Homogeneous Geodesics in Generalized Wallach Spaces

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Abstract

We classify generalized Wallach spaces which are g.o. spaces. We also investigate homogeneous geodesics in generalized Wallach spaces for any given invariant Riemannian metric and we give some examples.

Introduction

Let (M, g) be a homogeneous Riemannian manifold, i.e. a connected Riemannian manifold on which the largest connected group G of isometries acts transitively. Then M can be expressed as a homogeneous space $(G/K, g)$ where K is the isotropy group at a fixed point o of M, and g is a G-invariant metric. In this case the Lie algebra $\mathfrak{g}$ of G has an $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{m} \subset \mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ and $\mathfrak{k}$ is the Lie algebra of K. In general such decomposition is not unique. The $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{m}$ can be naturally identified with the tangent space T_oM via the projection $\pi : G \to G/K$.

A geodesic $\gamma(t)$ through the origin o of $M = G/K$ is called *homogeneous* if it is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tX)(o), \quad t \in \mathbb{R}, \tag{1}$$

where X is a non zero vector of $\mathfrak{g}$.

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A homogeneous Riemannian manifold is called a *g.o. space*, if all geodesics are homogeneous with respect to the largest connected group of isometries. All naturally reductive spaces are g.o. spaces ([15]), but the converse is not true in general. In [13] A. Kaplan proved the existence of g.o. spaces that are in no way naturally reductive. These are generalized Heisenberg groups with two-dimensional center. In [16] O. Kowalski, F. Prüfer and L. Vanhecke made an explicit classification of all naturally reductive spaces up to dimension five. In [18] O. Kowalski and L. Vanhecke gave a classification of all g.o. spaces, which are in no way naturally reductive, up to dimension six. In [11] C. Gordon described g.o. spaces which are nilmanifolds and in [29] H. Tamaru classified homogeneous g.o. spaces which are fibered over irreducible symmetric spaces. In [6] and [7] O. Kowalski and Z. Dušek investigated homogeneous geodesics in Heisenberh groups and some H-type groups. Examples of g.o. spaces in dimension seven were obtained by Dušek, O. Kowalski and S. Nikčević in ([9]). Also, in [1] the first author and D.V. Alekseevsky classified generalized flag manifolds which are g.o. spaces.

Concerning the existence of homogeneous geodesics in homogeneous Riemannian manifold, we recall the following. V.V. Kajzer proved that a Lie group endowed with a left-invariant metric admits at least one homogeneous geodesic ([12]). O. Kowalski and J. Szenthe extended this result to all homogeneous Riemannian manifolds ([17]). An extension of the result of [17] to reductive homogeneous pseudo-Riemannian manifolds has been also obtained ([7], [24]). Also, O. Kowalski, S. Nikčević and Z. Vlášek studied homogeneous geodesics in homogeneous Riemannian manifolds ([14]), and G. Calvaruso and R. Marinosci studied homogeneous geodesics in three-dimension Lie groups ([21], [5]). Homogeneous geodesics were studied by J. Szenthe ([25], [26], [27], [28]). In addition, D. Latifi studied homogeneous geodesics in homogeneous Finsler spaces ([19]), and the first author investigated homogeneous geodesics in the flag manifold $SO(2l+1)/U(l-m) \times SO(2m+1)$ ([3]).

Homogeneous geodesics in the affine setting were studied in [10]. Finally, D.V. Alekseevsky and Yu. G. Nikonorov in [2] studied the structure of compact g.o. spaces and gave some sufficient conditions for existence and non existence of an invariant metric with homogeneous geodesics on a homogeneous space of a compact Lie group G. They also gave a classification of compact simply connected g.o. spaces of positive Euler characteristic.

Because of these results, it is natural to study g.o. spaces as well as to describe homogeneous geodesics for other large classes of homogeneous spaces. In this paper, we study this problem for generalized Wallach spaces. These spaces were well known before as three-locally-symmetric spaces ([20]), however they were recently classified by Z. Chen, Y. Kang and K. Liang ([4]) and Yu.G. Nikonorov ([23]). We search for homogeneous geodesics in these spaces.

One of the main results in the present paper is Theorem 2, which classifies generalized Wallach spaces which are g.o. spaces. For those which are not, we show how to obtain homogeneous geodesics. We make explicit computations for the three dimensional Lie group $SU(2)$ (thus recovering a result of R.A. Marinosci), and for the Stiefel manifold $SO(4)/SO(2)$.

The paper is organized as follows: In Section 1 we recall the basic definitions and properties of homogeneous geodesics in a Riemannian manifold. In Section 2 we recall the definition of generalized Wallach spaces as well as their classification from [23]. In Section 3 we classify g.o. spaces among generalized Wallach spaces. For those which are not g.o. spaces, in Section 4 we discuss how to find all homogeneous geodesics for a given G-invariant Riemannian metric.

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1 Homogeneous geodesics in homogeneous Riemannian manifolds

Let $(M = G/K, g)$ be a homogeneous Riemannian manifold, where G is compact and semisimple. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{2}$$

be a reductive decomposition. The canonical projection $\pi : G \to G/K$ induces an isomorphism between the subspace $\mathfrak{m}$ and the tangent space T_oM at the identity $o = eK$. The G-invariant metric g induces a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ which is $\mathrm{Ad}(K)$-invariant. Let $B(\cdot, \cdot) = -$Killing form on $\mathfrak{g}$. Then any $\mathrm{Ad}(K)$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ can be expressed as $\langle x, y \rangle = B(\Lambda x, y)$ $(x, y \in \mathfrak{m})$, where Λ is an $\mathrm{Ad}(K)$-equivariant positive definite symmetric operator on $\mathfrak{m}$. Conversely, any such operator Λ determines an $\mathrm{Ad}(K)$-invariant scalar product $\langle x, y \rangle = B(\Lambda x, y)$ on $\mathfrak{m}$, which in turn determines a G-invariant Riemannian metric g on $\mathfrak{m}$. We say that Λ is the *operator associated* to the metric g, or simply the *associated operator*. Also, a Riemannian metric generated by inner product $B(\cdot, \cdot)$ is called *standard metric*.

Definition 1. *A nonzero vector $X \in \mathfrak{g}$ is called a geodesic vector if the curve (1) is a geodesic.*

Lemma 1 ([18]). *A nonzero vector $X \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0, \tag{3}$$

for all $Y \in \mathfrak{m}$. Here the subscript $\mathfrak{m}$ denotes the projection into $\mathfrak{m}$.

A useful description of homogeneous geodesics (1) is provided by the following :

Proposition 1 ([1]). *Let $(M = G/K, g)$ be a homogeneous Riemannian manifold and Λ be the associated operator. Let $a \in \mathfrak{k}$ and $x \in \mathfrak{m}$. Then the following are equivalent:*

(1) The orbit $\gamma(t) = \exp t(a+x) \cdot o$ of the one-parameter subgroup $\exp t(a+x)$ through the point $o = eK$ is a geodesic of M.

(2) $[a+x, \Lambda x] \in \mathfrak{k}$.

(3) $\langle [a,x], y\rangle = \langle x, [x,y]_{\mathfrak{m}}\rangle$ for all $y \in \mathfrak{m}$.

(4) $\langle [a+x, y]_{\mathfrak{m}}, x\rangle = 0$ for all $y \in \mathfrak{m}$.

An important corollary of Proposition 1 is the following:

Corollary 1 ([1]). *Let $(M = G/K, g)$ be a homogeneous Riemannian manifold. Then $(M = G/K, g)$ is a g.o. space if and only if for every $x \in \mathfrak{m}$ there exists an $a(x) \in \mathfrak{k}$ such that*

$$[a(x) + x, \Lambda x] \in \mathfrak{k}.$$

The following Proposition is very important for us to investigate g.o. spaces in generalized Wallach spaces.

Proposition 2. (*[2, Proposition 5]*) *Let $(M = G/K, g)$ be a compact g.o. space with associated operator Λ. Let $X, Y \in \mathfrak{m}$ be eigenvectors Λ with different eigenvalues λ, μ. Then*

$$[X,Y] = \frac{\lambda}{\lambda-\mu}[h,X] + \frac{\mu}{\lambda-\mu}[h,Y] \tag{4}$$

for some $h \in \mathfrak{k}$.

If we want to decide whether a homogeneous Riemannian manifold $(M = G/K, g)$ is a g.o. space or not, we need to find a decomposition of the form (2) and look for geodesic vectors of the form

$$X = \sum_{i=1}^{s} a_i e_i + \sum_{j=1}^{l} x_j A_j. \tag{5}$$

Here $\{e_i : i = 1, 2, \ldots, s\}$ is a basis of $\mathfrak{m}$ and $\{A_j : j = 1, 2, \ldots, l\}$ is a basis of $\mathfrak{k}$. By substituting X into equation (3) we obtain a system of linear algebraic equations for the parameters x_j. If for any $X \in \mathfrak{m} \setminus \{0\}$ this system for the variables x_j has real solutions, then it follows that the homogeneous Riemannian manifold $(M = G/K, g)$ is a g.o. space.

If, on the other hand, we need to find all homogeneous geodesics in the homogeneous Riemannian manifold $(M = G/K, g)$, then we have to calculate all geodesic vectors in the Lie algebra $\mathfrak{g}$. Condition (3) reduces to a system of s quadratic equations for the variables x_j and a_i. Then the geodesic vectors correspond to those solutions of this system for the variables $x_1, \ldots, x_l, a_1, \ldots, a_s$, which are not all equal to zero.

2 Generalized Wallach spaces

Let G/K be a compact homogeneous space with connected compact semisimple Lie group G and a compact subgroup K. Denoted by $\mathfrak{g}$ and $\mathfrak{k}$ Lie algebras of

G and K respectively. We assume that G/K is almost effective, i.e., there are no non-trivial ideals of the Lie algebra $\mathfrak{g}$ in $\mathfrak{k} \subset \mathfrak{g}$. Let $\langle \cdot, \cdot \rangle = B(\Lambda \cdot, \cdot)$ be an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{m}$, where Λ is the associated operator.

We assume that the homogeneous space G/K has the following property. The module $\mathfrak{m}$ decomposes into a direct sum of three $\mathrm{Ad}(K)$-invariant irreducible modules pairwise orthogonal with respect to B, i.e.

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3, \tag{6}$$

such that

$$[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{k} \quad i = 1, 2, 3. \tag{7}$$

A homogeneous space with this property is called *generalized Wallach space*.

Some examples of these spaces are the manifolds of complete flags in the complex, quaternionic, and Cayley projective planes, that is $SU(3)/T_{max}$, $Sp(3)/Sp(1) \times Sp(1) \times Sp(1)$, $F_4/Spin(8)$ (known as Wallach spaces), and the generalized flag manifolds $SU(n_1 + n_2 + n_3)/S(U(n_1) \times U(n_2) \times U(n_3))$, $SO(2n)/U(1) \times U(n-1)$ and $E_6/U(1) \times U(1) \times SO(8)$.

Every generalized Wallach space admits a three parameter family of invariant Riemannian metrics determined by $\mathrm{Ad}(K)$-invariant inner products

$$\langle \cdot, \cdot \rangle = \lambda_1 B(\cdot, \cdot) \mid_{\mathfrak{m}_1} + \lambda_2 B(\cdot, \cdot) \mid_{\mathfrak{m}_2} + \lambda_3 B(\cdot, \cdot) \mid_{\mathfrak{m}_3}, \tag{8}$$

where $\lambda_1, \lambda_2, \lambda_3$ are positive real numbers.

Let d_i be the dimension of $\mathfrak{m}_i$. Let $\{e_i^j\}$ be an orthogonal basis of $\mathfrak{m}_j$ with respect to B, where $j = 1, 2, 3$ and $1 \leq i \leq d_j$. Consider the expression $[ijk]$ defined by the equality

$$[ijk] = \sum_{\alpha, \beta, \gamma} B([e_\alpha^i, e_\beta^j], e_\gamma^k)^2, \tag{9}$$

where α, β, γ range from 1 to d_i, d_j and d_k respectively (cf. [30]). The symbols $[ijk]$ are symmetric in all three indices due to the bi-invariance of the metric B. Moreover, for the spaces under consideration we have $[ijk] = 0$, if two induces coincide.

We recall the classification of generalized Wallach spaces that was recently obtained by Yu.G. Nikoronov ([23]) and Z. Chen, Y. Kang, K. Liang ([4]):

Theorem 1 ([23], [4]). *Let G/K be a connected and simply connected compact homogeneous space. Then G/K is a generalized Wallach space if and only if one of the following types:*

1) G/K is a direct product of three irreducible symmetric spaces of compact type ($[ijk] = 0$ in this case).

2) The group is simple and the pair $(\mathfrak{g}, \mathfrak{k})$ is one of the pairs in Table 1.

3) $G = F \times F \times F \times F$ and $H = diag(F) \subset G$ for some connected simple connected compact simple Lie group F, with the following description on the Lie algebra level:

$$(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f}, diag(\mathfrak{f}) = \{(X, X, X, X) \mid X \in f\},$$

where $\mathfrak{f}$ *is the Lie algebra of* F*, and (up to permutation)* $\mathfrak{m}_1 = \{(X, X, -X, -X) \mid X \in f\}$, $\mathfrak{m}_2 = \{(X, -X, X, -X) \mid X \in f\}$, $\mathfrak{m}_3 = \{(X, -X, -X, X) \mid X \in f\}$.

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{g}$	$\mathfrak{k}$
$\mathfrak{so}(k+l+m)$	$\mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{so}(m)$	$\mathfrak{e}_7$	$\mathfrak{so}(8) \oplus 3\mathfrak{sp}(1)$
$\mathfrak{su}(k+l+m)$	$\mathfrak{su}(k) \oplus su(l) \oplus \mathfrak{su}(m)$	$\mathfrak{e}_7$	$\mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}$
$\mathfrak{sp}(k+l+m)$	$\mathfrak{sp}(k) \oplus \mathfrak{sp}(l) \oplus \mathfrak{sp}(m)$	$\mathfrak{e}_7$	$\mathfrak{so}(8)$
$\mathfrak{su}(2l), l \geq 2$	$\mathfrak{u}(l)$	$\mathfrak{e}_8$	$\mathfrak{so}(12) \oplus 2\mathfrak{sp}(1)$
$\mathfrak{so}(2l), l \geq 4$	$\mathfrak{u}(l) \oplus \mathfrak{u}(l-1)$	$\mathfrak{e}_8$	$\mathfrak{so}(8) \oplus \mathfrak{so}(8)$
$\mathfrak{e}_6$	$\mathfrak{su}(4) \oplus 2\mathfrak{sp}(1) \oplus \mathbb{R}$	$\mathfrak{f}_4$	$\mathfrak{so}(5) \oplus 2\mathfrak{sp}(1)$
$\mathfrak{e}_6$	$\mathfrak{so}(8) \oplus \mathbb{R}^2$	$\mathfrak{f}_4$	$\mathfrak{so}(8)$
$\mathfrak{e}_6$	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$		

Table 1. The pairs $(\mathfrak{g}, \mathfrak{k})$ corresponding to generalized Wallach spaces G/K with G simple.

3 g.o. generalized Wallach spaces

Let $(G/K, g)$ be a generalized Wallach space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, equipped with a G-invariant metric corresponding to a scalar product of the form (8). Let $l = \dim\mathfrak{k}$ and $d_i = \dim(\mathfrak{m}_i)$ $(i = 1, 2, 3)$, and let $\{e_i^0\}$ and $\{e_s^j\}$ be orthogonal bases of $\mathfrak{k}$ and $\mathfrak{m}_j$ respectively with respect to B, where $1 \leq i \leq l$, $j = 1, 2, 3$ and $1 \leq s \leq d_j$. For any $X \in \mathfrak{g} \setminus \{0\}$ we write

$$X = \sum_{i=1}^{l} x_i e_i^0 + \sum_{j=1}^{d_1} a_j e_j^1 + \sum_{k=1}^{d_2} b_k e_k^2 + \sum_{s=1}^{d_3} c_s e_s^3, \quad x_i, a_j, b_k, c_s \in \mathbb{R}. \tag{10}$$

Then by Lemma 1 it follows that X is a geodesic vector if and only if

$$\langle [\sum_{i=1}^{l} x_i e_i^0 + \sum_{j=1}^{d_1} a_j e_j^1 + \sum_{k=1}^{d_2} b_k e_k^2 + \sum_{s=1}^{d_3} c_s e_s^3, Y]_{\mathfrak{m}}, \sum_{j=1}^{d_1} a_j e_j^1 + \sum_{k=1}^{d_2} b_k e_k^2 + \sum_{s=1}^{d_3} c_s e_s^3 \rangle = 0, \tag{11}$$

for all $Y \in \mathfrak{m}$. Hence we obtain the following system of $d_1 + d_2 + d_3$ equations

$$\begin{cases} -\sum_{i=1}^{l}(\sum_{j=1}^{d_1} a_j B([e_i^0, e_1^1]_{\mathfrak{m}}, e_j^1)) x_i \lambda_1 = \sum_{k=1}^{d_2}\sum_{s=1}^{d_3} b_k c_s B([e_k^2, e_1^1]_{\mathfrak{m}}, e_s^3)(\lambda_3 - \lambda_2) \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ -\sum_{i=1}^{l}(\sum_{j=1}^{d_1} a_j B([e_i^0, e_{d_1}^1]_{\mathfrak{m}}, e_j^1)) x_i \lambda_1 = \sum_{k=1}^{d_2}\sum_{s=1}^{d_3} b_k c_s B([e_k^2, e_{d_1}^1]_{\mathfrak{m}}, e_s^3)(\lambda_3 - \lambda_2) \\ -\sum_{i=1}^{l}(\sum_{k=1}^{d_2} b_k B([e_i^0, e_1^2]_{\mathfrak{m}}, e_k^2)) x_i \lambda_2 = \sum_{j=1}^{d_1}\sum_{s=1}^{d_3} a_j c_s B([e_j^1, e_1^2]_{\mathfrak{m}}, e_s^3)(\lambda_3 - \lambda_1) \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ -\sum_{i=1}^{l}(\sum_{k=1}^{d_2} b_k B([e_i^0, e_{d_2}^2]_{\mathfrak{m}}, e_k^2)) x_i \lambda_2 = \sum_{j=1}^{d_1}\sum_{s=1}^{d_3} a_j c_s B([e_j^1, e_{d_2}^2]_{\mathfrak{m}}, e_s^3)(\lambda_3 - \lambda_1) \\ -\sum_{i=1}^{l}(\sum_{s=1}^{d_3} c_s B([e_i^0, e_1^3]_{\mathfrak{m}}, e_s^3)) x_i \lambda_3 = \sum_{j=1}^{d_1}\sum_{k=1}^{d_2} a_j b_k B([e_j^1, e_1^3]_{\mathfrak{m}}, e_k^2)(\lambda_2 - \lambda_1) \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ -\sum_{i=1}^{l} B(\sum_{s=1}^{d_3} c_s B([e_i^0, e_{d_3}^3]_{\mathfrak{m}}, e_s^3)) x_i \lambda_3 = \sum_{j=1}^{d_1}\sum_{k=1}^{d_2} a_j b_k B([e_j^1, e_{d_3}^3]_{\mathfrak{m}}, e_k^2)(\lambda_2 - \lambda_1). \end{cases} \tag{12}$$

If we set

$$A=-\begin{pmatrix} \sum_{j=1}^{d_1} a_j B([e_1^0,e_1^1]_{\mathfrak{m}},e_j^1)\lambda_1 & \cdots & \sum_{j=1}^{d_1} a_j B([e_l^0,e_1^1]_{\mathfrak{m}},e_j^1)\lambda_1 \\ \vdots & & \vdots \\ \sum_{j=1}^{d_1} a_j B([e_1^0,e_{d_1}^1]_{\mathfrak{m}},e_j^1)\lambda_1 & \cdots & \sum_{j=1}^{d_1} a_j B([e_l^0,e_{d_1}^1]_{\mathfrak{m}},e_j^1)\lambda_1 \\ \sum_{k=1}^{d_2} b_k B([e_1^0,e_1^2]_{\mathfrak{m}},e_k^2)\lambda_2 & \cdots & \sum_{k=1}^{d_2} b_k B([e_l^0,e_1^2]_{\mathfrak{m}},e_k^2)\lambda_2 \\ \vdots & & \vdots \\ \sum_{k=1}^{d_2} b_k B([e_1^0,e_{d_2}^2]_{\mathfrak{m}},e_k^2)\lambda_2 & \cdots & \sum_{k=1}^{d_2} b_k B([e_l^0,e_{d_2}^2]_{\mathfrak{m}},e_k^2)\lambda_2 \\ \sum_{s=1}^{d_3} c_s B([e_1^0,e_1^3]_{\mathfrak{m}},e_s^3)\lambda_3 & \cdots & \sum_{s=1}^{d_3} c_s B([e_l^0,e_1^3]_{\mathfrak{m}},e_s^3)\lambda_3 \\ \vdots & & \vdots \\ \sum_{s=1}^{d_3} c_s B([e_1^0,e_{d_3}^3]_{\mathfrak{m}},e_s^3)\lambda_3 & \cdots & \sum_{s=1}^{d_3} c_s B([e_l^0,e_{d_3}^3]_{\mathfrak{m}},e_s^3)\lambda_3 \end{pmatrix},$$

$$\mathtt{X}=\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_l\end{pmatrix},$$

and

$$B=\begin{pmatrix} \sum_{k=1}^{d_2}\sum_{s=1}^{d_3} b_k c_s B([e_k^2,e_1^1]_{\mathfrak{m}},e_s^3)(\lambda_3-\lambda_2) \\ \vdots \\ \sum_{k=1}^{d_2}\sum_{s=1}^{d_3} b_k c_s B([e_k^2,e_{d_1}^1]_{\mathfrak{m}},e_s^3)(\lambda_3-\lambda_2) \\ \sum_{j=1}^{d_1}\sum_{s=1}^{d_3} a_j c_s B([e_j^1,e_1^2]_{\mathfrak{m}},e_s^3)(\lambda_3-\lambda_1) \\ \vdots \\ \sum_{j=1}^{d_1}\sum_{s=1}^{d_3} a_j c_s B([e_j^1,e_{d_2}^2]_{\mathfrak{m}},e_s^3)(\lambda_3-\lambda_1) \\ \sum_{j=1}^{d_1}\sum_{k=1}^{d_2} a_j b_k B([e_j^1,e_1^3]_{\mathfrak{m}},e_k^2)(\lambda_2-\lambda_1) \\ \vdots \\ \sum_{j=1}^{d_1}\sum_{k=1}^{d_2} a_j b_k B([e_j^1,e_{d_3}^3]_{\mathfrak{m}},e_k^2)(\lambda_2-\lambda_1) \end{pmatrix},$$

then System (12) is equivalent to $A\mathtt{X}=B$. Then we have the following:

Proposition 3. *Let $(G/K,g)$ be a generalized Wallach space, where the metric g is determined by the scalar product (8). Then for any a_j $(j=1,\dots,d_1)$, b_k $(k=1,\dots,d_2)$ and c_s $(s=1,\dots,d_3)$ not all equal to zero, $(G/K,g)$ is a g.o. space if and only if* $\mathrm{rank}(A)=\mathrm{rank}(A,B)$.

Proof. By Proposition 1 and Corollary 1 it follows that $(G/K,g)$ is a g.o. space if and only if for any $X\in\mathfrak{m}\setminus\{0\}$ there exists an $a(X)$ such that the curve $\exp t(a(X)+X)\cdot o$ $(t\in\mathbb{R})$ is a geodesic. This means that system (12) of $d_1+d_2+d_3$ equations for the variables x_i $(i=1,\dots,l)$ has real solutions for any a_j,b_k,c_s $(j=1,\dots,d_1,k=1,\dots,d_2,s=1,\dots,d_3)$ not all equal to zero. Then system $A\mathtt{X}=B$ has real solutions if and only if $\mathrm{rank}(A)=\mathrm{rank}(A,B)$. ∎

Next, we shall investigate which of the families of spaces listed in Theorem 1 are g.o. spaces.

Theorem 2. *Let $(G/K, g)$ be a generalized Wallach space as listed in Theorem 1. Then*

1) If $(G/K, g)$ is a space of type 1) then this is a g.o. space for any $\mathrm{Ad}(K)$-invariant Riemannian metric.

2) If $(G/K, g)$ is a space of type 2) or 3) then this is a g.o. space if and only if g is the standard metric.

Proof. Assume that $(G/K, g)$ is a g.o. space, and the corresponding metric $(\lambda_1, \lambda_2, \lambda_3)$ is geodesic orbit with $\lambda_i \neq \lambda_j$. Proposition 2 implies that $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_i \oplus \mathfrak{m}_j$. Since $(G/K, g)$ is a generalized Wallach space, it follows that $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_k, k \notin \{i, j\}$, hence we get $[\mathfrak{m}_i, \mathfrak{m}_j] = 0$.

For case 1) it is $[ijk] = 0$ for all $i, j, k \in \{1, 2, 3\}$. Indeed, formula (9) implies that $B([e^i_\alpha, e^j_\beta], e^k_\gamma) = 0$ for any $i, j, k \in \{1, 2, 3\}$ and for any α, β, γ ranging from 1 to d_i, d_j and d_k respectively. Hence $[e^i_\alpha, e^j_\beta] = 0$ for $i \neq j$ and for any α, β ranging from 1 to d_i and d_j. Therefore, a generalized Wallach space $(G/K, g)$ of type 1) is naturally reductive for any $\mathrm{Ad}(K)$-invariant Riemannian metric $(\lambda_1, \lambda_2, \lambda_3)$, so it is a g.o. space.

For case 2) the only non zero $[ijk]$ is $[123]$ (cf. [22]). Then formula (9) implies that there exist α, β, γ such that $B([e^i_\alpha, e^j_\beta], e^k_\gamma) \neq 0$, where $i \neq j \neq k \neq i$. It follows that there exist α, β such that $[e^i_\alpha, e^j_\beta] \neq 0$ for $i \neq j$. This implies that $[\mathfrak{m}_i, \mathfrak{m}_j] \neq 0$, which is a contradiction. Hence we obtain that $\lambda_i = \lambda_j$. Therefore, a generalized Wallach space $(G/K, g)$ of type 2) or 3) is a g.o. space if and only if g is the standard metric. ∎

4 Homogeneous geodesics in generalized Wallach spaces

Let $(G/K, g)$ be a generalized Wallach space with B-orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ and $B = -$Killing form on $\mathfrak{g}$. Let $\{e^0_i\}$ and $\{e^j_s\}$ be the orthogonal bases of $\mathfrak{k}$ and $\mathfrak{m}_j$ respectively with respect to B ($1 \leq i \leq l$, $j = 1, 2, 3$, $1 \leq s \leq d_j$). For any $X \in \mathfrak{g} \setminus \{0\}$ we write

$$X = \sum_{i=1}^{l} x_i e^0_i + \sum_{j=1}^{d_1} a_j e^1_j + \sum_{k=1}^{d_2} b_k e^2_k + \sum_{s=1}^{d_3} c_s e^3_s, \qquad x_i, a_j, b_k, c_s \in \mathbb{R}.$$

In order to find all homogeneous geodesics in G/K, it suffices to find all the real solutions of the system (12), of $d_1 + d_2 + d_3$ quadratic equations for the variables x_i, a_j, b_k, c_s, which are not all equal to zero.

By Theorem 2 we only consider homogeneous geodesics in generalized Wallach spaces types 2) and 3) given in Theorem 1 for the metric $(\lambda_1, \lambda_2, \lambda_3)$, where at least two of $\lambda_1, \lambda_2, \lambda_3$ are different. Then the geodesic vectors correspond to those solutions of the system (12) for the variables x_i, a_j, b_k, c_s, which are not all equal to zero. However, for many generalized Wallach spaces it is difficult to find all the real solutions of the system (12).

Next, we will give two examples of generalized Wallach spaces and give all the homogeneous geodesics for any given metric.

Example 1. *We consider the generalized Wallach space $SU(2)/\{e\}$.*

Let $\{\sqrt{-1}h_\alpha, \frac{A_\alpha}{\sqrt{2}}, \frac{B_\alpha}{\sqrt{2}}\}$ be an orthogonal basis of $\mathfrak{su}(2)$ with respect to B, where α denotes a simple root of the Lie algebra $\mathfrak{su}(2)$, and $A_\alpha = E_\alpha - E_{-\alpha}$, $B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$. Here $\{E_\alpha\}$ denotes the Weyl basis of $\mathfrak{su}(2)$. We set $X_\alpha = \frac{A_\alpha}{\sqrt{2}}$, $Y_\alpha = \frac{B_\alpha}{\sqrt{2}}$. Then we have that

$$[\sqrt{-1}h_\alpha, X_\alpha] = \alpha(h_\alpha)Y_\alpha = Y_\alpha,$$
$$[\sqrt{-1}h_\alpha, Y_\alpha] = -\alpha(h_\alpha)X_\alpha = -X_\alpha,$$
$$[X_\alpha, Y_\alpha] = \sqrt{-1}h_\alpha.$$

The Lie algebra $\mathfrak{su}(2)$ has an orthogonal decomposition $\mathfrak{su}(2) = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, where $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ are spanned by $\sqrt{-1}h_\alpha, X_\alpha$ and Y_α respectively. For any $X \in \mathfrak{su}(2)$ we write

$$X = a\sqrt{-1}h_\alpha + bX_\alpha + cY_\alpha, \qquad a, b, c \in \mathbb{R}.$$

We will find all geodesic vectors of $SU(2)/\{e\}$ for a given metric $(\lambda_1, \lambda_2, \lambda_3)$. The system (12) for $SU(2)/\{e\}$ is

$$\begin{cases} bc(\lambda_3 - \lambda_2) = 0 \\ ac(\lambda_3 - \lambda_1) = 0 \\ ab(\lambda_2 - \lambda_1) = 0. \end{cases} \tag{13}$$

Case 1. $\lambda_1 = \lambda_2 \neq \lambda_3$. The solutions of the system (13) are $c = 0$ or $c \neq 0$, $a = b = 0$, so the geodesic vectors are $X = a\sqrt{-1}h_\alpha + bX_\alpha$ and $X = cY_\alpha$.
Case 2. $\lambda_1 = \lambda_3 \neq \lambda_2$. The solutions of the system (13) are $b = 0$ or $b \neq 0$, $a = c = 0$, so the geodesic vectors are $X = a\sqrt{-1}h_\alpha + cY_\alpha$ and $X = bX_\alpha$.
Case 3. $\lambda_2 = \lambda_3 \neq \lambda_1$. The solutions of the system (13) are $a = 0$ or $a \neq 0$, $b = c = 0$, so the geodesic vectors are $X = bX_\alpha + cY_\alpha$ and $X = a\sqrt{-1}h_\alpha$.
Case 4. $\lambda_1, \lambda_2, \lambda_3$ are distinct. Then the system (13) reduces to $ab = ac = bc = 0$, whose solutions are $a = b = 0$ or $a = c = 0$ or $b = c = 0$. In this case any vector $X \in \mathfrak{m}_i \backslash \{0\}$ $(i = 1, 2, 3)$ is a geodesic vector.

Therefore we obtain the following theorem, which recovers a result on R.A. Marinosci [21, p. 266]

Proposition 4. *For the generalized Wallach space $SU(2)/\{e\}$ the only geodesic vectors for a given metric $(\lambda_1, \lambda_2, \lambda_3)$ are the following:*
1) If $\lambda_i = \lambda_j \neq \lambda_k$ $(i, j, k \in \{1, 2, 3\})$, then any vector $X \in \mathfrak{m}_k \backslash \{0\}$ or $X \in (\mathfrak{m}_i \oplus \mathfrak{m}_j) \backslash \{0\}$.
2) If $\lambda_1, \lambda_2, \lambda_3$ are distinct, then any vector $X \in \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \mathfrak{m}_3$.

Example 2. *We consider the generalized Wallach space $SO(n)/SO(n-2)$, $(n \geq 4)$.*

This is the Stiefel manifold of orthogonal 2-frames in $\mathbb{R}^n$. Let $\mathfrak{so}(n)$ and $\mathfrak{so}(n-2)$ be the Lie algebras of $SO(n)$ and $SO(n-2)$ respectively. Let E_{ab} denote the $n \times n$ matrix with 1 in the (ab)-entry and 0 elsewhere. If $e_{ab} = E_{ab} - E_{ba}$, then the set $\mathcal{B} = \{e_{ij} = E_{ij} - E_{ji} : 1 \leq i < j \leq n\}$ is a B-orthogonal basis of $\mathfrak{so}(n)$. The multiplication table of the elements in $\mathcal{B}$ is given as follows:

Lemma 2. *If all four indices are distinct, then the Lie brackets in $\mathcal{B}$ are zero. Otherwise, it is $[e_{ij}, e_{jk}] = e_{ik}$, where i, j, k are distinct.*

Let $\mathfrak{so}(n) = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ be an orthogonal decomposition of $\mathfrak{so}(n)$ with respect to B, where $\mathfrak{k} = \mathrm{span}_{\mathbb{R}}\{e_{ij} : 3 \leq i < j \leq n\}$, $\mathfrak{m}_1 = \mathrm{span}_{\mathbb{R}}\{e_{12}\}$, $\mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{e_{1j} : 3 \leq j \leq n\}$, and $\mathfrak{m}_3 = \mathrm{span}_{\mathbb{R}}\{e_{2j} : 3 \leq j \leq n\}$. For any $X \in \mathfrak{so}(n)$ we write

$$X = \sum_{3\leq i<j\leq n} a_{ij}e_{ij} + a_{12}e_{12} + \sum_{3\leq j\leq n} a_{1j}e_{1j} + \sum_{3\leq j\leq n} a_{2j}e_{2j}, \qquad a_{ij} \in \mathbb{R}.$$

Then the system (12) for $SO(n)/SO(n-2)$ takes the form

$$\begin{cases}
(a_{13}a_{23} + a_{14}a_{24} + \cdots + a_{1n}a_{2n})(\lambda_3 - \lambda_2) = 0 \\
-(a_{14}a_{34} + a_{15}a_{35} + \cdots + a_{1,n}a_{3n})\lambda_2 = a_{12}a_{23}(\lambda_3 - \lambda_1) \\
(a_{13}a_{34} - a_{15}a_{45} - a_{16}a_{46} - \cdots - a_{1n}a_{4n})\lambda_2 = a_{12}a_{24}(\lambda_3 - \lambda_1) \\
(a_{13}a_{35} + a_{14}a_{45} - a_{16}a_{56} - a_{17}a_{57} - \cdots - a_{1n}a_{5n})\lambda_2 = a_{12}a_{25}(\lambda_3 - \lambda_1) \\
\qquad\vdots \\
(a_{13}a_{3,n-1} + \cdots + a_{1,n-2}a_{n-2,n-1} - a_{1n}a_{n-1,n})\lambda_2 = a_{12}a_{2,n-1}(\lambda_3 - \lambda_1) \\
(a_{13}a_{3n} + a_{14}a_{4,n} + \cdots + a_{1,n-1}a_{n-1,n})\lambda_2 = a_{12}a_{2n}(\lambda_3 - \lambda_1) \\
-(a_{24}a_{34} + a_{25}a_{35} + \cdots + a_{2n}a_{3n})\lambda_3 = a_{12}a_{13}(\lambda_1 - \lambda_2) \\
(a_{23}a_{34} - a_{25}a_{45} - a_{26}a_{46} - \cdots - a_{2n}a_{4n})\lambda_3 = a_{12}a_{14}(\lambda_1 - \lambda_2) \\
(a_{23}a_{35} + a_{24}a_{45} - a_{26}a_{56} - a_{27}a_{57} - \cdots - a_{2n}a_{5n})\lambda_3 = a_{12}a_{15}(\lambda_1 - \lambda_2) \\
\qquad\vdots \\
(a_{23}a_{3,n-1} + \cdots + a_{2,n-2}a_{n-2,n-1} - a_{2n}a_{n-1,n})\lambda_3 = a_{12}a_{1,n-1}(\lambda_1 - \lambda_2) \\
(a_{23}a_{3n} + a_{24}a_{4n} + \cdots + a_{2,n-1}a_{n-1,n})\lambda_3 = a_{12}a_{1,n}(\lambda_1 - \lambda_2).
\end{cases} \tag{14}$$

As the above system is difficult to handle, we restrict to the Stiefel manifold $SO(4)/SO(2)$ and look for geodesics $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} + a_{14}e_{14} + a_{23}e_{23} + a_{24}e_{24}$. Then the above system simplifies to

$$\begin{cases}
(a_{13}a_{23} + a_{14}a_{24})(\lambda_3 - \lambda_2) = 0 \\
a_{34}a_{14}\lambda_2 = -a_{12}a_{23}(\lambda_3 - \lambda_1) \\
a_{34}a_{13}\lambda_2 = a_{12}a_{24}(\lambda_3 - \lambda_1) \\
a_{34}a_{24}\lambda_3 = a_{13}a_{12}(\lambda_2 - \lambda_1) \\
a_{34}a_{23}\lambda_3 = -a_{14}a_{12}(\lambda_2 - \lambda_1).
\end{cases} \tag{15}$$

In order to find all geodesic vectors in the generalized Wallach space $SO(4)/SO(2)$ for a given metric, we should find all the non zero real solutions of the system (15).

Case 1. $\lambda_1 = \lambda_2 \neq \lambda_3$. Then the system (15) reduces to

$$\begin{cases}
a_{13}a_{23} + a_{14}a_{24} = 0 \\
a_{34}a_{14}\lambda_1 = -a_{12}a_{23}(\lambda_3 - \lambda_1) \\
a_{34}a_{13}\lambda_1 = a_{12}a_{24}(\lambda_3 - \lambda_1).
\end{cases} \tag{16}$$

If $a_{34} = 0$ and $a_{12} = 0$ then the geodesic vectors are $X = a_{13}e_{13} + a_{14}e_{14} + a_{23}e_{23} + a_{24}e_{24}$ with $a_{13}a_{23} + a_{14}a_{24} = 0$.

If $a_{34} = 0$ and $a_{12} \neq 0$ we get $a_{23} = a_{24} = 0$, so the geodesic vectors are $X = a_{12}e_{12} + a_{13}e_{13} + a_{14}e_{14}$.

If $a_{34} \neq 0$ and $a_{12} = 0$ we get $a_{13} = a_{14} = 0$, so the geodesic vectors are $X = a_{34}e_{34} + a_{23}e_{23} + a_{24}e_{24}$.

If $a_{34} \neq 0$, $a_{12} \neq 0$ and $a_{23} + a_{24} = 0$ we have $a_{13} = a_{14}$, geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} + a_{13}e_{14} + a_{23}e_{23} - a_{23}e_{24}$.

If $a_{34} \neq 0$, $a_{12} \neq 0$ and $a_{23} + a_{24} \neq 0$ we have $a_{13} \neq a_{14}$. If $a_{13} = -a_{14}$ we have $a_{23} = a_{24}$, so geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} - a_{13}e_{14} + a_{23}e_{23} + a_{23}e_{24}$. If $a_{13} \neq -a_{14}$ we have $a_{23} \neq a_{24}$, so geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} + a_{14}e_{14} + a_{23}e_{23} + a_{24}e_{24}$.

Case 2. $\lambda_1 = \lambda_3 \neq \lambda_2$. Then the system (15) reduces to

$$\begin{cases} a_{13}a_{23} + a_{14}a_{24} = 0 \\ a_{34}a_{24}\lambda_1 = a_{13}a_{12}(\lambda_2 - \lambda_1) \\ a_{34}a_{23}\lambda_1 = -a_{12}a_{14}(\lambda_2 - \lambda_1). \end{cases} \tag{17}$$

If $a_{34} = 0$ and $a_{12} = 0$, geodesic vectors are $X = a_{13}e_{13} + a_{14}e_{14} + a_{23}e_{23} + a_{24}e_{24}$ with $a_{13}a_{23} + a_{14}a_{24} = 0$.

If $a_{34} = 0$ and $a_{12} \neq 0$ we get $a_{13} = a_{14} = 0$, geodesic vectors are $X = a_{12}e_{12} + a_{23}e_{23} + a_{24}e_{24}$.

If $a_{34} \neq 0$ and $a_{12} = 0$ we get $a_{23} = a_{24} = 0$, geodesic vectors are $X = a_{34}e_{34} + a_{13}e_{13} + a_{14}e_{14}$.

If $a_{34} \neq 0$, $a_{12} \neq 0$ and $a_{23} + a_{24} = 0$ we have $a_{13} = a_{14}$, geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} + a_{13}e_{14} + a_{23}e_{23} - a_{23}e_{24}$.

If $a_{34} \neq 0$, $a_{12} \neq 0$ and $a_{23} + a_{24} \neq 0$ we have $a_{13} \neq a_{14}$. If $a_{13} = -a_{14}$ we have $a_{23} = a_{24}$, so geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} - a_{13}e_{14} + a_{23}e_{23} + a_{23}e_{24}$. If $a_{13} \neq -a_{14}$ we have $a_{23} \neq a_{24}$, so geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} + a_{14}e_{14} + a_{23}e_{23} + a_{24}e_{24}$.

Case 3. $\lambda_3 = \lambda_2 \neq \lambda_1$. Then the system (12) reduces to

$$\begin{cases} a_{34}a_{14}\lambda_2 = -a_{12}a_{23}(\lambda_2 - \lambda_1) \\ a_{34}a_{13}\lambda_2 = a_{12}a_{24}(\lambda_2 - \lambda_1) \\ a_{34}a_{24}\lambda_2 = a_{13}a_{12}(\lambda_2 - \lambda_1) \\ a_{34}a_{23}\lambda_2 = -a_{14}a_{12}(\lambda_2 - \lambda_1). \end{cases} \tag{18}$$

If $a_{34} = 0$ and $a_{12} = 0$, geodesic vectors are $X = a_{13}e_{13} + a_{14}e_{14} + a_{23}e_{23} + a_{24}e_{24}$.

If $a_{34} = 0$ and $a_{12} \neq 0$ we have $a_{13} = a_{14} = a_{23} = a_{24} = 0$, geodesic vectors are $X = a_{12}e_{12}$.

If $a_{34} \neq 0$ and $a_{12} = 0$ we have $a_{13} = a_{14} = a_{23} = a_{24} = 0$, geodesic vectors are $X = a_{34}e_{34}$.

If $a_{34} \neq 0$, $a_{12} \neq 0$ and $a_{23} + a_{24} = 0$ we have $a_{13} = a_{14}$, geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} + a_{13}e_{14} + a_{23}e_{23} - a_{23}e_{24}$.

If $a_{34} \neq 0$, $a_{12} \neq 0$ and $a_{23} + a_{24} \neq 0$ we have $a_{13} \neq a_{14}$. If $a_{13} = -a_{14}$ we have $a_{23} = a_{24}$, so geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} - a_{13}e_{14} + a_{23}e_{23} +$

$a_{23}e_{24}$. If $a_{13} \neq -a_{14}$ we have $a_{23} \neq a_{24}$, so geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} + a_{14}e_{14} + a_{23}e_{23} + a_{24}e_{24}$.

Case 4. $\lambda_1, \lambda_2, \lambda_3$ are all different.

If $a_{34} = 0$ and $a_{12} = 0$, geodesic vectors are $X = a_{13}e_{13} + a_{14}e_{14} + a_{23}e_{23} + a_{24}e_{24}$ with $a_{13}a_{23} + a_{14}a_{24} = 0$.

If $a_{34} = 0$ and $a_{12} \neq 0$ we have $a_{13} = a_{14} = a_{23} = a_{24} = 0$, geodesic vectors are $X = a_{12}e_{12}$.

If $a_{34} \neq 0$ and $a_{12} = 0$ we have $a_{13} = a_{14} = a_{23} = a_{24} = 0$, geodesic vectors are $X = a_{34}e_{34}$.

If $a_{34} \neq 0$, $a_{12} \neq 0$ and $a_{23} + a_{24} = 0$ we have $a_{13} = a_{14}$, geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} + a_{13}e_{14} + a_{23}e_{23} - a_{23}e_{24}$.

If $a_{34} \neq 0$, $a_{12} \neq 0$ and $a_{23} + a_{24} \neq 0$ we have $a_{13} \neq a_{14}$. If $a_{13} = -a_{14}$ we have $a_{23} = a_{24}$, so geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} - a_{13}e_{14} + a_{23}e_{23} + a_{23}e_{24}$. If $a_{13} \neq -a_{14}$ we have $a_{23} \neq a_{24}$, so geodesic vectors are $X = a_{34}e_{34} + a_{12}e_{12} + a_{13}e_{13} + a_{14}e_{14} + a_{23}e_{23} + a_{24}e_{24}$.

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