Solvability of a fractional integral equation with the concept of measure of noncompactness

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Abstract

We study the existence of solutions for a new class of integral equations involving a fractional integral with respect to another function. Our techniques are based on the measure of non-compactness concept combined with a generalized version of Darbo's theorem. Some examples are presented to illustrate the obtained results.

1 Introduction

Many problems in several branches of science such as engineering, physics, biology and other disciplines can be modeled as an integral equation (see, for example [10, 23, 32, 34, 35]). The most used techniques to study the existence of solutions to an integral equation are based on fixed point arguments. For example, Banach contraction principle was used by several authors in order to establish existence results for different kinds of integral equations (see, for example [27, 29, 30, 33]). Similarly, many solvability results were derived via Schauder's fixed point theorem (see, for example [14, 25, 26]). On the other hand, in the absence of compacity and the Lipschitz condition, the above mentioned theorems cannot be applied. In such situation, the measure of non-compactness concept can be useful in order to avoid the mentioned problems.

Several existence results for integral equations were established using fixed point theorems involving measures of non-compactness. For more details, we

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refer the reader to [1, 3, 4, 5, 7, 8, 9, 12, 13, 15, 20, 21, 28] and the references therein. On the other hand, due to the importance of fractional calculus in modelling various problems from many fields of science and engineering, a great attention has been focused recently on the study of fractional integral equations (see, for example [2, 6, 11, 16, 17, 18, 24]).

In this paper, we deal with a new class of fractional integral equations involving a fractional integral with respect to another function. Using a fixed point theorem involving measures of non-compactness, an existence result is established. Some numerical examples are also provided.

2 Problem formulation and preliminaries

The main purpose of this paper is to study the following integral equation

$$x(t) = f_1(t, x(t)) + \frac{f_2(t, x(t))}{\Gamma(\alpha)} \int_a^t \frac{g'(\tau)k(\tau, x(\tau))}{(g(t) - g(\tau))^{1-\alpha}}\, d\tau, \quad t \in [a, T], \tag{2.1}$$

where $\alpha \in (0, 1)$, $0 \leq a < T$, $k : [a, T] \times \mathbb{R} \to \mathbb{R}$, $g : [a, T] \to \mathbb{R}$ and $f_i : [a, T] \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$, from the point of view of the theory of existence of solutions. Eq.(2.1) can be written in the form

$$x(t) = f_1(t, x(t)) + f_2(t, x(t)) I^\alpha_{a,g} k(\cdot, x(\cdot))(t), \quad t \in [a, T],$$

where $I^\alpha_{a,g}$ is the fractional integral of order α with respect to the function g defined by

$$I^\alpha_{a,g}\psi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(\tau)\psi(\tau)}{(g(t) - g(\tau))^{1-\alpha}}\, d\tau, \quad t \in [a, T].$$

In the particular case $g(\tau) = \tau$, Eq.(2.1) models some problems in the queuing theory and biology (see [19]). In such case, $I^\alpha_{a,g}$ is the Riemann-Liouville fractional integral defined by

$$I^\alpha_a\psi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\psi(\tau)}{(t - \tau)^{1-\alpha}}\, d\tau, \quad t \in [a, T]. \tag{2.2}$$

In the case $g(\tau) = \tau^\beta$ $(\beta > 0)$, $I^\alpha_{a,g}$ is the Erdélyi-Kober fractional integral defined by

$$I^\alpha_{a,\beta}\psi(t) = \frac{\beta}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\beta-1}\psi(\tau)}{(t^\beta - \tau^\beta)^{1-\alpha}}\, d\tau, \quad t \in [a, T]. \tag{2.3}$$

However, in the case $g(\tau) = \ln \tau$ and $a > 0$, $I^\alpha_{a,g}$ is the Hadamard fractional integral defined by

$$J^\alpha_a\psi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{\psi(\tau)}{\tau}\, d\tau, \quad t \in [a, T]. \tag{2.4}$$

For more details on fractional calculus, we refer the reader to [31].

Using a measure of non-compactness in the set $C([a,T];\mathbb{R})$ of real continuous functions in $[a,T]$ introduced by Banaś and Goebel [8], we give sufficient conditions for the existence of at least one solution to Eq.(2.1). To the best of our knowledge, an integral equation of the type (2.1) has not been considered earlier.

Before presenting and proving the main results of this paper, we need to recall some basic concepts and present some preliminaries results that will be useful later.

Let $(H, \|\cdot\|)$ be a given Banach space. We denote by B_H the family of all nonempty bounded subsets of H. A mapping $\eta : B_H \to [0,\infty)$ is said to be a measure of non-compactness (see [8]) if it satisfies the following axioms:
(A1) for all $B \in B_H$, we have

$$\eta(B) = 0 \implies B \text{ is precompact;}$$

(A2) for every pair $(B_1, B_2) \in B_H \times B_H$, we have

$$B_1 \subseteq B_2 \Longrightarrow \eta(B_1) \le \eta(B_2);$$

(A3) for every $B \in B_H$,

$$\eta(\overline{B}) = \eta(B) = \eta(\overline{\text{co}}B),$$

where $\overline{\text{co}}B$ denotes the closed convex hull of B;
(A4) for every pair $(B_1, B_2) \in B_H \times B_H$ and $\lambda \in (0,1)$, we have

$$\eta(\lambda B_1 + (1-\lambda)B_2) \le \lambda\eta(B_1) + (1-\lambda)\eta(B_2);$$

(A5) if $\{B_n\}$ is a sequence of closed and decreasing (w.r.t $\subseteq$) sets in B_H such that $\eta(B_n) \to 0$ as $n \to \infty$, then $B_\infty := \bigcap_{n=1}^{\infty} B_n$ is nonempty and compact.

A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be a comparison function if it satisfies the following properties:
(i) φ is non-decreasing;
(ii) $\lim_{n\to\infty} \varphi^{(n)}(t) = 0$, for all $t > 0$, where $\varphi^{(n)}$ denotes the n-iteration of φ.
Examples of comparison functions are

$$\varphi(t) = \lambda\, t, \lambda \in [0,1);\ \varphi(t) = \arctan t;\ \varphi(t) = \ln(1+t);\ \varphi(t) = \frac{t}{1+t}.$$

Through this paper, we denote by Φ the set of comparison functions.

In [5], Aghajani *et al.* proved the following useful result in order to prove that a function is a comparison function.

Lemma 2.1. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a non-decreasing and upper semi-continuous function. Then the following conditions are equivalent:*
(i) $\lim_{n\to\infty} \varphi^{(n)}(t) = 0$ for all $t > 0$.
(ii) $\varphi(t) < t$ for all $t > 0$.

In the same paper, the authors proved the following generalization of Darbo's theorem (see [1]).

Lemma 2.2. *Let C be a nonempty, bounded, closed and convex subset of a Banach space H. Let $U : C \to C$ be a continuous mapping such that*

$$\eta(UW) \leq \varphi(\eta(W)), \quad W \subseteq C,$$

where $\varphi \in \Phi$ and η is a measure of non-compactness in H. Then U has at least one fixed point.

For given functions $\varphi, \psi : [0, \infty) \to \mathbb{R}$ and a real number λ, define the operators $T_{\max}$ and T by

$$T_{\max}(\varphi, \psi)(t) = \max\{\varphi(t), \psi(t)\}, \quad t \geq 0$$

and

$$T(\lambda, \varphi)(t) = \lambda\varphi(t), \quad t \geq 0.$$

Denote by $C([0, \infty); \mathbb{R})$ the set of real continuous functions in $[0, \infty)$ and by $\widetilde{\Phi}$ the set $\Phi \cap C([0, \infty); \mathbb{R})$.

The following lemma is interesting for our purpose.

Lemma 2.3. *The following properties hold:*
(i) $T_{\max}(\widetilde{\Phi} \times \widetilde{\Phi}) \subseteq \widetilde{\Phi}$.
(ii) $T([0, 1] \times \widetilde{\Phi}) \subseteq \widetilde{\Phi}$.

Proof. Let $(\varphi, \psi) \in \widetilde{\Phi} \times \widetilde{\Phi}$. We have to prove that $T_{\max}(\varphi, \psi) : [0, \infty) \to [0, \infty)$ belongs to $\widetilde{\Phi}$. Since the maximum of a finite number of continuous functions is continuous, we have $T_{\max}(\varphi, \psi) \in C([0, \infty); \mathbb{R})$. Moreover, since $(\varphi, \psi) \in \widetilde{\Phi} \times \widetilde{\Phi} \subseteq \Phi \times \Phi$, then φ and ψ are non-decreasing functions, which yields $T_{\max}(\varphi, \psi)$ is also a non-decreasing function. From Lemma 2.1, we have

$$\varphi(t) < t, \ \psi(t) < t, \quad t > 0.$$

Then

$$T_{\max}(\varphi, \psi)(t) < t, \quad t > 0.$$

Again, by Lemma 2.1, we get

$$\lim_{n \to \infty} [T_{\max}(\varphi, \psi)]^{(n)}(t) = 0, \quad t > 0.$$

Therefore, $T_{\max}(\varphi, \psi) \in \Phi$. As consequence, $T_{\max}(\varphi, \psi) \in \Phi \cap C([0, \infty); \mathbb{R}) = \widetilde{\Phi}$, which proves (i). A similar argument can be used to prove (ii). ∎

Let $H = C([a, T]; \mathbb{R})$ be the set of real continuous functions in $[a, T]$. Such a set is a Banach space with respect to the norm

$$\|v\| = \max\{|v(t)| : t \in [a, T]\}, \quad v \in C([a, T]; \mathbb{R}).$$

We will use the following notations through this paper. Let $B \in B_H$. For $v \in B$ and $\rho \geq 0$, set

$$\omega(v, \rho) = \sup\{|v(t) - v(s)| : t, s \in [a, T], |t - s| \leq \rho\}.$$

We define the mapping $\Omega : B_H \times [0,\infty) \to [0,\infty)$ by

$$\Omega(B,\rho) = \sup\{\omega(v,\rho) : v \in B\}, \quad (B,\rho) \in B_H \times [0,\infty).$$

It was proved in [8] that the mapping $\eta : B_H \to [0,\infty)$ defined by

$$\eta(B) = \lim_{\rho \to 0^+} \Omega(B,\rho), \quad B \in B_H$$

is a measure of non-compactness in the Banach space H.

Now, we are ready to present our main results. This is the aim of the next section.

3 Main results

Our first existence result is obtained under the following hypotheses:
(H1) The functions $f_i : [a,T] \times \mathbb{R} \to \mathbb{R}$ $(i = 1,2)$ are continuous.
(H2) There exist functions $\varphi_1, \varphi_2 : [0,\infty) \to [0,\infty)$ such that
(i) $\varphi_i(0) = 0, i = 1,2$.
(ii) φ_1 is non-decreasing and continuous.
(iii) $\varphi_2 \in \widetilde{\Phi}$.
(iv) There exist $\delta \geq 0$ and $\theta \in \widetilde{\Phi}$ such that

$$\varphi_1(t) \leq \delta\theta(t), \quad t \geq 0.$$

(v) For all $(t,u,v) \in [a,T] \times \mathbb{R} \times \mathbb{R}$, we have

$$|f_i(t,u) - f_i(t,v)| \leq \varphi_i(|u-v|), \quad i = 1,2.$$

(H3) The function $k : [a,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and it satisfies

$$|k(t,u)| \leq \varphi(|u|), \quad (t,u) \in [a,T] \times \mathbb{R},$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a non-decreasing function.
(H4) $g \in C([a,T];\mathbb{R}) \cap C^1((a,T];\mathbb{R})$ with $g'(t) > 0$, for all $t \in (a,T]$.
(H5) There exists $r_0 > 0$ such that

$$(\varphi_1(r_0) + f_1^*)\Gamma(\alpha+1) + (\varphi_2(r_0) + f_2^*)\varphi(r_0)(g(T) - g(a))^\alpha \leq r_0\Gamma(\alpha+1)$$

and, moreover

$$\delta + \frac{\varphi(r_0)}{\Gamma(\alpha+1)}(g(T) - g(a))^\alpha \leq 1,$$

where

$$f_i^* = \max\{|f_i(t,0)| : t \in [a,T]\}, \quad i = 1,2.$$

Our main result is the following.

Theorem 3.1. *Under the assumptions* (H1)-(H5), Eq.(2.1) has at least one solution $y^* \in C([a,T];\mathbb{R})$. Moreover, we have $\|y^*\| \leq r_0$.

Proof. Let us consider the operator U defined on $H = C([a,T];\mathbb{R})$ by

$$(Ux)(t) = f_1(t,x(t)) + \frac{f_2(t,x(t))}{\Gamma(\alpha)} \int_a^t \frac{g'(\tau)k(\tau,x(\tau))}{(g(t)-g(\tau))^{1-\alpha}}\, d\tau, \quad (x,t) \in H \times [a,T]. \tag{3.1}$$

At first, we show that the operator U maps H into itself. Set

$$(Gx)(t) = \int_a^t \frac{g'(\tau)k(\tau,x(\tau))}{(g(t)-g(\tau))^{1-\alpha}}\, d\tau, \quad (x,t) \in H \times [a,T]. \tag{3.2}$$

From assumption (H1), we have just to prove that G maps H into itself, that is, $Gx : [a,T] \to \mathbb{R}$ is continuous for every $x \in H$. Observe that Gx is well-defined. In fact, for all $t \in [a,T]$, from assumption (H3), we have

$$|(Gx)(t)| \leq \varphi(\|x\|) \int_a^t \frac{g'(\tau)}{(g(t)-g(\tau))^{1-\alpha}}\, d\tau = \frac{\varphi(\|x\|)}{\alpha}(g(t)-g(a))^{\alpha} < \infty. \tag{3.3}$$

Let us check the continuity of Gx at the point a. Let $\{t_n\}$ be a sequence in $[a,T]$ such that $t_n \to a^+$ as $n \to \infty$. Using (3.3), for all $n \in \mathbb{N}$, we have

$$|(Gx)(t_n)| \leq \varphi(\|x\|) \int_a^{t_n} \frac{g'(\tau)}{(g(t_n)-g(\tau))^{1-\alpha}}\, d\tau = \frac{\varphi(\|x\|)}{\alpha}(g(t_n)-g(a))^{\alpha}.$$

Letting $n \to \infty$ in the above inequality and using the continuity of g, we get

$$\lim_{n\to\infty} (Gx)(t_n) = 0 = (Gx)(a),$$

which proves the continuity of Gx at the point a. Now, let $t \in (a,T]$ be fixed and $\{t_n\}$ be a sequence in $(a,T]$ such that $t_n \to t$ as $n \to \infty$. Without restriction of the generality, we may assume that $t_n \geq t$ for n large enough. We have

$$\begin{aligned} &|(Gx)(t_n) - (Gx)(t)| = \\ &\qquad \left| \int_a^{t_n} \frac{g'(\tau)k(\tau,x(\tau))}{(g(t_n)-g(\tau))^{1-\alpha}}\, d\tau - \int_a^{t} \frac{g'(\tau)k(\tau,x(\tau))}{(g(t)-g(\tau))^{1-\alpha}}\, d\tau \right|, \quad n \in \mathbb{N}. \end{aligned}$$

For n large enough, we can write

$$\begin{aligned} &|(Gx)(t_n) - (Gx)(t)| \\ &\quad \leq \left| \int_a^t \left(\frac{g'(\tau)k(\tau,x(\tau))}{(g(t_n)-g(\tau))^{1-\alpha}} - \frac{g'(\tau)k(\tau,x(\tau))}{(g(t)-g(\tau))^{1-\alpha}} \right) d\tau \right| \\ &\qquad + \left| \int_t^{t_n} \frac{g'(\tau)k(\tau,x(\tau))}{(g(t_n)-g(\tau))^{1-\alpha}}\, d\tau \right| \\ &\quad \leq \varphi(\|x\|) \int_a^t \left(\frac{g'(\tau)}{(g(t)-g(\tau))^{1-\alpha}} - \frac{g'(\tau)}{(g(t_n)-g(\tau))^{1-\alpha}} \right) d\tau \\ &\qquad + \varphi(\|x\|) \int_t^{t_n} \frac{g'(\tau)}{(g(t_n)-g(\tau))^{1-\alpha}}\, d\tau \\ &\quad = \frac{\varphi(\|x\|)}{\alpha} \left((g(t)-g(a))^{\alpha} + (g(t_n)-g(t))^{\alpha} - (g(t_n)-g(a))^{\alpha} \right) \\ &\qquad + \frac{\varphi(\|x\|)}{\alpha}(g(t_n)-g(t))^{\alpha}. \end{aligned}$$

Since g is continuous in $[a,T]$, we have

$$\lim_{n\to\infty} \frac{\varphi(\|x\|)}{\alpha}\left((g(t)-g(a))^{\alpha}+(g(t_n)-g(t))^{\alpha}-(g(t_n)-g(a))^{\alpha}\right)+ \frac{\varphi(\|x\|)}{\alpha}(g(t_n)-g(t))^{\alpha}=0,$$

which yields $\lim_{n\to\infty}|(Gx)(t_n)-(Gx)(t)| = 0$. Then Gx is continuous at t. As consequence, $Gx \in H$, for all $x\in H$, and $U: H\to H$ is well-defined.

On the other hand, for an arbitrarily fixed $x\in H$ and $t\in[a,T]$, we have

$$\begin{aligned}
|(Ux)(t)| &\le |f_1(t,x(t))| + \frac{|f_2(t,x(t))|}{\Gamma(\alpha)}\int_a^t \frac{g'(\tau)|k(\tau,x(\tau))|}{(g(t)-g(\tau))^{1-\alpha}}\,d\tau \\
&\le |f_1(t,x(t))-f_1(t,0)|+|f_1(t,0)| \\
&\quad + \frac{|f_2(t,x(t))-f_2(t,0)|+|f_2(t,0)|}{\Gamma(\alpha)}\int_a^t \frac{g'(\tau)\varphi(|x(\tau)|)}{(g(t)-g(\tau))^{1-\alpha}}\,d\tau \\
&\le \varphi_1(|x(t)|)+f_1^* + \frac{(\varphi_2(|x(t)|)+f_2^*)\varphi(\|x\|)}{\Gamma(\alpha+1)}(g(t)-g(a))^{\alpha} \\
&\le \varphi_1(\|x\|)+f_1^* + \frac{(\varphi_2(\|x\|)+f_2^*)\varphi(\|x\|)}{\Gamma(\alpha+1)}(g(T)-g(a))^{\alpha}.
\end{aligned}$$

Then

$$\|Ux\| \le \varphi_1(\|x\|)+f_1^* + \frac{(\varphi_2(\|x\|)+f_2^*)\varphi(\|x\|)}{\Gamma(\alpha+1)}(g(T)-g(a))^{\alpha}, \quad x\in H.$$

From the above inequality, the fact that the functions $\varphi_1,\varphi_2,\varphi : [0,\infty)\to[0,\infty)$ are non-decreasing, and assumption (H5), we infer that the operator U maps $\overline{B(0,r_0)}$ into itself, where

$$\overline{B(0,r_0)} = \{x\in H:\ \|x\|\le r_0\}.$$

Now, we claim that the operator $U : \overline{B(0,r_0)} \to \overline{B(0,r_0)}$ is continuous. We write U in the form

$$Ux = F_1x + \frac{1}{\Gamma(\alpha)}F_2x + GX, \quad x\in \overline{B(0,r_0)},$$

where

$$(F_ix)(t) = f_i(t,x(t)), \quad (x,t)\in \overline{B(0,r_0)}\times[a,T],\ i=1,2$$

and G is defined by (3.2). In order to prove our claim, it is sufficient to show that the operators F_i ($i=1,2$) and G are continuous on $\overline{B(0,r_0)}$. Firstly, we show that F_i are continuous operators on $\overline{B(0,r_0)}$. To do this, we take a sequence $\{x_n\}\subset \overline{B(0,r_0)}$ and $x\in\overline{B(0,r_0)}$ such that $\|x_n-x\|\to 0$ as $n\to\infty$, and we have to prove that $\|F_ix_n-F_ix\|\to 0$ as $n\to\infty$. In fact, for all $t\in[a,T]$, we have

$$\begin{aligned}
|(F_ix_n)(t)-(F_ix)(t)| &= |f_i(t,x_n(t))-f_i(t,x(t))| \\
&\le \varphi_i(|x_n(t)-x(t)|) \\
&\le \varphi_i(\|x_n-x\|).
\end{aligned}$$

Then

$$\|F_i x_n - F_i x\| \leq \varphi_i(\|x_n - x\|), \quad n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality, using the continuity of the functions φ_i ($i = 1, 2$) and the fact that $\varphi_i(0) = 0$, we get

$$\lim_{n\to\infty} \|F_i x_n - F_i x\| \leq \varphi_i\left(\lim_{n\to\infty} \|x_n - x\|\right) = \varphi_i(0) = 0.$$

This proves that F_i is a continuous operator on $\overline{B(0, r_0)}$, for all $i = 1, 2$. Next, we show that G is a continuous operator on $\overline{B(0, r_0)}$. To do this, we fix a real number $\varepsilon > 0$ and we take arbitrary functions $x, y \in \overline{B(0, r_0)}$ such that $\|x - y\| < \varepsilon$. For all $t \in [a, T]$, we have

$$\begin{aligned} |(Gx)(t) - (Gy)(t)| &\leq \int_a^t \frac{g'(\tau)|k(\tau, x(\tau)) - k(\tau, y(\tau))|}{(g(t) - g(\tau))^{1-\alpha}}\, d\tau \\ &\leq u(r_0, \varepsilon) \int_a^t \frac{g'(\tau)}{(g(t) - g(\tau))^{1-\alpha}}\, d\tau \\ &= \frac{u(r_0, \varepsilon)}{\alpha}(g(t) - g(a))^\alpha \\ &\leq \frac{u(r_0, \varepsilon)}{\alpha}(g(T) - g(a))^\alpha, \end{aligned}$$

where

$$u(r_0, \varepsilon) = \sup\{|k(\tau, v) - k(\tau, w)| : \tau \in [a, T],\, v, w \in [-r_0, r_0],\, |v - w| < \varepsilon\}.$$

Therefore,

$$\|Gx - Gy\| \leq \frac{u(r_0, \varepsilon)}{\alpha}(g(T) - g(a))^\alpha.$$

Since k is uniform continuous on the compact $[a, T] \times [-r_0, r_0]$, we have $u(r_0, \varepsilon) \to 0$ as $\varepsilon \to 0^+$ and, therefore, the last inequality gives us

$$\lim_{\varepsilon\to 0^+} \|Gx - Gy\| = 0.$$

Then G is continuous on $\overline{B(0, r_0)}$ and U maps continuously the set $\overline{B(0, r_0)}$ into itself.

Further, let W be a nonempty subset of $\overline{B(0, r_0)}$. Let $\rho > 0$ be fixed, $x \in W$ and $t_1, t_2 \in [a, T]$ be such that $|t_1 - t_2| \leq \rho$. Without restriction of the generality, we may assume $t_1 \geq t_2$. We have

$$
\begin{aligned}
&|(Ux)(t_1) - (Ux)(t_2)| \\
&\leq |f_1(t_1, x(t_1)) - f_1(t_2, x(t_2))| \\
&+ \left| \frac{f_2(t_1, x(t_1))}{\Gamma(\alpha)} \int_a^{t_1} \frac{g'(\tau)k(\tau, x(\tau))}{(g(t_1) - g(\tau))^{1-\alpha}} \, d\tau - \frac{f_2(t_2, x(t_2))}{\Gamma(\alpha)} \int_a^{t_2} \frac{g'(\tau)k(\tau, x(\tau))}{(g(t_2) - g(\tau))^{1-\alpha}} \, d\tau \right| \\
&\leq |f_1(t_1, x(t_1)) - f_1(t_2, x(t_2))| \\
&+ \left| \frac{f_2(t_1, x(t_1))}{\Gamma(\alpha)} \int_a^{t_1} \frac{g'(\tau)k(\tau, x(\tau))}{(g(t_1) - g(\tau))^{1-\alpha}} \, d\tau - \frac{f_2(t_2, x(t_2))}{\Gamma(\alpha)} \int_a^{t_1} \frac{g'(\tau)k(\tau, x(\tau))}{(g(t_1) - g(\tau))^{1-\alpha}} \, d\tau \right| \\
&+ \frac{|f_2(t_2, x(t_2))|}{\Gamma(\alpha)} \left| \int_a^{t_1} \frac{g'(\tau)k(\tau, x(\tau))}{(g(t_1) - g(\tau))^{1-\alpha}} \, d\tau - \int_a^{t_2} \frac{g'(\tau)k(\tau, x(\tau))}{(g(t_2) - g(\tau))^{1-\alpha}} \, d\tau \right| \\
&\leq |f_1(t_1, x(t_1)) - f_1(t_1, x(t_2))| + |f_1(t_1, x(t_2)) - f_1(t_2, x(t_2))| \\
&+ \frac{|f_2(t_1, x(t_1)) - f_2(t_1, x(t_2))| + |f_2(t_1, x(t_2)) - f_2(t_2, x(t_2))|}{\Gamma(\alpha)} \int_a^{t_1} \frac{g'(\tau)|k(\tau, x(\tau))|}{(g(t_1) - g(\tau))^{1-\alpha}} \, d\tau \\
&+ \frac{|f_2(t_2, x(t_2)) - f_2(t_2, 0)| + |f_2(t_2, 0)|}{\Gamma(\alpha)} \left| \int_a^{t_1} \frac{g'(\tau)k(\tau, x(\tau))}{(g(t_1) - g(\tau))^{1-\alpha}} \, d\tau - \int_a^{t_2} \frac{g'(\tau)k(\tau, x(\tau))}{(g(t_1) - g(\tau))^{1-\alpha}} \, d\tau \right| \\
&+ \frac{|f_2(t_2, x(t_2)) - f_2(t_2, 0)| + |f_2(t_2, 0)|}{\Gamma(\alpha)} \left| \int_a^{t_2} \frac{g'(\tau)k(\tau, x(\tau))}{(g(t_1) - g(\tau))^{1-\alpha}} \, d\tau - \int_a^{t_2} \frac{g'(\tau)k(\tau, x(\tau))}{(g(t_2) - g(\tau))^{1-\alpha}} \, d\tau \right| \\
&\leq \varphi_1(|x(t_1) - x(t_2)|) + \omega_{f_1}(r_0, \rho) + \frac{\varphi_2(|x(t_1) - x(t_2)|) + \omega_{f_2}(r_0, \rho)}{\Gamma(\alpha + 1)} \varphi(r_0)(g(t_1) - g(a))^{\alpha} \\
&+ \frac{(\varphi_2(r_0) + f_2^*)\varphi(r_0)}{\Gamma(\alpha)} \int_{t_2}^{t_1} \frac{g'(\tau)}{(g(t_1) - g(\tau))^{1-\alpha}} \, d\tau \\
&+ \frac{(\varphi_2(r_0) + f_2^*)\varphi(r_0)}{\Gamma(\alpha + 1)} \left((g(t_2) - g(a))^{\alpha} + (g(t_1) - g(t_2))^{\alpha} - (g(t_1) - g(a))^{\alpha}\right) \\
&\leq \varphi_1(\omega(x, \rho)) + \omega_{f_1}(r_0, \rho) + \frac{\varphi_2(\omega(x, \rho)) + \omega_{f_2}(r_0, \rho)}{\Gamma(\alpha + 1)} \varphi(r_0)(g(T) - g(a))^{\alpha} \\
&+ \frac{2(\varphi_2(r_0) + f_2^*)\varphi(r_0)}{\Gamma(\alpha + 1)} (g(t_1) - g(t_2))^{\alpha} \\
&\leq \varphi_1(\Omega(W, \rho)) + \omega_{f_1}(r_0, \rho) + \frac{\varphi_2(\Omega(W, \rho)) + \omega_{f_2}(r_0, \rho)}{\Gamma(\alpha + 1)} \varphi(r_0)(g(T) - g(a))^{\alpha} \\
&+ \frac{2(\varphi_2(r_0) + f_2^*)\varphi(r_0)}{\Gamma(\alpha + 1)} [\omega(g, \rho)]^{\alpha},
\end{aligned}
$$

where

$$\omega_{f_i}(r_0, \rho) = \sup\{|f_i(t, y) - f_i(s, u)| : u \in [-r_0, r_0], t, s \in [a, T], |t - s| \leq \rho\}, \quad i = 1, 2.$$

Therefore,

$$
\begin{aligned}
\omega(Ux, \rho) \leq \varphi_1(\Omega(W, \rho)) + \omega_{f_1}(r_0, \rho) + \\
\frac{\varphi_2(\Omega(W, \rho)) + \omega_{f_2}(r_0, \rho)}{\Gamma(\alpha + 1)} \varphi(r_0)(g(T) - g(a))^{\alpha} + \frac{2(\varphi_2(r_0) + f_2^*)\varphi(r_0)}{\Gamma(\alpha + 1)} [\omega(g, \rho)]^{\alpha}.
\end{aligned}
$$

Passing to the limit as $\rho \to 0^+$, we obtain

$$\eta(UW) \leq \varphi_1(\eta(W)) + \frac{\varphi_2(\eta(W))}{\Gamma(\alpha + 1)} \varphi(r_0)(g(T) - g(a))^{\alpha}.$$

Taking in consideration assumption (H2)-(iv), we get

$$\begin{aligned} \eta(UW) &\leq \left(\delta + \frac{\varphi(r_0)(g(T)-g(a))^\alpha}{\Gamma(\alpha+1)}\right) T_{\max}(\theta, \varphi_2)(\eta(W)) \\ &= T(\lambda, T_{\max}(\theta, \varphi_2))(\eta(W)), \end{aligned}$$

where

$$\lambda = \delta + \frac{\varphi(r_0)(g(T)-g(a))^\alpha}{\Gamma(\alpha+1)}.$$

Since $\lambda \in [0,1]$ (see (H5)), from Lemma 2.3, $T(\lambda, T_{\max}(\theta, \varphi_2)) \in \Phi$. Finally, by Lemma 2.2, the operator U has at least one fixed point $y^* \in \overline{B(0,r_0)}$, which is a solution to Eq.(2.1) satisfying $\|y^*\| \leq r_0$. This completes the proof. ∎

If in Theorem 3.1 we add the following assumptions:
(H6) For $i = 1,2$, $f_i([a,T] \times [0,\infty)) \subseteq [0,\infty)$ and f_i is non-decreasing with respect to each variable on $[a,T] \times [0,\infty)$.
(H7) $k([a,T] \times [0,\infty)) \subseteq [0,\infty)$ and k is non-decreasing with respect to each variable on $[a,T] \times [0,\infty)$.
Then we obtain the following result.

Theorem 3.2. *Under the assumptions* (H1)-(H7), Eq.(2.1) has at least one solution $y^* \in C([a,T];\mathbb{R})$ which is non-negative and non-decreasing on the interval $[a,T]$.

Proof. Let us consider the operator U defined on $H = C([a,T];\mathbb{R})$ by (3.1). Following the proof of Theorem 3.1, we know that the operator U maps H into itself, and that, for $x \in H$, we have the estimate

$$\|Ux\| \leq \varphi_1(\|x\|) + f_1^* + \frac{(\varphi_2(\|x\|) + f_2^*)\varphi(\|x\|)}{\Gamma(\alpha+1)}(g(T)-g(a))^\alpha, \quad x \in H,$$

which yields $U(\overline{B(0,r_0)}) \subseteq \overline{B(0,r_0)}$. Now, we consider the subset $B_{r_0}^+$ of $\overline{B(0,r_0)}$ defined by

$$B_{r_0}^+ = \{x \in \overline{B(0,r_0)} : x(t) \geq 0,\, t \in [a,T]\}.$$

In virtue (H6) and (H7), we may infer that the operator U maps $B_{r_0}^+$ into itself. Following the same argument used in the proof of Theorem 3.1, we obtain that U is a continuous operator on $B_{r_0}^+$. Next, we take the subset Q of $B_{r_0}^+$ defined by

$$Q = \{x \in B_{r_0}^+ : x \text{ is non-decreasing}\}.$$

Obviously, the set Q is nonempty, bounded, closed and convex. Let us prove that the operator U maps Q into itself. To do this, let $x \in Q$ be fixed and (t,s) be a pair of elements in $[a,T] \times [a,T]$ such that $t \geq s$. By assumption (H6), we have

$$f_i(t,x(t)) \geq f_i(s,x(s)), \quad i = 1,2.$$

So we have just to prove that

$$\int_a^t \frac{g'(\tau)k(\tau,x(\tau))}{(g(t)-g(\tau))^{1-\alpha}}\,d\tau \geq \int_a^s \frac{g'(\tau)k(\tau,x(\tau))}{(g(s)-g(\tau))^{1-\alpha}}\,d\tau. \tag{3.4}$$

We have

$$\begin{aligned}
&\int_a^t \frac{g'(\tau)k(\tau,x(\tau))}{(g(t)-g(\tau))^{1-\alpha}}\,d\tau - \int_a^s \frac{g'(\tau)k(\tau,x(\tau))}{(g(s)-g(\tau))^{1-\alpha}}\,d\tau \\
&= \int_a^s \left(\frac{g'(\tau)}{(g(t)-g(\tau))^{1-\alpha}} - \frac{g'(\tau)}{(g(s)-g(\tau))^{1-\alpha}}\right) k(\tau,x(\tau))\,d\tau \\
&+ \int_s^t \frac{g'(\tau)k(\tau,x(\tau))}{(g(t)-g(\tau))^{1-\alpha}}\,d\tau.
\end{aligned}$$

Using assumption (H7), we obtain

$$\begin{aligned}
&\int_a^t \frac{g'(\tau)k(\tau,x(\tau))}{(g(t)-g(\tau))^{1-\alpha}}\,d\tau - \int_a^s \frac{g'(\tau)k(\tau,x(\tau))}{(g(s)-g(\tau))^{1-\alpha}}\,d\tau \\
&\geq k(s,x(s)) \int_a^s \left(\frac{g'(\tau)}{(g(t)-g(\tau))^{1-\alpha}} - \frac{g'(\tau)}{(g(s)-g(\tau))^{1-\alpha}}\right) d\tau \\
&+k(s,x(s)) \int_s^t \frac{g'(\tau)}{(g(t)-g(\tau))^{1-\alpha}}\,d\tau \\
&= k(s,x(s)) \left(\int_a^t \frac{g'(\tau)}{(g(t)-g(\tau))^{1-\alpha}}\,d\tau - \int_a^s \frac{g'(\tau)}{(g(s)-g(\tau))^{1-\alpha}}\,d\tau\right) \\
&= \frac{k(s,x(s))}{\alpha}\left((g(t)-g(a))^\alpha - (g(s)-g(a))^\alpha\right) \geq 0.
\end{aligned}$$

Then (3.4) holds and Ux is a non-decreasing function. Therefore, u maps Q into itself. Finally, using the same argument of the proof of Theorem 3.1, we obtain

$$\eta(UW) \leq T(\lambda, T_{\max}(\theta,\varphi_2))(\eta(W)), \quad W \subseteq Q,$$

where

$$\lambda = \delta + \frac{\varphi(r_0)(g(T)-g(a))^\alpha}{\Gamma(\alpha+1)} \in [0,1].$$

An application of Lemma 2.2 gives us the desired result. ■

4 Particular cases and examples

In this section, using Theorem 3.1, we give some existence results for some functional equations involving various types of fractional integrals. We present also some illustrative examples.

4.1 A functional equation involving Riemann-Liouville fractional integral

Let us consider the integral equation

$$x(t) = f_1(t,x(t)) + \frac{f_2(t,x(t))}{\Gamma(\alpha)} \int_a^t \frac{k(\tau,x(\tau))}{(t-\tau)^{1-\alpha}}\,d\tau, \quad t \in [a,T], \tag{4.1}$$

where $\alpha \in (0,1)$, $0 \le a < T$, $k : [a,T] \times \mathbb{R} \to \mathbb{R}$ and $f_i : [a,T] \times \mathbb{R} \to \mathbb{R}$ $(i = 1,2)$. Eq.(4.1) can be written as

$$x(t) = f_1(t, x(t)) + f_2(t, x(t)) I_a^\alpha k(\cdot, x(\cdot))(t), \quad t \in [a,T],$$

where I_a^α is the Riemann-Liouville fractional integral of order α defined by (2.2).

Take

$$g(t) = t, \quad t \in [a,T]$$

in Theorem 3.1, we obtain the following existence result.

Corollary 4.1. *Suppose that all the assumptions* (H1)-(H3) are satisfied. Moreover, suppose that there exists $r_0 > 0$ such that

$$(\varphi_1(r_0) + f_1^*)\Gamma(\alpha+1) + (\varphi_2(r_0) + f_2^*)\varphi(r_0)(T-a)^\alpha \le r_0 \Gamma(\alpha+1)$$

and

$$\delta + \frac{\varphi(r_0)}{\Gamma(\alpha+1)}(T-a)^\alpha \le 1.$$

Then (4.1) has at least one solution $y^* \in C([a,T];\mathbb{R})$. Moreover, we have $\|y^*\| \le r_0$.

We present the following example to illustrate the above result.

Example 4.2. Consider the integral equation

$$x(t) = \frac{t}{2} + \frac{\sqrt{\pi}}{8}\left(\frac{t}{3} + |x(t)|\right) \int_0^t \frac{x(\tau)}{(\tau^2+1)\sqrt{t-\tau}}\, d\tau, \quad t \in [0,1]. \tag{4.2}$$

The above equation can be written in the form (4.1) with $a = 0$, $T = 1$, $\alpha = \frac{1}{2}$,

$$\begin{aligned} f_1(t,u) &= \frac{t}{2}, \quad (t,u) \in [0,1] \times \mathbb{R}, \\ f_2(t,u) &= \frac{\sqrt{\pi}}{8}\left(\frac{t}{3} + |u|\right), \quad (t,u) \in [0,1] \times \mathbb{R}, \\ k(t,u) &= \frac{\sqrt{\pi}}{2} \frac{u}{(t^2+1)}, \quad (t,u) \in [0,1] \times \mathbb{R}. \end{aligned}$$

We can check easily that all the assumptions (H1)-(H3) are satisfied with $\varphi_1(t) = \theta(t) = \delta = 0$, $\varphi_2(t) = \frac{\sqrt{\pi}}{8} t$ and $\varphi(r) = \frac{\sqrt{\pi}}{2} r$. Moreover, in this case, we have $f_1^* = \frac{1}{2}$ and $f_2^* = \frac{\sqrt{\pi}}{24}$. Observe that for $r_0 = 1$, we have

$$\delta + \frac{\varphi(r_0)}{\Gamma(\alpha+1)}(T-a)^\alpha = 1.$$

Moreover,

$$(\varphi_1(r_0) + f_1^*)\Gamma(\alpha+1) + (\varphi_2(r_0) + f_2^*)\varphi(r_0)(T-a)^\alpha = \frac{\sqrt{\pi}}{4} + \frac{\pi}{12} \le \frac{\sqrt{\pi}}{2} = r_0 \Gamma(\alpha+1).$$

Then all the required assumptions of Corollary 4.1 are satisfied. As consequence, Eq.(4.2) has at least one solution $y^* \in C([0,1];\mathbb{R})$ satisfying $\|y^*\| \le 1$.

4.2 A functional equation involving Erdélyi-Kober fractional integral

Let us consider the integral equation

$$x(t) = f_1(t, x(t)) + \frac{\beta f_2(t, x(t))}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\beta-1} k(\tau, x(\tau))}{(t^\beta - \tau^\beta)^{1-\alpha}} \, d\tau, \quad t \in [a, T], \tag{4.3}$$

where $\alpha \in (0,1)$, $\beta > 0$, $0 \leq a < T$, $k : [a, T] \times \mathbb{R} \to \mathbb{R}$ and $f_i : [a, T] \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$. Eq.(4.3) can be written as

$$x(t) = f_1(t, x(t)) + f_2(t, x(t)) I^\alpha_{a,\beta} k(\cdot, x(\cdot))(t), \quad t \in [a, T],$$

where $I^\alpha_{a,\beta}$ is the Erdélyi-Kober fractional integral defined by (2.3).

Take

$$g(t) = t^\beta, \quad t \in [a, T]$$

in Theorem 3.1, we obtain the following existence result.

Corollary 4.3. *Suppose that all the assumptions* (H1)-(H3) are satisfied. Moreover, suppose that there exists $r_0 > 0$ such that

$$(\varphi_1(r_0) + f_1^*)\Gamma(\alpha + 1) + (\varphi_2(r_0) + f_2^*)\varphi(r_0) \left(T^\beta - a^\beta\right)^\alpha \leq r_0 \Gamma(\alpha + 1)$$

and

$$\delta + \frac{\varphi(r_0)}{\Gamma(\alpha + 1)} \left(T^\beta - a^\beta\right)^\alpha \leq 1.$$

Then (4.3) has at least one solution $y^* \in C([a, T]; \mathbb{R})$. Moreover, we have $\|y^*\| \leq r_0$.

We present the following example to illustrate the above result.

Example 4.4. We consider the integral equation

$$x(t) = \frac{e^{-1} t}{10} + \frac{\beta e^{\frac{37}{19}}}{76\Gamma(\alpha)} (t + 1 + \arctan |x(t)|) \int_0^t \frac{\tau^{\beta-1} x(\tau)}{\left(t^\beta - \tau^\beta\right)^{1-\alpha} \sqrt{\tau + 1}} \, d\tau, \quad t \in [0, 1], \tag{4.4}$$

where $\alpha \in (0,1)$ and $\beta > 0$. The above equation can be written in the form (4.3) with $a = 0$, $T = 1$,

$$\begin{aligned} f_1(t, u) &= \frac{e^{-1} t}{10}, \quad (t, u) \in [0, 1] \times \mathbb{R}, \\ f_2(t, u) &= \frac{t + 1 + \arctan |u|}{2}, \quad (t, u) \in [0, 1] \times \mathbb{R}, \\ k(t, u) &= \frac{e^{\frac{37}{19}} u}{38\sqrt{t + 1}}, \quad (t, u) \in [0, 1] \times \mathbb{R}. \end{aligned}$$

We can check easily that all the assumptions (H1)-(H3) are satisfied with $\varphi_1(t) = \theta(t) = \delta = 0$, $\varphi_2(t) = \frac{t}{2}$ and $\varphi(r) = \frac{e^{\frac{37}{19}}}{38} r$. Moreover, in this case, we have $f_1^* = \frac{e^{-1}}{10}$ and $f_2^* = 1$. On the other hand, we have [22]

$$\frac{e^{\frac{37}{19}}}{76} = 0.0922... \leq 0.8856 \leq \Gamma(\alpha+1), \text{ for } \alpha \in (0,1).$$

Then for $r_0 = \frac{1}{2}$, we have

$$\delta + \frac{\varphi(r_0)}{\Gamma(\alpha+1)} \left(T^\beta - a^\beta\right)^\alpha = \frac{\frac{e^{\frac{37}{19}}}{76}}{\Gamma(\alpha+1)} \leq 1.$$

Next, we have

$$\begin{aligned} &(\varphi_1(r_0) + f_1^*) + \frac{(\varphi_2(r_0) + f_2^*)\varphi(r_0)\left(T^\beta - a^\beta\right)^\alpha}{\Gamma(\alpha+1)} \\ &= \frac{e^{-1}}{10} + \frac{5e^{\frac{37}{19}}}{304\Gamma(\alpha+1)} \leq 0.16698... \leq \frac{1}{2} = r_0. \end{aligned}$$

Then all the required assumptions of Corollary 4.3 are satisfied. As consequence, Eq.(4.4) has at least one solution $y^* \in C([0,1];\mathbb{R})$ satisfying $\|y^*\| \leq \frac{1}{2}$.

4.3 A functional equation involving Hadamard fractional integral

We consider the integral equation

$$x(t) = f_1(t, x(t)) + \frac{f_2(t, x(t))}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{k(\tau, x(\tau))}{\tau} \, d\tau, \quad t \in [a, T], \tag{4.5}$$

where $0 < a < T$, $\alpha \in (0,1)$, $k : [a,T] \times \mathbb{R} \to \mathbb{R}$ and $f_i : [a,T] \times \mathbb{R} \to \mathbb{R}$ ($i = 1,2$). Eq.(4.5) can be written as

$$x(t) = f_1(t, x(t)) + f_2(t, x(t)) J_a^\alpha k(\cdot, x(\cdot))(t), \quad t \in [a, T],$$

where J_a^α is the Hadamard fractional integral of order α defined by (2.4).

Take

$$g(t) = \ln t, \quad t \in [a, T]$$

in Theorem 3.1, we obtain the following existence result.

Corollary 4.5. *Suppose that all the assumptions* (H1)-(H3) are satisfied. Moreover, suppose that there exists $r_0 > 0$ such that

$$(\varphi_1(r_0) + f_1^*)\Gamma(\alpha+1) + (\varphi_2(r_0) + f_2^*)\varphi(r_0) \left(\ln \frac{T}{a}\right)^\alpha \leq r_0 \Gamma(\alpha+1) \tag{4.6}$$

and

$$\delta + \frac{\varphi(r_0)}{\Gamma(\alpha+1)} \left(\ln \frac{T}{a}\right)^\alpha \leq 1. \tag{4.7}$$

Then (4.5) has at least one solution $y^* \in C([a,T];\mathbb{R})$. Moreover, we have $\|y^*\| \leq r_0$.

We end the paper with the following example that illustrates the above result.

Example 4.6. Let us consider the integral equation

$$x(t) = e^{-2}t + 2\alpha e^{-3}x(t)\int_1^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} x(\tau)\,d\tau, \quad t \in [1,e^1], \tag{4.8}$$

where $\alpha \in (0,1)$. Eq.(4.8) can be written in the form (4.5) with $a = 1$, $T = e^1$,

$$\begin{array}{rcl} f_1(t,u) & = & e^{-2}t, \quad (t,u) \in [1,e^1] \times \mathbb{R}, \\ f_2(t,u) & = & \dfrac{u}{2}, \quad (t,u) \in [1,e^1] \times \mathbb{R}, \\ k(t,u) & = & 4e^{-3}\Gamma(\alpha+1)tu, \quad (t,u) \in [1,e^1] \times \mathbb{R}. \end{array}$$

We can check easily that all the assumptions (H1)-(H3) are satisfied with $\varphi_1(t) = \theta(t) = \delta = 0$, $\varphi_2(t) = \frac{t}{2}$ and $\varphi(r) = 4e^{-2}\Gamma(\alpha+1)r$. In this case, we have $f_1^* = e^{-1}$ and $f_2^* = 0$. Moreover, for $r_0 = \frac{e^2}{4}$, the inequalities (4.6) and (4.7) are satisfied. Therefore, by Corollary 4.5, Eq.(4.8) has at least one solution $y^* \in C([1,e^1];\mathbb{R})$ such that $\|y^*\| \leq \frac{e^2}{4}$.

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References

[1] R. Agarwal, M. Meehan, D. O'Regan, Fixed Point Theory and Applications, Cambridge University Press, 2004.

[2] R.P. Agarwal, B. Samet, An existence result for a class of nonlinear integral equations of fractional orders, Nonlinear Analysis: Modelling and Control. 21 (5) (2016) 616–629.

[3] A. Aghajani, R. Allahyari, M. Mursaleen, A generalization of Darbo's theorem with application to the solvability of systems of integral equations, J. Comput. Appl. Math. 260 (2014) 68–77.

[4] A. Aghajani, J. Banaś, Y. Jalilian, Existence of solutions for a class of nonlinear Volterra singular integral equations, Comput. Math. Appl. 62 (2011) 1215–1227.

[5] A. Aghajani, J. Banaś, N. Sabzali, Some generalizations of Darbo fixed point theorem and applications, Bull. Belg. Math. Soc. Simon Stevin. 20 (2) (2013) 345–358.

[6] A. Aghajani, E. Pourhadi, J.J. Trujillo, Application of measure of noncompactness to a cauchy problem for fractional differential equation in Banach spaces, Fract. Calc. Appl. Anal. 16 (2013) 962–977.

[7] J. Banaś, Measures of noncompactness in the study of solutions of nonlinear differential and integral equations, Cent. Eur. J. Math. 10 (6) (2012) 2003–2011.

[8] J. Banaś, K. Goebel, Measures of Noncompactness in Banach Space. Lecture Notes in Pure and Applied Mathematics, Vol. 60. Dekker, New York, 1980.

[9] J. Banaś, R. Rzepka, An application of a measure of noncompactness in the study of asymptotic stability, Appl. Math. Lett. 16 (2003) 1–6.

[10] D. Beglov, B. Roux, An integral equation to describe the solvation of polar molecules in liquid water, J. Phys. Chem. B. 101 (1997) 7821–7826.

[11] M. Benchohra, D. Seba, Integral equations of fractional order with multiple time delays in banach spaces, Electron. J. Differential Equations. 56 (2012) 1–8.

[12] U. Çakan, I. Ozdemir, An application of Darbo fixed-point theorem to a class of functional integral equations, Numer. Funct. Anal. Optim. 36 (2015) 29–40.

[13] M. Cichon, M.A. Metwali, On quadratic integral equations in Orlicz spaces, J. Math. Anal. Appl. 387 (2012) 419–432.

[14] B.C. Dhage, D. O'Regan, A fixed point theorem in Banach algebras with applications to nonlinear integral equation, Funct. Differ. Equ. 7 (3-4) (2000) 259–267.

[15] M.A. Darwish, On monotonic solutions of a singular quadratic integral equation with supremum, Dynam. Systems Appl. 17 (2008) 539–550.

[16] M.A. Darwish, On quadratic integral equation of fractional orders, J. Math. Anal. Appl. 311 (2005) 112–119.

[17] M.A. Darwish, On Erdélyi-Kober fractional Urysohn-Volterra quadratic integral equation, Appl. Math. Comput. 273 (2016) 562–569.

[18] M.A. Darwish, K. Sadarangani, On a quadratic integral equation with supremum involving Erdélyi-Kober fractional order, Mathematische Nachrichten. 288 (5-6) (2015) 566–576.

[19] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, 1985.

[20] B.C. Dhage, M. Imdad, Asymptotic behaviour of nonlinear quadratic functional integral equations involving Carathodory, Nonlinear Anal. 71 (2009) 1285–1291.

[21] B.C. Dhage, S.S. Bellale, Local asymptotic stability for nonlinear quadratic functional integral equations, Electron. J. Qual. Theory Differ. Equ. 10 (2008) 1–13.

[22] G.M. Fichtenholz, Differential and Integral Calculus II, PWN, Warsaw, 1980 (in Polish).

[23] H.W. Hethcote, D.W. Tudor, Integral equation models for endemic infectious diseases, J. Math. Biology. 9 (1980) 37–47.

[24] M. Jleli, M. Mursaleen, B. Samet, On a class of q-integral equations of fractional orders, Electron. J. Differ. Equ. 17 (2016) 1–14.

[25] S. Kablin, L. Nibenbebg, On a theorem of P. Nowosad, J. Math. Anal. Appl. 17 (1967) 61–67.

[26] J. Klamka, Schauder's fixed point theorem in nonlinear controllability, Control and Cybernetics, 29 (1) (2000) 153-165. problems

[27] V. Muresan, Volterra integral equations with iterations of linear modification of the argument, Novi Sad J. Math. 33(2003) 1–10.

[28] M. Mursaleen, S.A. Mohiuddine, Applications of measures of noncompactness to the infinite system of differential equations in l^p spaces, Nonlinear Anal. 75 (2012) 2111–2115.

[29] I.M. Olaru, An integral equation via weakly Picard operators, Fixed Point Theory. 11(2010) 97–106.

[30] I.M. Olaru, Generalization of an integral equation related to some epidemic models, Carpathian J. Math. 26 (1) (2010) 92–96.

[31] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Yverdon et alibi, 1993.

[32] HA. Schenck, Improved integral formulation for acoustic radiation problems, J Acoust Soc Am. 44 (1968) 41–58.

[33] M.A. Serban, I.A. Rus, A. Petrusel, A class of abstract Volterra equations via weakly Picard operators technique, Math. Ineq. Appl. 13(2010) 255–269.

[34] J. Sladek, V. Sladek, S.N. Atluri, Local boundary integral equation (LBIE) method for solving problems of elasticity with nonhomogeneous material properties, Computational Mechanics. 24 (2000) 456–462.

[35] T. Terai, On calculation of sound fields around three dimensional objects by integral equation methods, J Sound Vib. 69 (1) (1980) 71–100.

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