–9}$$

That is, $RT(f,\tilde{f},g,\tilde{g})$ and $RT(f,\beta\tilde{f},g,\tilde{g})$ are the same, except all Reidmeister representatives are multiplied on the right by β.

By the same reason, we have

$$RT(\bar{\beta}\bar{f},\beta\tilde{f},\bar{g},\tilde{g}) = \rho_\gamma(RT(\bar{\beta}\bar{f},\gamma\beta\tilde{f},\bar{g},\tilde{g})), \quad \forall\gamma\in\Gamma_2.$$

2 Main result

The following averaging formula for the coincidence Reidemeister trace is our main result. Because the sum of the coefficients in the coincidence Reidemeister trace equals the coincidence Lefschetz number, this is a direct generalization of McCord's averaging formula for $L(f,g)$.

Theorem 2.1. *Let M_1, M_2 be closed orientable manifolds of the same dimension. Given finite regular covers $\bar{M}_1, \bar{M}_2$ and universal covers $\tilde{M}_1, \tilde{M}_2$ and the commutative diagram*

$$\begin{array}{ccc}
\tilde{M}_1 & \underset{\tilde{g}}{\overset{\tilde{f}}{\longrightarrow}} & \tilde{M}_2 \\
\downarrow & & \downarrow \\
\bar{M}_1 & \underset{\bar{g}}{\overset{\bar{f}}{\longrightarrow}} & \bar{M}_2 \\
\downarrow & & \downarrow \\
M_1 & \underset{g}{\overset{f}{\longrightarrow}} & M_2
\end{array}$$

we have

$$RT(f, \tilde{f}, g, \tilde{g}) = \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar{\beta} \in \Pi_2/\Gamma_2} \rho_\beta \circ \hat{i}_2^\beta \left(RT(\bar{\beta}\bar{f}, \beta\tilde{f}, \bar{g}, \tilde{g}) \right).$$

Proof. Our proof follows the results and methods from [8]. For brevity in the following, we will omit subscripts on Reidemeister classes when they are obvious from context. For example $[\delta]_{\tau_\beta\phi,\psi} \in \mathcal{R}[\tau_\beta\phi, \psi]$ will simply be written $[\delta] \in \mathcal{R}[\tau_\beta\phi, \psi]$.

Recall that

$$\begin{aligned}
RT(f, \tilde{f}, g, \tilde{g}) &= \sum_{[\alpha] \in \mathcal{R}[\phi,\psi]} \operatorname{ind}(f, g; \mathbb{S}_{[\alpha]})[\alpha] \\
&= \sum_{[\alpha] \in \mathcal{R}[\phi,\psi]} \operatorname{ind}(f, g; p_1(\operatorname{Coin}(\alpha\tilde{f}, \tilde{g}))) \, [\alpha]
\end{aligned}$$

by the definition of RT and $\mathbb{S}_{[\alpha]}$. Relabeling classes as in Lemma 1.2, for each $[\bar{\beta}] \in \mathcal{R}[\bar{\phi}, \bar{\psi}]$ we fix a preimage $[\beta] \in \mathcal{R}[\phi, \psi]$, and the above becomes:

$$RT(f, \tilde{f}, g, \tilde{g}) = \sum_{[\bar{\beta}] \in \mathcal{R}[\bar{\phi},\bar{\psi}]} \left(\sum_{[\gamma]_{\tau_\beta\phi,\psi} \in \operatorname{im}(\hat{i}_2^\beta)} \operatorname{ind}(f, g; \mathbb{S}_{[\gamma\beta]})[\gamma\beta]_{\phi,\psi} \right).$$

Rather than the inner sum over elements $[\gamma]$ of $\operatorname{im}(\hat{i}_2^\beta)$, we can sum over elements $[\gamma']$ of $\mathcal{R}[\tau_\beta\phi', \psi']$ with $\hat{i}_2^\beta([\gamma']) = [\gamma]$. Since a single $[\gamma] \in \operatorname{im}(\hat{i}_2^\beta)$ may come from several elements in $\mathcal{R}[\tau_\beta\phi', \psi']$, we must insert the fraction below. We obtain:

$$RT(f, \tilde{f}, g, \tilde{g}) = \sum_{[\bar{\beta}] \in \mathcal{R}[\bar{\phi},\bar{\psi}]} \left(\sum_{[\gamma'] \in \mathcal{R}[\tau_\beta\phi',\psi']} \frac{1}{\#(\hat{i}_2^\beta)^{-1}([\gamma])} \operatorname{ind}(f, g; \mathbb{S}_{[\gamma\beta]})[\gamma\beta]_{\phi,\psi} \right).$$

By [8, Lemma 2.1], we also have

$$\begin{aligned}
\#(\hat{i}_2^\beta)^{-1}([\gamma]) &= [\operatorname{coin}(\tau_{\bar{\beta}}\bar{\phi}, \bar{\psi}) : u_{\gamma\beta}(\operatorname{coin}(\tau_{\gamma\beta}\phi, \psi))] \\
&= \frac{\#\operatorname{coin}(\tau_{\bar{\beta}}\bar{\phi}, \bar{\psi})}{\#u_{\gamma\beta}(\operatorname{coin}(\tau_{\gamma\beta}\phi, \psi))}
\end{aligned}$$

and thus we can rewrite the above as

$$
\begin{aligned}
&RT(f,\tilde{f},g,\tilde{g})\\
&= \sum_{[\bar{\beta}]\in\mathcal{R}[\bar{\phi},\bar{\psi}]} \left(\sum_{[\gamma']\in\mathcal{R}[\tau_\beta\phi',\psi']} \frac{\#u_{\gamma\beta}(\mathrm{coin}(\tau_{\gamma\beta}\phi,\psi))}{\#\mathrm{coin}(\tau_{\bar{\beta}}\bar{\phi},\bar{\psi})} \mathrm{ind}(f,g;\mathbb{S}_{[\gamma\beta]})[\gamma\beta]_{\phi,\psi} \right)\\
&= \sum_{\bar{\beta}\in\Pi_2/\Gamma_2} \left(\sum_{[\gamma']\in\mathcal{R}[\tau_\beta\phi',\psi']} \frac{\#u_{\gamma\beta}(\mathrm{coin}(\tau_{\gamma\beta}\phi,\psi))}{\#[\bar{\beta}]\cdot\#\mathrm{coin}(\tau_{\bar{\beta}}\bar{\phi},\bar{\psi})} \mathrm{ind}(f,g;\mathbb{S}_{[\gamma\beta]})[\gamma\beta]_{\phi,\psi} \right).
\end{aligned}
$$

On each Reidemeister class $[\bar{\beta}]$ of $\mathcal{R}[\bar{\phi},\bar{\psi}]$, the group Π_1/Γ_1 acts transitively by the rule $\bar{\beta}\mapsto\bar{\psi}(\bar{\delta})\bar{\beta}\bar{\phi}(\bar{\delta})^{-1}$. The isotropy subgroup at $\bar{\beta}$ is

$$\left\{\bar{\delta} \mid \bar{\psi}(\bar{\delta})\bar{\beta}\bar{\phi}(\bar{\delta})^{-1}=\bar{\beta}\right\} = \mathrm{coin}(\tau_{\bar{\beta}}\bar{\phi},\bar{\psi}).$$

Thus

$$[\Pi_1:\Gamma_1] = \#[\bar{\beta}]\cdot\#\mathrm{coin}(\tau_{\bar{\beta}}\bar{\phi},\bar{\psi}),\ \forall[\bar{\beta}]\in\mathcal{R}[\bar{\phi},\bar{\psi}],$$

and so

$$
\begin{aligned}
&RT(f,\tilde{f},g,\tilde{g})\\
&= \frac{1}{[\Pi_1:\Gamma_1]} \sum_{\bar{\beta}\in\Pi_2/\Gamma_2} \left(\sum_{[\gamma']\in\mathcal{R}[\tau_\beta\phi',\psi']} \#u_{\gamma\beta}(\mathrm{coin}(\tau_{\gamma\beta}\phi,\psi))\mathrm{ind}(f,g;\mathbb{S}_{[\gamma\beta]})[\gamma\beta]_{\phi,\psi} \right).
\end{aligned}
$$

Now we remark that the coincidence class $p_1'(\mathrm{Coin}(\gamma\beta\tilde{f},\tilde{g}))$ of $(\bar{\beta}\bar{f},\bar{g})$ covers by $\bar{p}_1$ the coincidence class $\mathbb{S}_{[\gamma\beta]} = p_1(\mathrm{Coin}(\gamma\beta\tilde{f},\tilde{g}))$ of (f,g). By Lemma 4.2 of [8] we see that this cover is $\#u_1(\mathrm{coin}(\tau_{\gamma\beta}\phi,\psi))$-fold. Since the projections p_1' and p_1 are orientation preserving local homeomorphisms and the index is a local invariant, we have

$$\mathrm{ind}(\beta\bar{f},\bar{g};p_1'(\mathrm{Coin}(\gamma\beta\tilde{f},\tilde{g}))) = \#u_{\gamma\beta}(\mathrm{coin}(\tau_{\gamma\beta}\phi,\psi))\cdot\mathrm{ind}(f,g;\mathbb{S}_{[\gamma\beta]}), \tag{2–1}$$

and consequently, we obtain

$$RT(f,\tilde{f},g,\tilde{g}) = \frac{1}{[\Pi_1:\Gamma_1]} \sum_{\bar{\beta}\in\Pi_2/\Gamma_2} \left(\sum_{[\gamma']\in\mathcal{R}[\tau_\beta\phi',\psi']} \mathrm{ind}(\bar{\beta}\bar{f},\bar{g};\mathbb{S}_{[\gamma\beta]})[\gamma\beta]_{\phi,\psi} \right).$$

We know that $[\gamma\beta]_{\phi,\psi} = \rho_\beta([\gamma]_{\tau_\beta\phi,\psi})$, and by the definition of γ' we have $[\gamma]_{\tau_\beta\phi,\psi} = \hat{i}_2^{\beta}([\gamma']_{\tau_\beta\phi',\psi'})$. Therefore, the above identity reduces to:

$$
\begin{aligned}
RT(f,\tilde{f},g,\tilde{g}) &= \frac{1}{[\Pi_1:\Gamma_1]} \sum_{\bar{\beta}\in\Pi_2/\Gamma_2} \left(\sum_{[\gamma']\in\mathcal{R}[\tau_\beta\phi',\psi']} \mathrm{ind}(\bar{\beta}\bar{f},\bar{g};\mathbb{S}_{[\gamma\beta]})\rho_\beta(\hat{i}_2^{\beta}([\gamma']_{\tau_\beta\phi',\psi'})) \right)\\
&= \frac{1}{[\Pi_1:\Gamma_1]} \sum_{\bar{\beta}\in\Pi_2/\Gamma_2} \rho_\beta\circ\hat{i}_2^{\beta} \left(\sum_{[\gamma']\in\mathcal{R}[\tau_\beta\phi',\psi']} \mathrm{ind}(\bar{\beta}\bar{f},\bar{g};\mathbb{S}_{[\gamma\beta]})[\gamma']_{\phi',\psi'} \right)\\
&= \frac{1}{[\Pi_1:\Gamma_1]} \sum_{\bar{\beta}\in\Pi_2/\Gamma_2} \rho_\beta\circ\hat{i}_2^{\beta}\left(RT(\bar{\beta}\bar{f},\beta\tilde{f},\bar{g},\tilde{g})\right).
\end{aligned}
$$

∎

Consider the case $\Pi_2 = \Gamma_2$. That is, the following diagram is commutative

$$\begin{tikzcd}
\bar{M} \arrow[d, "\bar{p}"] \arrow[dr, "\bar{f}", "\bar{g}"'] & \\
M \arrow[r, "f", "g"'] & M'
\end{tikzcd}$$

so that $\bar{f} = f \circ \bar{p}$ and $\bar{g} = g \circ \bar{p}$. Then the formula in Theorem 2.1 reads as

$$\begin{aligned}
RT(f, \tilde{f}, g, \tilde{g}) &= \frac{1}{[\Pi_1 : \Gamma_1]} \ \hat{\imath}_2\big(RT(\bar{f}, \tilde{f}, \bar{g}, \tilde{g})\big) \\
&= \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{[\gamma] \in \mathcal{R}[\phi', \psi']} \operatorname{ind}(\bar{f}, \bar{g}; p_1'(\operatorname{Coin}(\gamma\tilde{f}, \tilde{g}))) \ \hat{\imath}_2([\gamma]).
\end{aligned}$$

The sum of the coefficients of both sides gives

$$L(f, g) = \frac{1}{[\Pi_1 : \Gamma_1]} \ L(\bar{f}, \bar{g}).$$

Since $\deg(\bar{p}) = [\Pi : \Gamma]$, we obtain a result

$$L(f \circ \bar{p}, g \circ \bar{p}) = \deg(\bar{p}) L(f, g).$$

This is already known for example in [1, Corollary 14.6].

As another special case, we consider the case where $f = g = \mathrm{id}_M$, $\bar{f} = \bar{g} = \mathrm{id}_{\bar{M}}$ and $\tilde{f} = \tilde{g} = \mathrm{id}_{\tilde{M}}$. Then the formula in Theorem 2.1 reads as

$$RT(\mathrm{id}_M, \mathrm{id}_{\tilde{M}}, \mathrm{id}_M, \mathrm{id}_{\tilde{M}}) = \frac{1}{[\Pi : \Gamma]} \sum_{\bar{\beta} \in \Pi/\Gamma} \rho_\beta \circ \hat{\imath}_2^{\beta} \left(RT(\bar{\beta}, \beta, \mathrm{id}_M, \mathrm{id}_{\tilde{M}}) \right).$$

The sum of the coefficients of both sides gives

$$L(\mathrm{id}_M, \mathrm{id}_M) = \frac{1}{[\Pi : \Gamma]} \sum_{\bar{\beta} \in \Pi/\Gamma} L(\bar{\beta}, \mathrm{id}_{\bar{M}})$$

or

$$L(\mathrm{id}_M) = \chi(M) = \frac{1}{[\Pi : \Gamma]} \sum_{\bar{\beta} \in \Pi/\Gamma} L(\bar{\beta}).$$

Since $\chi(M) = \dfrac{1}{[\Pi : \Gamma]} \chi(\bar{M})$, we obtain a new formula

$$\chi(\bar{M}) = \sum_{\bar{\beta} \in \Pi/\Gamma} L(\bar{\beta}).$$

Of course, this identity can be obtained from the averaging formula for Lefschetz number

$$L(f) = \frac{1}{[\Pi : \Gamma]} \sum_{\bar{\beta} \in \Pi/\Gamma} L(\bar{\beta}\bar{f})$$

by taking M to be any compact connected polyhedron X and f to be the identity map id_X.

Corollary 2.2. *Let X be a compact connected polyhedron with a finite group G of deck transformations. Then*
$$\chi(X) = \sum_{g\in G} L(g).$$

If we consider the case where $M = M_1 = M_2$, $g = \mathrm{id}_M$ and $\tilde{g} = \mathrm{id}_{\tilde{M}}$, then the coincidence theory becomes the fixed point theory. In particular, our main result, Theorem 2.1, gives rise to the fixed point version on closed orientable manifolds M. However, the fixed point theory is available on more general topological spaces, for example compact connected polyhedra. In fact, using arguments from [7] we can prove the fixed point version of Theorem 2.1 on compact connected polyhedra. This will provide another proof of the following main result of [9].

Theorem 2.3 ([9, Theorem 4.1]). *Let X be a compact connected polyhedron. Given the commutative diagram*
$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\
\downarrow & & \downarrow \\
\bar{X} & \xrightarrow{\bar{f}} & \bar{X} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & X
\end{array}$$
we have
$$RT(f,\tilde{f}) = \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\alpha}\in\Pi/\Gamma} \rho_\alpha \circ \hat{\imath}^\alpha \left(RT(\bar{\alpha}\bar{f}, \alpha\tilde{f})\right).$$

3 An axiomatic proof

The averaging formula for the coincidence Reidemeister trace can also be obtained as an application of a uniqueness theorem from [4]. This uniqueness result was a direct generalization of earlier work in [3, 12, 13] concerning the local fixed point and coincidence index. To demonstrate the basic idea, we begin with an axiomatic proof of an averaging formula for the local fixed point index.

Recall that the local fixed point index $\mathrm{ind}(f;U)$ is an integer invariant which is defined for any open set U such that $\mathrm{Fix}(f)\cap U$ is compact.

Let M be a compact differentiable manifold with universal covering $p:\tilde{M}\to M$ and a finite covering $\bar{p}:\bar{M}\to M$ with the natural map $p':\tilde{M}\to\bar{M}$ with $p = p'\circ\bar{p}$.

When $U\subset M$ is an open set, and $f:U\to M$ is a continuous map, say $(f;U)$ is *admissible* when $\mathrm{Fix}(f)\cap U$ is compact, and let $\mathcal{C}(M)$ be the set of all admissible pairs on M. If $H: M\times[0,1]\to M$ is a homotopy, we say that H is *admissible in U* if the set
$$\{(x,t) \mid H(x,t) = x\}$$
is compact in $M\times[0,1]$.

The following is a result of Furi, Pera, and Spadini [3].

Theorem 3.1. *Let M be a compact differentiable manifold, and let $\iota : \mathcal{C}(M) \to \mathbb{R}$ be any function satisfying the following 3 axioms:*

- *(Homotopy) If there is a homotopy from $(f;U)$ to $(f';U)$ which is admissible in U, then $\iota(f;U) = \iota(f';U)$.*
- *(Additivity) If U_1, U_2 are disjoint open subsets of U with $\mathrm{Fix}(f) \cap U \subset U_1 \sqcup U_2$, then*
$$\iota(f;U_1) + \iota(f;U_2) = \iota(f;U).$$
- *(Normalization) If c is a constant map, then $\iota(c;M) = 1$*

Then ι is the local fixed point index.

We will use the uniqueness theorem above to obtain the following averaging formula for the local fixed point index.

Theorem 3.2. *Let $\mathrm{ind}(f;U)$ be the local fixed point index of an admissible pair, and write $\bar{U} = p^{-1}(U)$. For any specific lift $\bar{f} : \bar{M} \to \bar{M}$ of f, we have*

$$\mathrm{ind}(f;U) = \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} \mathrm{ind}(\bar{\beta}\bar{f};\bar{U}). \tag{3–1}$$

Proof. Before proving formula (3–1) we should note that when $\mathrm{Fix}(f) \cap U$ is compact in M, since q is a covering map we will have $\mathrm{Fix}(\bar{\beta}\bar{f}) \cap \bar{U}$ compact in $\bar{M}$ for any lift $\bar{f}$ of f, and thus the pair $(\bar{f};\bar{U})$ is admissible and so the index on the right side of (3–1) is defined.

For any admissible pair $(f;U)$, let

$$\iota(f;U) = \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} \mathrm{ind}(\bar{\beta}\bar{f};\bar{U}),$$

where $\bar{f}$ is any lift of f. It suffices to show that ι satisfies the three axioms of Theorem 3.1.

First we remark that ι is well defined (independent of the choice of lift $\bar{f}$). It suffices to show that if $\bar{f}$ and $\hat{f}$ are two different choices of lifts, then

$$\sum_{\bar{\beta}\in\Pi/\Gamma} \mathrm{ind}(\bar{\beta}\bar{f};\bar{U}) = \sum_{\bar{\beta}\in\Pi/\Gamma} \mathrm{ind}(\bar{\beta}\hat{f};\bar{U}).$$

There is some $\bar{\alpha} \in \Pi/\Gamma$ with $\bar{\alpha}\hat{f} = \bar{f}$, and so we have:

$$\sum_{\bar{\beta}\in\Pi/\Gamma} \mathrm{ind}(\bar{\beta}\bar{f};\bar{U}) = \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} \mathrm{ind}(\bar{\beta}\bar{\alpha}\hat{f};\bar{U}) = \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} \mathrm{ind}(\bar{\beta}\hat{f};\bar{U})$$

as desired. Now we show that ι satisfies the three axioms of Theorem 3.1.

For the homotopy axiom, let H be an admissible homotopy from $(f;U)$ to $(f';U)$. Then for any lift $\bar{\beta}\bar{f}$ of f, the homotopy H will lift to a unique homotopy

$\bar{H}$ from $\bar{\beta}\bar{f}$ to some lift $\bar{\beta}\bar{f}'$ of f', and it can be checked that this $\bar{H}$ is admissible in $\bar{U}$. Thus we have

$$\iota(f;U) = \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} \operatorname{ind}(\bar{\beta}\bar{f};\bar{U}) = \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} \operatorname{ind}(\bar{\beta}\bar{f}';\bar{U}) = \iota(f';U)$$

and so ι satisfies the homotopy axiom.

For the additivity axiom, let $(f;U) \in \mathcal{C}(M)$ with $\mathrm{Fix}(f) \cap U \subset U_1 \sqcup U_2$. Then for any lift $\bar{f}$ of f we have:

$$\mathrm{Fix}(\bar{f}) \cap \bar{U} \subset \bar{p}^{-1}(\mathrm{Fix}(f) \cap U) \subset \bar{p}^{-1}(U_1 \sqcup U_2) = \bar{U}_1 \sqcup \bar{U}_2.$$

Thus the additivity axiom for $\operatorname{ind}(\bar{\beta}\bar{f};\bar{U})$ gives:

$$\begin{aligned}
\iota(f;U) &= \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} \operatorname{ind}(\bar{\beta}\bar{f};\bar{U}) \\
&= \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} (\operatorname{ind}(\bar{\beta}\bar{f};\bar{U}_1) + \operatorname{ind}(\bar{\beta}\bar{f};\bar{U}_2)) \\
&= \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} \operatorname{ind}(\bar{\beta}\bar{f};\bar{U}_1) + \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} \operatorname{ind}(\bar{\beta}\bar{f};\bar{U}_2) \\
&= \iota(f;U_1) + \iota(f;U_2),
\end{aligned}$$

and thus ι satisfies the additivity axiom.

For the normalization axiom, let $c : M \to M$ be a constant map. Then any lift $\bar{c} : \bar{M} \to \bar{M}$ is a constant map on $\bar{M}$, and so $\bar{\beta}\bar{c}$ is a constant map for any $\bar{\beta}$. Thus using the normalization axiom on $\operatorname{ind}(\bar{\beta}\bar{c}, \bar{M})$ gives:

$$\iota(c;M) = \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} \operatorname{ind}(\bar{\beta}\bar{c};\bar{M}) = \frac{1}{[\Pi:\Gamma]} \sum_{\bar{\beta}\in\Pi/\Gamma} 1 = 1.$$

Thus ι satisfies the normalization axiom.

Since ι satisfies the three axioms, it must be the fixed point index. ■

If we take $U = M$ above we obtain a new proof of the classical Lefschetz averaging formula, since $\operatorname{ind}(f;M) = L(f)$.

An averaging formula like the one in Theorem 3.2 can be obtained for the local coincidence index, by substituting the uniqueness theorem for the coincidence index from [12] in place of Theorem 3.1. The uniqueness theorems in [13] can be used to obtain similar averaging formulas for the local Reidmeister trace and local coincidence Reidemeister trace. We will prove only the most general of these, the local coincidence Reidemeister trace.

Theorem 3.1 was generalized to the coincidence Reidemeister trace in [13] as follows: Let M_1 and M_2 be oriented closed manifolds of the same dimension, let $\tilde{M}_1$ and $\tilde{M}_2$ be their universal covers, and let $\mathcal{C}(M_1, M_2)$ be the set of *admissible tuples* $(f, \tilde{f}, g, \tilde{g}; U)$ where $f, g : M_1 \to M_2$ are continuous, $\tilde{f}, \tilde{g} : \tilde{M}_1 \to \tilde{M}_2$ are lifts of f and g, and U is an open set with $\mathrm{Coin}(f, g) \cap U$ compact. If H, G :

$M_1 \times [0,1] \to M_2$ are homotopies, we say the pair (H, G) is admissible in U when the set

$$\{(x,t) \in M_1 \times [0,1] \mid H(x,t) = G(x,t)\}$$

is compact.

The local coincidence Reidemeister trace is defined as in Definition 1–3, but restricting only to the subset U:

$$RT(f, \tilde{f}, g, \tilde{g}; U) = \sum_{[\beta] \in \mathcal{R}[\phi,\psi]} \mathrm{ind}(f|_U, g|_U; \mathbb{S}_{[\beta]}|_U)[\beta],$$

where $f|_U$ and $g|_U$ denote the restrictions of f and g to U, and

$$\mathbb{S}_{[\beta]}|_U = p_1(\mathrm{Coin}(\beta\tilde{f}|_{\tilde{U}}, \tilde{g}|_{\tilde{U}})), \qquad \tilde{U} = p_1^{-1}(U).$$

It is clear from the definition that $RT(f, \tilde{f}, g, \tilde{g}; M_1) = RT(f, \tilde{f}, g, \tilde{g})$.

The manifolds in [3, 12, 13] are always assumed to be differentiable. This assumption is made in order to use transversality arguments, but is not necessary. Jezierski proves a "topological transversality lemma" in [6] which provides the same results for topological manifolds. The work in [4] generalizes all results from [3, 12, 13] without the differentiability assumption, so we will not require differentiability in this paper. We will also use the normalization axiom used in [4], which is weaker than the one from [13] based on the Lefschetz number. In our normalization property we assume that M_1 and M_2 have fixed chosen orientations.

When S is any set, let $\epsilon : \mathbb{Z}S \to \mathbb{Z}$ be the augmentation map (sum of coefficients).

Theorem 3.3 ([13], Theorem 3; [4], Theorem 44). *Let σ be a function with domain $\mathcal{C}(M_1, M_2)$ such that $\sigma(f, \tilde{f}, g, \tilde{g}; U) \in \mathbb{Z}\mathcal{R}[\phi, \psi]$, where ϕ and ψ are the induced homomorphisms of f and g respectively. Assume that σ satisfies the following axioms:*

- *(Homotopy) If there is a pair of homotopies from f to f' and g to g' which is admissible in U, then $\sigma(f, \tilde{f}, g, \tilde{g}; U) = \sigma(f', \tilde{f}', g', \tilde{g}'; U)$, where $\tilde{f}'$ and $\tilde{g}'$ are the unique lifts of f' and g' obtained by lifting the homotopies to $\tilde{f}$ and $\tilde{g}$.*
- *(Additivity) If U_1, U_2 are disjoint open subsets of U with $\mathrm{Coin}(f,g) \cap U \subset U_1 \sqcup U_2$, then*

$$\sigma(f, \tilde{f}, g, \tilde{g}; U) = \sigma(f, \tilde{f}, g, \tilde{g}; U_1) + \sigma(f, \tilde{f}, g, \tilde{g}; U_2)$$

- *(Normalization) Let $c : M_1 \to M_2$ be a constant map with constant value $c \in M_2$, and $g : U \to M_2$ be an orientation preserving embedding with $g(x) = c$. (That is, g carries the positive orientation on U to the positive orientation on $g(U)$.) Then:*

$$\epsilon(\sigma(c, \tilde{c}, g, \tilde{g}; U)) = 1.$$

- *(Lift invariance) For any $\alpha, \beta \in \Pi_2$, we have*

$$\epsilon(\sigma(f, \tilde{f}, g, \tilde{g}; U)) = \epsilon(\sigma(f, \alpha\tilde{f}, g, \beta\tilde{g}; U))$$

- *(Coincidence of lifts) If* $[\alpha]$ *appears in* $\sigma(f, \tilde{f}, g, \tilde{g}; U)$ *with nonzero coefficient, then* $\alpha\tilde{f}$ *and* $\tilde{g}$ *have a coincidence on* $p_1^{-1}(U)$.

Then σ *is the local coincidence Reidemeister trace.*

Our averaging formula for the local coincidence Reidemeister trace is proved similarly to Theorem 3.2.

Theorem 3.4. *For any* $U \subset M_1$*, let* $\bar{U} = \bar{p}_1^{-1}(U) \subset \bar{M}_1$ *and let* $\bar{f}, \bar{g}$ *be specific chosen lifts of* f, g*. Then we have*

$$RT(f, \tilde{f}, g, \tilde{g}; U) = \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar{\beta} \in \Pi_2/\Gamma_2} \rho_\beta \circ \hat{i}_2^\beta \left(RT(\bar{\beta}\bar{f}, \beta\tilde{f}, \bar{g}, \tilde{g}; \bar{U})\right),$$

where $\beta \in \Pi_2$ *is any element whose image in* Π_2/Γ_2 *is* $\bar{\beta}$.

Proof. We will follow the proof of Theorem 3.2. Throughout the proof, for $U \subset M_1$, let $\bar{U} = \bar{p}_1^{-1}(U) \subset \bar{M}_1$. For any tuple $(f, \tilde{f}, g, \tilde{g}; U) \in \mathcal{C}(M_1, M_2)$, let $\bar{f}, \bar{g} : \bar{M}_1 \to \bar{M}_2$ be given by the diagram (1–5), and let

$$\sigma(f, \tilde{f}, g, \tilde{g}; U) = \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar{\beta} \in \Pi_2/\Gamma_2} \rho_\beta \circ \hat{i}_2^\beta \left(RT(\bar{\beta}\bar{f}, \beta\tilde{f}, \bar{g}, \tilde{g}; \bar{U})\right). \tag{3–2}$$

We will show that σ satisfies the 5 axioms of Theorem 3.3.

First we show that σ is well defined, that is, independent of the element $\beta \in \Pi_2$ which projects to $\bar{\beta} \in \Pi_2/\Gamma_2$. It suffices to show that if β and β' each project to $\bar{\beta}$, then

$$\rho_\beta \circ \hat{i}_2^\beta(RT(\bar{\beta}\bar{f}, \beta\tilde{f}, \bar{g}, \tilde{g}; \bar{U})) = \rho_{\beta'} \circ \hat{i}_2^{\beta'}(RT(\bar{\beta}\bar{f}, \beta'\tilde{f}, \bar{g}, \tilde{g}; \bar{U})).$$

Let $\gamma \in \Gamma_2$ be some element with $\gamma\beta = \beta'$. Then we must show

$$\rho_\beta \circ \hat{i}_2^\beta(RT(\bar{\beta}\bar{f}, \beta\tilde{f}, \bar{g}, \tilde{g}; \bar{U})) = \rho_{\gamma\beta} \circ \hat{i}_2^{\gamma\beta}(RT(\bar{\beta}\bar{f}, (\gamma\beta)\tilde{f}, \bar{g}, \tilde{g}; \bar{U})),$$

This follows from the following observation: For $\beta \in \Pi_2$ and $\gamma \in \Gamma_2$, we have the following commutative diagram

$$\rho_{\gamma\beta} \circ \hat{i}_2^{\gamma\beta} = \rho_\beta \circ \hat{i}_2^\beta \circ \rho_\gamma$$

$$\begin{array}{ccccc}
\mathcal{R}[\tau_{\gamma\beta}\phi', \psi'] & \xrightarrow{\hat{i}_2^{\gamma\beta}} & \mathcal{R}[\tau_{\gamma\beta}\phi, \psi] & \xrightarrow{\rho_{\gamma\beta}} & \mathcal{R}[\phi, \psi] \\
\big\downarrow{=} & & & & \big\uparrow{\rho_\beta} \\
\mathcal{R}[\tau_\gamma(\tau_\beta\phi)', \psi'] & \xrightarrow{\rho_\gamma} & \mathcal{R}[\tau_\beta\phi', \psi'] & \xrightarrow{\hat{i}_2^\beta} & \mathcal{R}[\tau_\beta\phi, \psi]
\end{array}$$

$$\begin{array}{cccccc}
 & [\delta] & \xrightarrow{\hat{i}_2^{\gamma\beta}} & [\delta] & \xrightarrow{\rho_{\gamma\beta}} & [\delta(\gamma\beta)] = [(\delta\gamma)\beta] \\
\forall\delta \in \Gamma_2, & \big\downarrow{=} & & & & \big\uparrow{\rho_\beta} \\
 & [\delta] & \xrightarrow{\rho_\gamma} & [\delta\gamma] & \xrightarrow{\hat{i}_2^\beta} & [\delta\gamma]
\end{array}$$

Then we have

$$\begin{aligned}\rho_{\gamma\beta} \circ \hat{i}_2^{\gamma\beta}(RT(\bar\beta\bar f, (\gamma\beta)\tilde f, \bar g, \tilde g; \bar U)) &= \rho_\beta \circ \hat{i}_2^{\beta} \circ \rho_\gamma(RT(\bar\beta\bar f, (\gamma\beta)\tilde f, \bar g, \tilde g; \bar U)) \\ &= \rho_\beta \circ \hat{i}_2^{\beta}(RT(\bar\beta\bar f, \beta\tilde f, \bar g, \tilde g; \bar U)),\end{aligned}$$

as required, where the last equality is by (1–9), which holds for the local coincidence Reidemeister trace (this is [13, Theorem 6]).

Now we verify that our five axioms hold for σ. The homotopy axiom is satisfied exactly as in Theorem 3.2. Any pair of admissible homotopies (H, G) from f to f' and g to g' will uniquely determine a pair of admissible homotopies $(\bar\beta\bar H, \bar G)$ from $\bar\beta\bar f$ to $\bar\beta\bar f'$ and $\bar g$ to $\bar g'$, and also a pair of admissible homotopies from $\tilde f$ to $\tilde f'$ and $\tilde g$ to $\tilde g'$. Then by the homotopy property for the Reidemeister trace we will have:

$$\begin{aligned}\sigma(f, \tilde f, g, \tilde g; U) &= \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\beta\in\Pi_2/\Gamma_2} \rho_\beta \circ \hat{i}_2^{\beta}\left(RT(\bar\beta\bar f, \beta\tilde f, \bar g, \tilde g; \bar U)\right) \\ &= \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\beta\in\Pi_2/\Gamma_2} \rho_\beta \circ \hat{i}_2^{\beta}\left(RT(\bar\beta\bar f', \beta\tilde f', \bar g', \tilde g'; \bar U)\right) \\ &= \sigma(f', \tilde f', g', \tilde g'; U)\end{aligned}$$

For the additivity axiom again we follow exactly the argument from Theorem 3.2. We omit the details.

For the normalization axiom, let c be a constant map and g be an orientation preserving embedding of some small set U with $g(x) = c$ for some $x \in U$. Then there is some $\bar\alpha \in \Pi_2/\Gamma_2$ with $\bar g(\bar x) = \bar\alpha\bar c$.

We will assume that U is sufficiently small so that $\bar U$ is a union of finitely many components homeomorphic to U by p_1. Let $\bar U_0$ be the connected component of $\bar U$ which contains $\bar x$. Then we have a disjoint union

$$\bar U = \bigsqcup_{\bar\gamma\in\Pi_1/\Gamma_1} \bar\gamma(\bar U_0),$$

and

$$\begin{aligned}\epsilon(\sigma(c, \tilde c, g, \tilde g; U)) &= \epsilon\left(\frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\beta\in\Pi_2/\Gamma_2} \rho_\beta \circ \hat{i}_2^{\beta}\left(RT(\bar\beta\bar c, \beta\tilde c, \bar g, \tilde g; \bar U)\right)\right) \\ &= \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\beta\in\Pi_2/\Gamma_2} \epsilon\left(RT(\bar\beta\bar c, \beta\tilde c, \bar g, \tilde g; \bar U)\right) \\ &= \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\beta\in\Pi_2/\Gamma_2} \sum_{\bar\gamma\in\Pi_1/\Gamma_1} \epsilon\left(RT(\bar\beta\bar c, \beta\tilde c, \bar g, \tilde g; \bar\gamma(\bar U_0))\right).\end{aligned}$$

The set $\mathrm{Coin}(\bar\beta\bar c, \bar g) \cap \bar\gamma(\bar U_0)$ is nonempty when $\bar g(\bar\gamma\bar x) = \bar\beta\bar c$. Since $\bar g(\bar x) = \bar\alpha\bar c$, this is equivalent to $\bar\psi(\bar\gamma)\bar g(\bar x) = \bar\beta\bar\alpha^{-1}\bar g(\bar x)$, which is to say that $\bar\psi(\bar\gamma) = \bar\beta\bar\alpha^{-1}$.

In the case where $\mathrm{Coin}(\bar\beta\bar c, \bar g) \cap \bar\gamma(\bar U_0)$ is nonempty, it contains a single coincidence point $\bar\gamma\bar x$, and $\bar g$ is an orientation preserving embedding of $\bar\gamma(\bar U_0)$, because covering transformations of covering spaces of orientable manifolds are orientation preserving. Thus by the normalization property for RT we have:

$$\epsilon\left(RT(\bar\beta\bar c, \beta\tilde c, \bar g, \tilde g; \bar\gamma(\bar U_0))\right) = \begin{cases} 1 & \text{if } \bar\psi(\bar\gamma) = \bar\beta\bar\alpha^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the above summation gives:

$$\begin{aligned}
\epsilon(\sigma(c, \tilde c, g, \tilde g; U)) &= \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\beta \in \Pi_2/\Gamma_2} \#\{\bar\gamma \in \Pi_1/\Gamma_1 \mid \bar\psi(\bar\gamma) = \bar\beta\bar\alpha^{-1}\} \\
&= \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\beta \in \Pi_2/\Gamma_2} \#\bar\psi^{-1}(\{\bar\beta\bar\alpha^{-1}\}) \\
&= \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\delta \in \Pi_2/\Gamma_2} \#\bar\psi^{-1}(\{\delta\}).
\end{aligned}$$

The set inside the sum has 0 elements when $\delta \in \mathrm{im}\bar\psi$, and otherwise has size $\#\ker\bar\psi$. Thus we have:

$$\epsilon(\sigma(c, \tilde c, g, \tilde g; U)) = \frac{1}{[\Pi_1 : \Gamma_1]} \#\mathrm{im}\bar\psi \cdot \#\ker\bar\psi = 1$$

by the First Isomorphism Theorem.

For the lift invariance axiom, we have

$$\begin{aligned}
\epsilon(\sigma(f, \alpha\tilde f, g, \gamma\tilde g; U)) &= \epsilon\left(\frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\beta \in \Pi_2/\Gamma_2} \rho_\beta \circ \hat{i}_2^\beta \left(RT(\bar\beta\bar f, \beta\alpha\tilde f, \bar g, \gamma\tilde g; \bar U)\right)\right) \\
&= \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\beta \in \Pi_2/\Gamma_2} \epsilon\left(RT(\bar\beta\bar f, \beta\alpha\tilde f, \bar g, \gamma\tilde g; \bar U)\right) \\
&= \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\beta \in \Pi_2/\Gamma_2} \epsilon\left(RT(\bar\beta\bar f, \beta\tilde f, \bar g, \tilde g; \bar U)\right) \\
&= \epsilon(\sigma(f, \tilde f, g, \tilde g; U)).
\end{aligned}$$

Finally we must demonstrate the coincidence of lifts axiom. Say that $[\alpha]$ appears in $\sigma(f, \tilde f, g, \tilde g; U)$ with nonzero coefficient, and we must show that $\alpha\tilde f$ and $\tilde g$ have a coincidence in $p_1^{-1}(U)$. By (3–2), if $[\alpha]$ has nonzero coefficient in $\sigma(f, \tilde f, g, \tilde g; U)$, then there is some $\bar\beta$ and some γ such that $\rho_\beta(\hat{i}_2^\beta([\gamma])) = [\alpha]$, and $[\gamma]$ has nonzero coefficient in $RT(\bar\beta\bar f, \beta\tilde f, \bar g, \tilde g; \bar U)$.

Then by the coincidence of lifts property applied to $RT(\bar\beta\bar f, \beta\tilde f, \bar g, \tilde g; \bar U)$, the maps $\gamma\beta\tilde f$ and $\tilde g$ have a coincidence on $\bar U$. But since $\rho_\beta(\hat{i}_2^\beta([\gamma])) = [\alpha]$, we have $\alpha = \gamma\beta$, and thus $\alpha\tilde f$ and $\tilde g$ have a coincidence on $\bar U$ as desired.

We have shown that σ satisfies the 5 axioms of Theorem 3.3, and thus it must equal the coincidence Reidemeister trace. ∎

4 Examples

Example 4.1. Let M_1 be a 3-dimensional orientable flat manifold $\Pi_1\backslash\mathbb{R}^3$ where Π_1 is the 3-dimensional orientable Bieberbach group $\mathfrak{G}_2$ [14, Theorem 3.5.5]:

$$\Pi_1 = \langle t_1, t_2, t_3, \alpha \mid [t_i, t_j] = 1,\ \alpha^2 = t_1,\ \alpha t_2 \alpha^{-1} = t_2^{-1},\ \alpha t_3 \alpha^1 = t_3^{-1}\rangle.$$

We can embed this group into $\mathrm{Aff}(\mathbb{R}^3)$ by taking $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as the standard basis for $\mathbb{R}^3$ and

$$t_i = (\mathbf{e}_i, I_3)\ (i = 1, 2, 3), \quad \alpha = \left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right).$$

Then the translation lattice is

$$\Gamma_1 = \Pi_1 \cap \mathbb{R}^3 = \langle t_1, t_2, t_3 \rangle = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle = \mathbb{Z}^3$$

and the holonomy group is $\Pi_1/\Gamma_1 \cong \mathbb{Z}_2$.

Let $M_2 = \mathbb{R}P^3$. Assume that there is a commutative diagram

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "\tilde{f}", "\tilde{g}"'] \arrow[d, "p_1'"] & S^3 \arrow[d, "p_2'"] \\
T^3 = \Gamma_1\backslash\mathbb{R}^3 \arrow[r, "\bar{f}", "\bar{g}"'] \arrow[d, "\bar{p}_1"] & S^3 \arrow[d, "\bar{p}_2"] \\
M_1 = \Pi_1\backslash\mathbb{R}^3 \arrow[r, "f", "g"'] & M_2 = \mathbb{R}P^3
\end{tikzcd}$$

inducing the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_1 \arrow[r, "i_1"] \arrow[d, "\tau_\beta\phi'"', "\psi'"] & \Pi_1 \arrow[r, "u_1"] \arrow[d, "\tau_\beta\phi"', "\psi"] & \Pi_1/\Gamma_1 \cong \mathbb{Z}_2 \arrow[r] \arrow[d, "\tau_{\bar{\beta}}\bar{\phi}"', "\bar{\psi}"] & 1 \\
1 \arrow[r] & \Gamma_2 = \{1\} \arrow[r, "i_2"] & \Pi_2 \arrow[r, "u_2"] & \Pi_2/\Gamma_2 \cong \mathbb{Z}_2 \arrow[r] & 1
\end{tikzcd}$$

where $\bar{\phi}$ is an isomorphism and $\bar{\psi}$ is a trivial homomorphism.

Let β be the antipodal map of S^3. Then $\Pi_2 = \langle \beta \rangle$. Since Π_2 is abelian, we have $\tau_\beta\phi = \phi$ and $\tau_{\bar{\beta}}\bar{\phi} = \bar{\phi}$.

Remark that since ψ' and $\bar{\psi}$ are trivial homomorphisms, it follows that ψ is a trivial homomorphism. Since ϕ' is a trivial homomorphism and $\bar{\phi}$ is an isomorphism, it follows that $\phi(\alpha) = \beta$.

A simple computation shows that

$$\begin{aligned}
1 &\to \mathrm{coin}(\phi', \psi') = \Gamma_1 \to \mathrm{coin}(\phi, \psi) = \Gamma_1 \to \mathrm{coin}(\bar{\phi}, \bar{\psi}) = \{\bar{1}\} \\
&\to \mathcal{R}[\phi', \psi'] = \{[1]\} \to \mathcal{R}[\phi, \psi] = \{[1] = [\beta]\} \to \mathcal{R}[\bar{\phi}, \bar{\psi}] = \{[\bar{1}] = [\bar{\beta}]\} \to 1.
\end{aligned}$$

There is only one coincidence class of (f,g):

$$\mathbb{S}_{[1]} = p_1\big(\mathrm{Coin}(\tilde{f},\tilde{g})\big)\,.$$

Hence

$$RT(f,\tilde{f},g,\tilde{g}) = \mathrm{ind}(f,g;\mathbb{S}_{[1]})[1].$$

On the other hand, because $\tau_\beta\phi = \phi$, we have

$$RT(\bar{\beta}\bar{f},\beta\tilde{f},\bar{g},\tilde{g}) = RT(\bar{f},\tilde{f},\bar{g},\tilde{g}) = \mathrm{ind}(\bar{f},\bar{g};\mathbb{S}_{[1]})[1] \in \mathbb{Z}\mathcal{R}[\phi',\psi'].$$

Thus

$$\begin{aligned}
\frac{1}{[\Pi_1:\Gamma_1]} & \sum_{\bar{\beta}\in\Pi_2/\Gamma_2} \rho_\beta \circ \hat{i}^\beta \left(RT(\bar{\beta}\bar{f},\beta\tilde{f},\bar{g},\tilde{g})\right) \\
&= \frac{1}{2}\Big(\rho_1 \circ \hat{i}_2^1 \left(RT(\bar{f},\tilde{f},\bar{g},\tilde{g})\right) + \rho_\beta \circ \hat{i}_2^\beta \left(RT(\bar{\beta}\bar{f},\beta\tilde{f},\bar{g},\tilde{g})\right)\Big) \\
&= \frac{1}{2}\Big(\rho_1 \circ \hat{i}_2^1 \Big(\mathrm{ind}(\bar{f},\bar{g};\mathbb{S}_{[1]})[1]\Big) + \rho_\beta \circ \hat{i}_2^\beta \Big(\mathrm{ind}(\bar{f},\bar{g};\mathbb{S}_{[1]})[1]\Big)\Big) \\
&= \mathrm{ind}(f,g;\mathbb{S}_{[1]})[1].
\end{aligned}$$

This example demonstrates how the averaging formula for the coincidence Reidemeister trace holds. Notice also in this example that the averaging formula for the Nielsen coincidence number also holds. This is expected from from [8, Theorem 4.6] since $\mathrm{coin}(\phi,\psi) \subset \Gamma_1$.

Example 4.2. Let $M_1 = \mathbb{R}P^3$. Let M_2 be a 3-dimensional orientable flat manifold $\Pi_2\backslash\mathbb{R}^3$ where Π_2 is the 3-dimensional orientable Bieberbach group $\mathfrak{G}_2$:

$$\Pi_2 = \langle t_1,t_2,t_3,\alpha \mid [t_i,t_j]=1,\ \alpha^2 = t_1,\ \alpha t_2\alpha^{-1} = t_2^{-1},\ \alpha t_3\alpha^1 = t_3^{-1}\rangle.$$

Let $\Gamma_2 = \langle t_1,t_2,t_3\rangle = \mathbb{Z}^3$.

Assume that there is a commutative diagram

$$\begin{array}{ccc}
S^3 & \xrightarrow[\tilde{g}]{\tilde{f}} & \mathbb{R}^3 \\
\big\downarrow{\scriptstyle p_1'} & & \big\downarrow{\scriptstyle p_2'} \\
S^3 & \xrightarrow[\bar{g}]{\bar{f}} & T^3 = \Gamma_2\backslash\mathbb{R}^3 \\
\big\downarrow{\scriptstyle \bar{p}_1} & & \big\downarrow{\scriptstyle \bar{p}_2} \\
M_1 = \mathbb{R}P^3 & \xrightarrow[g]{f} & M_2 = \Pi_2\backslash\mathbb{R}^3
\end{array}$$

inducing the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_1 = \{1\} & \xrightarrow{i_1} & \Pi_1 & \xrightarrow{u_1} & \Pi_1/\Gamma_1 \cong \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & {\scriptstyle \tau_\gamma\phi'}\big\downarrow{\scriptstyle \psi'} & & {\scriptstyle \tau_\gamma\phi}\big\downarrow{\scriptstyle \psi} & & {\scriptstyle \tau_{\bar{\gamma}}\bar{\phi}}\big\downarrow{\scriptstyle \bar{\psi}} & & \\
1 & \longrightarrow & \Gamma_2 & \xrightarrow{i_2} & \Pi_2 & \xrightarrow{u_2} & \Pi_2/\Gamma_2 \cong \mathbb{Z}_2 & \longrightarrow & 1
\end{array}$$

Let β be the antipodal map of S^3. Then $\Pi_1 = \langle\beta\rangle$. Notice that every homomorphism $\xi : \Pi_1 \to \Pi_2$ must be a trivial homomorphism. Otherwise, $\xi(\beta)$ is of the form

$$\xi(\beta) = t_1^{n_1} t_2^{n_2} t_3^{n_3} \alpha.$$

Then $1 = \xi(\beta^2) = \xi(\beta)^2 = (t_1^{n_1} t_2^{n_2} t_3^{n_3} \alpha)^2 = t_1^{2n_1+1}$, which is impossible. Thus $\tau_\gamma\phi$ and ψ are trivial, and the remaining homomorphisms are also trivial.

A simple computation shows that

$$\begin{aligned} 1 &\to \operatorname{coin}(\phi', \psi') = \Gamma_1 \to \operatorname{coin}(\phi, \psi) = \Pi_1 \to \operatorname{coin}(\bar\phi, \bar\psi) = \Pi_1/\Gamma_1 \\ &\to \mathcal{R}[\phi', \psi'] = \Gamma_2 \to \mathcal{R}[\phi, \psi] = \Pi_2 \to \mathcal{R}[\bar\phi, \bar\psi] = \Pi_2/\Gamma_2 \to 1. \end{aligned}$$

The coincidence classes of (f, g) are

$$\mathbb{S}_\gamma = p_1\big(\operatorname{Coin}(\gamma\tilde f, \tilde g)\big), \quad \forall\gamma \in \Pi_2.$$

Hence

$$RT(f, \tilde f, g, \tilde g) = \sum_{\gamma\in\Pi_2} \operatorname{ind}(f, g; \mathbb{S}_\gamma)\gamma \in \mathbb{Z}\Pi_2.$$

On the other hand,

$$\begin{aligned} &\frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\gamma\in\Pi_2/\Gamma_2} \rho_\gamma \circ \hat{i}^\gamma \left(RT(\bar\gamma\bar f, \gamma\tilde f, \bar g, \tilde g)\right) \\ &= \frac{1}{2}\Big(\rho_1 \circ \hat{i}_2^1 \left(RT(\bar f, \tilde f, \bar g, \tilde g)\right) + \rho_\alpha \circ \hat{i}_2^\alpha \left(RT(\bar\alpha\bar f, \alpha\tilde f, \bar g, \tilde g)\right)\Big) \\ &= \frac{1}{2}\left(\rho_1 \circ \hat{i}_2^1 \left(\sum_{\gamma\in\Gamma_2} \operatorname{ind}(\bar f, \bar g; \mathbb{S}_\gamma)\gamma\right) + \rho_\alpha \circ \hat{i}_2^\alpha \left(\sum_{\gamma\in\Gamma_2} \operatorname{ind}(\bar\alpha\bar f, \bar g; \mathbb{S}_\gamma)\gamma\right)\right) \\ &= \sum_{\gamma\in\Gamma_2} \frac{1}{2}\operatorname{ind}(\bar f, \bar g; p_1'\big(\operatorname{Coin}(\gamma\tilde f, \tilde g)\big)\,\gamma + \sum_{\gamma\in\Gamma_2} \frac{1}{2}\operatorname{ind}(\bar\alpha\bar f, \bar g; p_1'\big(\operatorname{Coin}(\gamma\alpha\tilde f, \tilde g)\big)\,\gamma\alpha \\ &= \sum_{\gamma\in\Gamma_2} \operatorname{ind}(f, g; p_1\big(\operatorname{Coin}(\gamma\tilde f, \tilde g)\big)\,\gamma + \sum_{\gamma\in\Gamma_2} \operatorname{ind}(f, g; p_1\big(\operatorname{Coin}(\gamma\alpha\tilde f, \tilde g)\big)\,\gamma\alpha. \end{aligned}$$

Consequently, we have

$$RT(f, \tilde f, g, \tilde g) = \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar\gamma\in\Pi_2/\Gamma_2} \rho_\gamma \circ \hat{i}^\gamma \left(RT(\bar\gamma\bar f, \gamma\tilde f)\right).$$

5 Open questions in the nonorientable case

Nielsen coincidence theory for nonorientable manifolds of the same dimension is generally defined in terms of the "semi-index" of Dobrenko and Jezierski [2]. We denote the semi-index of a coincidence class $\mathbb{S}$ by $|\operatorname{ind}|(f, g; \mathbb{S})$. The semi-index can be used to define a coincidence Reidemeister trace as in (1–3).

$$|RT|(f, \tilde f, g, \tilde g) = \sum_{[\beta]\in\mathcal{R}[\phi,\psi]} |\operatorname{ind}|(f, g; \mathbb{S}_{[\beta]})[\beta] \in \mathbb{Z}\mathcal{R}[\phi, \psi].$$

When the manifolds M_1 and M_2 are orientable, it is well known that $|\mathrm{ind}|(f,g;\mathbb{S}) = |\mathrm{ind}(f,g;\mathbb{S})|$, the absolute value of the classical coincidence index, which implies that

$$|RT|(f,\tilde{f},g,\tilde{g}) = |RT(f,\tilde{f},g,\tilde{g})|,$$

where the absolute values on the right denote taking absolute values of each integer coefficient, after grouping together all terms.

It is natural to ask whether an averaging formula will hold for this semi-index Reidemeister trace.

Question. Does the following formula hold?

$$|RT|(f,\tilde{f},g,\tilde{g}) = \frac{1}{[\Pi_1:\Gamma_1]} \sum_{\bar{\beta}\in\Pi_2/\Gamma_2} \rho_\beta \circ \hat{i}_2^{\beta}\left(|RT|(\bar{\beta}\bar{f},\beta\tilde{f},\bar{g},\tilde{g})\right)$$

In the orientable case semi-index will average appropriately– recall (2–1) which states that

$$\mathrm{ind}(\beta\bar{f},\bar{g};p_1'(\mathrm{Coin}(\gamma\beta\tilde{f},\tilde{g}))) = \#u_{\gamma\beta}(\mathrm{coin}(\tau_{\gamma\beta}\phi,\psi))\cdot \mathrm{ind}(f,g;\mathbb{S}_{[\gamma\beta]}).$$

This implies that for finite regular covers between orientable manifolds we have

$$|\mathrm{ind}|(\beta\bar{f},\bar{g};p_1'(\mathrm{Coin}(\gamma\beta\tilde{f},\tilde{g}))) = \#u_{\gamma\beta}(\mathrm{coin}(\tau_{\gamma\beta}\phi,\psi))\cdot |\mathrm{ind}|(f,g;\mathbb{S}_{[\gamma\beta]}),$$

which will imply the averaging formula above for $|RT|$ in the orientable case.

It is not clear if the formula holds when the manifolds are nonorientable. In the following example, the averaging formula does hold.

Example 5.1. Consider

$$\begin{array}{ccc} S^2 & \xrightarrow{\tilde{f}} & S^2 \\ \big\downarrow p & & \big\downarrow p \\ \mathbb{R}P^2 & \xrightarrow{f} & \mathbb{R}P^2 \end{array}$$

with $f = \mathrm{id}_{\mathbb{R}P^2}$ and $\tilde{f} = \mathrm{id}_{S^2}$. Then the diagram induces the identity endomorphism ϕ of the group $\Pi = \{1,\alpha\}$ of transformations of p. Observe that

- $\mathcal{R}[\phi,\phi] = \{[1],[\alpha]\}$,
- $p\left(\mathrm{Coin}(\tilde{f},\tilde{f})\right) = \mathbb{R}P^2$ and $p\left(\mathrm{Coin}(\alpha\tilde{f},\tilde{f})\right) = \emptyset$,
- $|\mathrm{ind}|(f,f;p\left(\mathrm{Coin}(\tilde{f},\tilde{f})\right)) = 1$
  (by changing one f to be a small rotation), and
  $|\mathrm{ind}|(f,f;p\left(\mathrm{Coin}(\alpha\tilde{f},\tilde{f})\right)) = 0$.

Then we have

$$\begin{aligned} |RT|(f,\tilde{f},f,\tilde{f}) &= |\mathrm{ind}|(f,f;p(\mathrm{Coin}(\tilde{f},\tilde{f})))[1] + |\mathrm{ind}|(f,f;p(\mathrm{Coin}(\alpha\tilde{f},\tilde{f})))[\alpha] \\ &= 1[1] + 0[\alpha] = 1[1] \in \mathbb{Z}\mathcal{R}[\phi,\phi], \\ N(f,f) &= 1. \end{aligned}$$

On the other hand we can compute $|RT|(\tilde{f},\tilde{f},\tilde{f},\tilde{f})$ and $|RT|(\alpha\tilde{f},\alpha\tilde{f},\tilde{f},\tilde{f})$. Because the manifold involved is an orientable manifold S^2, $|RT|$ is the absolute value of the classical RT. For the computation, we remark that $\Gamma = \{1\}$ and so ϕ' is trivial, hence $\bar{\Pi} = \Pi$ and $\bar{\phi} = \phi$. Indeed, we have

$$\begin{aligned}|RT|(\tilde{f},\tilde{f},\tilde{f},\tilde{f}) &= |\mathrm{ind}(\tilde{f},\tilde{f};S^2)|[1] = 2[1] \in \mathbb{Z}\mathcal{R}[\phi',\phi'],\\ |RT|(\alpha\tilde{f},\alpha\tilde{f},\tilde{f},\tilde{f}) &= |\mathrm{ind}(\alpha\tilde{f},\tilde{f};\emptyset)|[1] = 0[1] \in \mathbb{Z}\mathcal{R}[\tau_\alpha\phi',\phi'],\end{aligned}$$

hence

$$\frac{1}{2}\left\{\rho_1 \circ \hat{\iota}_2^1(|RT|(\tilde{f},\tilde{f},\tilde{f},\tilde{f})) + \rho_\alpha \circ \hat{\iota}_2^\alpha(|RT|(\alpha\tilde{f},\alpha\tilde{f},\tilde{f},\tilde{f}))\right\} = \frac{1}{2}\{2[1]+0[1]\}.$$

Consequently, we have

$$|RT|(f,\tilde{f},f,\tilde{f}) = \frac{1}{2}\left\{\rho_1 \circ \hat{\iota}_2^1(|RT|(\tilde{f},\tilde{f},\tilde{f},\tilde{f})) + \rho_\alpha \circ \hat{\iota}_2^\alpha(|RT|(\alpha\tilde{f},\alpha\tilde{f},\tilde{f},\tilde{f}))\right\}.$$

Note that in this example the semi-index Nielsen number does not average:

$$N(f,f) = 1 \neq \frac{1}{2}(1+0) = \frac{1}{2}(N(\tilde{f},\tilde{f}) + N(\alpha\tilde{f},\tilde{f})).$$

This was expected already by [8, Theorem 4.5] as $\mathrm{coin}(\phi,\phi) = \Pi \not\subset \Gamma$.

The real issue in deriving an averaging formula for the semi-index Reidemeister trace seems to be the semi-index itself. The following question seems likely to be easier, but we still do not have an answer:

Question. When M_1 and M_2 are possibly nonorientable, does the following formula hold?

$$|\mathrm{ind}|(f,g;\mathbb{S}) = \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar{\beta}\in\Pi_2/\Gamma_2} |\mathrm{ind}|(\bar{\beta}f,\bar{g};\bar{\mathbb{S}}) \qquad (5\text{–}1)$$

The axiomatic approach is not likely to succeed in answering these questions, since there is no axiomatic formulation of the semi-index. In particular the semi-index is defined only for a coincidence class, not an arbitrary (compact) coincidence set. Therefore the semi-index will not have an additivity property like the one used in [3, 12, 13, 4].

A nonorientable version of Theorem 3.3 is proved in [4] using an index with values in $\mathbb{Z} \oplus \mathbb{Z}_2$ which is closely related to the semi-index. Roughly, the $\mathbb{Z}$-part of the index is analogous to the classical index of nondegenerate coincidence points, while the $\mathbb{Z}_2$-part carries information about any degenerate coincidences. The $\mathbb{Z} \oplus \mathbb{Z}_2$-index of a coincidence class is zero if and only if the semi-index is zero, so a Nielsen number using the $\mathbb{Z} \oplus \mathbb{Z}_2$-index will agree with the traditional semi-index Nielsen number. Unlike the semi-index, the $\mathbb{Z} \oplus \mathbb{Z}_2$ index can be defined for any coincidence set, and has the appropriate additivity property.

The fact that the values are not integers introduces new complications for a formulation of an averaging formula like (5–1) for the $\mathbb{Z} \oplus \mathbb{Z}_2$-index. In particular the factor of $1/[\Pi_1 : \Gamma_1]$ needs to be reinterpreted if the sum gives a non-integer value, and also some adjustment must be made for the fact that when the covering spaces are all orientable, the index of all lifts will have values in $\mathbb{Z}$, while the index of the maps themselves may still have nontrivial $\mathbb{Z}_2$ part.

Still, the axioms for the $\mathbb{Z} \oplus \mathbb{Z}_2$-index may provide a simple way to prove an averaging formula, if it can be properly formulated. Therefore we ask:

Question. Is there an averaging formula for the $\mathbb{Z} \oplus \mathbb{Z}_2$-index analogous to (5–1)?

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