A simplicial approach to multiplier bimonoids

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Abstract

Although multiplier bimonoids in general are not known to correspond to comonoids in any monoidal category, we classify them in terms of maps from the Catalan simplicial set to another suitable simplicial set; thus they can be regarded as (co)monoids in something more general than a monoidal category (namely, the simplicial set itself). We analyze the particular simplicial maps corresponding to that class of multiplier bimonoids which can be regarded as comonoids.

1 Introduction

The recent paper [6] showed that monoids, as well as many generalizations, including monads, monoidal categories, skew monoidal categories [9], and internal versions of these, can be classified as simplicial maps from the *Catalan simplicial set* C to appropriately chosen simplicial sets. For the constructions in the current paper the most relevant observation in [6] is a bijective correspondence between monoids in a monoidal category $\mathcal{M}$ and simplicial maps from C to the nerve $\mathsf{N}(\mathcal{M})$ of $\mathcal{M}$.

Bialgebras — over a field or, more generally, in a braided monoidal category $\mathcal{C}$— can be defined as comonoids in the monoidal category $\mathcal{M}$ of monoids in $\mathcal{C}$. Thus applying the results of [6], they are classified by simplicial maps from C to the nerve $\mathsf{N}(\mathcal{M}^{\mathsf{op}})$ of the category $\mathcal{M}$ regarded with the opposite composition.

Classically, *Hopf algebras* over a field are defined as bialgebras with a further property. *Multiplier Hopf algebras* [10] generalize Hopf algebras beyond the case when the algebra has a unit. The typical motivating example of a multiplier Hopf algebra consists of finitely supported functions on an infinite group with values

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in the base field. The analogous notion of *multiplier bialgebra* was introduced later, in [2], together with its 'weak' generalization.

For many applications it is important to work with Hopf algebras and bialgebras not only over fields but, more generally, in braided monoidal categories which are different from the symmetric monoidal category of vector spaces. The *formulation of multiplier bialgebras in braided monoidal categories* is our longstanding project initiated in [3].

Generalizing some constructions in [8] to any braided monoidal category $\mathcal{C}$, we described in [4] how *certain* multiplier bimonoids in $\mathcal{C}$ can be seen as *certain* comonoids in an appropriately constructed monoidal category $\mathcal{M}$. In light of the findings of [6], this gives rise to a correspondence between these multiplier bialgebras and certain simplicial maps from the Catalan simplicial set C to $\mathsf{N}(\mathcal{M}^{\mathsf{op}})$.

The aim of this paper is to go beyond that characterization and prove a bijection between *arbitrary* multiplier bimonoids in $\mathcal{C}$ and *arbitrary* simplicial maps from C to a suitable simplicial set M_{12}. The simplicial maps $\mathsf{C} \to \mathsf{N}(\mathcal{M}^{\mathsf{op}})$ that correspond, via the results of [4] and [6], to nice enough multiplier bimonoids, turn out to factorize through a canonical embedding of a simplicial subset of M_{12} into $\mathsf{N}(\mathcal{M}^{\mathsf{op}})$.

Regular multiplier bimonoids constitute a distinguished class of multiplier bimonoids. In order to classify them as well, we also present a simplicial set M with the property that simplicial maps $\mathsf{C} \to \mathsf{M}$ correspond bijectively to regular multiplier bimonoids in $\mathcal{C}$.

Notation

Throughout the paper, $\mathcal{C}$ denotes a braided monoidal category. We do not assume that it is strict but — relying on coherence — we omit explicitly denoting the associativity and unit isomorphisms. The monoidal unit is denoted by I and the monoidal product is denoted by juxtaposition (we also use the power notation for the iterated monoidal product of the same object). The braiding is denoted by c. The composite of morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathcal{C}$ is denoted by $g.f\colon\ : A \to C$ and we write 1 for the identity morphisms in $\mathcal{C}$.

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2 Preliminaries on the Catalan simplicial set

In this section we briefly recall from [6] an explicit description of the Catalan simplicial set and its role in the classification of monads in bicategories; thus in particular of monoids in monoidal categories.

2.1 Simplicial sets

Consider the *simplex category* Δ whose objects are non-empty finite ordinals and whose morphisms are the order preserving functions. By definition, a *simplicial set* is a presheaf on Δ. Explicitly, a simplicial set W is given by a collection $\{\mathsf{W}_n\}$ of sets labelled by the natural numbers n — the sets of *n-simplices* — together with the *face maps* $d_i \colon \mathsf{W}_n \to \mathsf{W}_{n-1}$ and the *degeneracy maps* $s_i \colon \mathsf{W}_n \to \mathsf{W}_{n+1}$, for $0 \leq i \leq n$, obeying the *simplicial relations*:

$$d_i d_j = d_{j-1} d_i \quad \text{if } i < j \qquad\qquad s_i s_j = s_{j+1} s_i \quad \text{if } i \leq j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } i \in \{j, j+1\} \\ s_j d_{i-1} & \text{if } i > j+1. \end{cases}$$

An n-simplex is said to be *degenerate* if it belongs to the image of one of the degeneracy maps, otherwise it is *non-degenerate*.

We often draw an n-simplex w as an n-dimensional oriented geometric simplex whose $n-1$ dimensional faces are $d_i(w)$. For example, for $n = 2$ we draw an oriented triangle

$$\begin{tikzcd}
& w_1 := d_0 d_2(w) = d_1 d_0(w) \arrow[dr, "w_{12} := d_0(w)"] & \\
w_0 := d_1 d_2(w) = d_1 d_1(w) \arrow[ur, "w_{01} := d_2(w)"] \arrow[rr, "w_{02} := d_1(w)"'] \arrow[rr, phantom, bend left, "w"] & & w_2 := d_0 d_1(w) = d_0 d_0(w),
\end{tikzcd}$$

for $n = 3$ we draw an oriented tetrahedron, and so on.

A *simplicial map* is a natural transformation between such presheaves. Explicitly, a simplicial map $\mathsf{W} \to \mathsf{V}$ is a collection of functions $\{f_n \colon \mathsf{W}_n \to \mathsf{V}_n\}$ labelled by the natural numbers n which commute with the face and degeneracy maps in the sense that $s_i f_n = f_{n+1} s_i$ and $d_i f_n = f_{n-1} d_i$ for all possible values of i.

2.2 The Catalan simplicial set

The *Catalan simplicial set* C has a single 0-simplex $*$. It has two 1-simplices: $s_0(*)$ and a non-degenerate one to be called α. There are three degenerate 2-simplices $s_0 s_0(*) = s_1 s_0(*)$, $s_0(\alpha)$ and $s_1(\alpha)$ and two non-degenerate ones to be denoted by

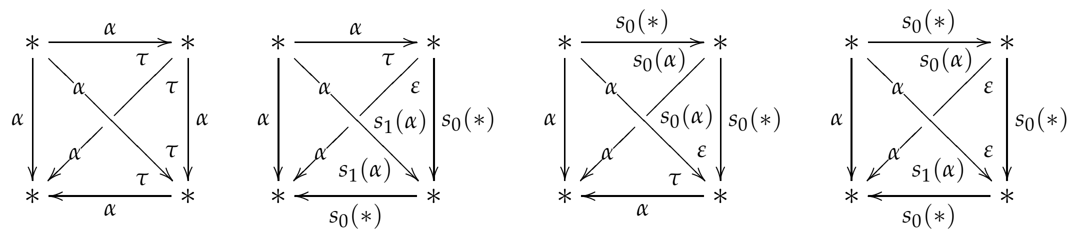
$$\begin{tikzcd}
& * \arrow[dr, "\alpha"] & \\
* \arrow[ur, "\alpha"] \arrow[rr, "\alpha"'] \arrow[rr, phantom, bend left, "\tau"] & & *
\end{tikzcd}
\qquad
\begin{tikzcd}
& * \arrow[dr, "s_0(*)"] & \\
* \arrow[ur, "s_0(*)"] \arrow[rr, "\alpha"'] \arrow[rr, phantom, bend left, "\varepsilon"] & & *.
\end{tikzcd}$$

All higher simplices are generated *coskeletally*, meaning that for any natural number $n > 2$, and for any *n-boundary* (that is, $n+1$-tuple $\{w_0, \ldots, w_n\}$ of $n-1$-simplices such that $d_j(w_i) = d_i(w_{j+1})$ for all $0 \le i \le j < n$) there is a unique *filler* (that is, an n-simplex w obeying $d_i(w) = w_i$ for all $0 \le i \le n$). In this situation we write $w = (w_0, \ldots, w_n)$.

From this property of the Catalan simplicial set one can deduce that there are four non-degenerate 3-simplices

$$\phi = (\tau,\tau,\tau,\tau),\ \lambda = (\varepsilon, s_1(\alpha), \tau, s_1(\alpha)),$$
$$\varrho = (s_0(\alpha), \tau, s_0(\alpha), \varepsilon),\ \kappa = (\varepsilon, s_1(\alpha), s_0(\alpha), \varepsilon)$$

corresponding to the four tetrahedra drawn below.



For some equivalent, more conceptual, descriptions of C consult [6].

2.3 The nerve of a bicategory

Any bicategory $\mathcal{B}$ determines a simplicial set $\mathsf{N}(\mathcal{B})$ known as the *nerve* of $\mathcal{B}$. The 0-simplices of $\mathsf{N}(\mathcal{B})$ are the objects of $\mathcal{B}$. The 1-simplices are the 1-cells of $\mathcal{B}$, with faces provided by the source and the target maps. For a given 2-boundary $\{w_{12}, w_{02}, w_{01}\}$, the 2-simplices

$$\begin{tikzcd} & w_1 \arrow[dr, "w_{12}"] & \\ w_0 \arrow[ur, "w_{01}"] \arrow[rr, "w_{02}"'] & \arrow[u, phantom, "w"] & w_2 \end{tikzcd}$$

are 2-cells $w\colon w_{12}w_{01} \to w_{02}$ in $\mathcal{B}$. For a given 3-boundary $\{w_{123}, w_{023}, w_{013}, w_{012}\}$, there is precisely one filler if the diagram

$$\begin{tikzcd} (w_{23}w_{12})w_{01} \arrow[r, "\cong"] \arrow[d, "w_{123}1"'] & w_{23}(w_{12}w_{01}) \arrow[r, "1w_{012}"] & w_{23}w_{02} \arrow[d, "w_{023}"] \\ w_{13}w_{01} \arrow[rr, "w_{013}"'] & & w_{03} \end{tikzcd}$$

commutes and there is no filler otherwise. If the filler exists then it is denoted by $(w_{123}, w_{023}, w_{013}, w_{012})$. All higher simplices are determined coskeletally.

The degenerate 1-simplex on a 0-simplex A is the identity 1-cell $1\colon A \to A$. The degenerate 2-simplices $s_0(a)$ and $s_1(a)$ on a 1-simplex a are the coherence isomorphism 2-cells $a1 \to a$ and $1a \to a$, respectively. On higher simplices the degeneracy maps are determined by the uniqueness of the filler for a given boundary.

As was observed in [6], a simplicial map $\mathsf{C} \to \mathsf{N}(\mathcal{B})$ is the same thing as a monad in $\mathcal{B}$. The 1-cell underlying the monad is the image of α, with multiplication and unit provided by the images of τ and ε, respectively.

Monoidal categories can be seen as bicategories with a single object. Thus the above considerations apply in particular to them. In particular, the above used symbol $\mathsf{N}(\mathcal{M})$ stands for the *monoidal nerve* of a monoidal category $\mathcal{M}$ (rather than the nerve of the underlying ordinary category). Since a monad in a one-object bicategory is the same as a monoid in the corresponding monoidal category, simplicial maps $\mathsf{C} \to \mathsf{N}(\mathcal{M})$ classify the monoids in $\mathcal{M}$.

3 A simplicial description of multiplier bimonoids

Multiplier bimonoids in braided monoidal categories are defined as compatible pairs of counital fusion morphisms [3]. Thus it is not too surprising that the first step in our simplicial characterization of multiplier bimonoids is a simplicial treatment of counital fusion morphisms. In Section 3.2 we shall associate to the braided monoidal category $\mathcal{C}$ a simplicial set M_{12} such that a simplicial map $\mathsf{C} \to \mathsf{M}_{12}$ is the same thing as a multiplier bimonoid in $\mathcal{C}$. As a preparation for that, first we construct in Section 3.1 a simplicial set M_1 and analyze the relation of simplicial maps $\mathsf{C} \to \mathsf{M}_1$ to counital fusion morphisms in $\mathcal{C}$. The simplicial set M_1 and its symmetric counterpart M_2 will be used as building blocks of M_{12}.

3.1 Counital fusion morphisms

Recall that a *fusion morphism* on an object A of $\mathcal{C}$ is a morphism $t\colon A^2 \to A^2$ making commutative the first diagram of

$$
\begin{tikzcd}
A^3 \arrow[r, "t1"] \arrow[d, "1t"'] & A^3 \arrow[r, "1t"] & A^3 \\
A^3 \arrow[r, "c1"'] & A^3 \arrow[r, "1t"'] & A^3 \arrow[r, "c^{-1}1"'] & A^3 \arrow[ul, "t1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A^2 \arrow[r, "t"] \arrow[dr, "1e"'] & A^2 \arrow[d, "1e"] \\
& A.
\end{tikzcd}
\tag{3.1}
$$

The morphism $e\colon A \to I$ is a *counit* of t if it makes commutative the second diagram of (3.1).

The simplicial set M_1 has a single 0-simplex $*$. Its 1-simplices are the semigroups in $\mathcal{C}$; that is, objects A equipped with an associative multiplication $m\colon A^2 \to A$. The 2-simplices with given 2-boundary $\{A_{12}, A_{02}, A_{01}\}$ are morphisms $\varphi\colon A_{02}A_{12} \to A_{01}A_{12}$ in $\mathcal{C}$ rendering commutative the diagrams

$$
\begin{tikzcd}
A_{02}A_{12}A_{12} \arrow[r, "\varphi 1"] \arrow[d, "1m_{12}"'] & A_{01}A_{12}A_{12} \arrow[d, "1m_{12}"] \\
A_{02}A_{12} \arrow[r, "\varphi"'] & A_{01}A_{12}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{02}A_{02}A_{12} \arrow[rr, "m_{02}1"] \arrow[d, "1\varphi"'] & & A_{02}A_{12} \arrow[rr, "\varphi"] & & A_{01}A_{12} \\
A_{02}A_{01}A_{12} \arrow[r, "c1"'] & A_{01}A_{02}A_{12} \arrow[rr, "1\varphi"'] & & A_{01}A_{01}A_{12} \arrow[r, "c^{-1}1"'] & A_{01}A_{01}A_{12}. \arrow[u, "m_{01}1"']
\end{tikzcd}
$$

For a given 3-boundary $\{\varphi_{123}, \varphi_{023}, \varphi_{013}, \varphi_{012}\}$ there is precisely one filler if the fusion equation

$$\begin{tikzcd}
A_{03}A_{13}A_{23} \arrow[rrr, "\varphi_{013}1"] \arrow[d, "1\varphi_{123}"'] & & & A_{01}A_{13}A_{23} \arrow[rr, "1\varphi_{123}"] & & A_{01}A_{12}A_{23} \\
A_{03}A_{12}A_{23} \arrow[r, "c1"'] & A_{12}A_{03}A_{23} \arrow[rrr, "1\varphi_{023}"'] & & & A_{12}A_{02}A_{23} \arrow[r, "c^{-1}1"'] & A_{02}A_{12}A_{23} \arrow[u, "\varphi_{012}1"']
\end{tikzcd} \tag{3.2}$$

commutes — in which case we write $(\varphi_{123}, \varphi_{023}, \varphi_{013}, \varphi_{012})$ for the filler — and there is no filler otherwise. All higher simplices are generated coskeletally.

The action of the face maps should be clear from the above presentation. The unique degenerate 1-simplex is the monoidal unit I of $\mathcal{C}$ (regarded as a trivial semigroup), while for a 1-simplex (A, m) the degenerate 2-simplices $s_0(A, m)$ and $s_1(A, m)$ are given by

$$\begin{tikzcd}
 & * \arrow[dr, "A"] \arrow[d, phantom, "m"] & \\
* \arrow[ur, "I"] \arrow[rr, "A"'] & {} & *
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
 & * \arrow[dr, "I"] \arrow[d, phantom, "1"] & \\
* \arrow[ur, "A"] \arrow[rr, "A"'] & {} & *
\end{tikzcd}$$

respectively. On higher simplices the degeneracy maps are determined by the uniqueness of the filler for a given boundary.

Remark 3.1. If the simplicial set M_1 looks contrived, we can motivate it as follows, based on the fusion equation (3.2). Suppose we were to try to define a simplicial set with a unique 0-simplex, with objects of $\mathcal{C}$ as 1-simplices, with morphisms of the form $\varphi\colon A_{02}A_{12} \to A_{01}A_{12}$ as 2-simplices, and with 4-tuples $(\varphi_{123}, \varphi_{023}, \varphi_{013}, \varphi_{012})$ of 2-simplices satisfying the fusion equation as 3-simplices, and with higher simplices defined coskeletally. Then we can define the last degeneracy map in each degree as in the definition of M_1, but the other degeneracy maps will not exist. For a 1-simplex A, the degenerate 2-simplex $s_0(A)$ should be a morphism $AA \to IA$ in $\mathcal{C}$, so it will be available if we require that, instead of being mere objects A of $\mathcal{C}$, each 1-simplex be equipped with a morphism $m\colon A^2 \to A$. For a 2-simplex φ as above, the existence of 3-simplices with the boundary appropriate for $s_0(\varphi)$ and $s_1(\varphi)$ amounts to the commutativity of the two diagrams in the definition of 2-simplices in M_1. Furthermore the morphism $AA \to IA$ induced by m will satisfy these equations if and only if m is associative. Thus in this sense, the form of the simplicial set M_1 is forced upon us by the fusion equation.

More formally, we could proceed as follows. Any morphism in the simplex category Δ has an epi-mono factorization. Considering only those morphisms in Δ in whose factorization only the last fibre of the epimorphism has more than one element, we obtain a subcategory to be denoted by ∇. The construction of the previous paragraph determines a presheaf $\mathcal{P}_{\mathcal{C}}$ on ∇.

The inclusion functor $J\colon \nabla \to \Delta$ induces a functor $J^*\colon [\Delta^{\mathsf{op}}, \mathsf{Set}] \to [\nabla^{\mathsf{op}}, \mathsf{Set}]$ between the presheaf categories, possessing a right adjoint given by the right Kan extension Ran_J. Explicitly, J^* takes a simplicial set to the presheaf on ∇ obtained by forgetting all but the last degeneracy map; and Ran_J takes a presheaf

X on ∇ to the simplicial set whose n-simplices are families $\{x_f \in \mathsf{X}_j\}$ labelled by morphisms $f\colon j \to n$ in Δ such that for all morphisms w of codomain j in ∇, the identity $w^*(x_f) = w_{f.w}$ holds (where w^* denotes the image of w under the functor $\mathsf{X}\colon \nabla^{\mathsf{op}} \to \mathsf{Set}$).

The value of Ran_J at the presheaf $\mathcal{P}_{\mathcal{C}}$ on ∇ is precisely M_1.

Proposition 3.2. *For any braided monoidal category $\mathcal{C}$, consider the associated simplicial set M_1 above. To give a simplicial map $\mathsf{C} \to \mathsf{M}_1$ is the same as specifying an object A in $\mathcal{C}$ equipped with both a semigroup structure with multiplication $m\colon A^2 \to A$ and a fusion morphism $t\colon A^2 \to A^2$ with counit $e\colon A \to I$, subject to the following compatibility relations.*

$$\begin{tikzcd}
A^3 \arrow[r, "t1"] \arrow[d, "1m"'] \arrow[dr, phantom, "(a)"] & A^3 \arrow[d, "1m"] \\
A^2 \arrow[r, "t"'] & A^2
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[rr, "m1"] \arrow[d, "1t"'] & & A^2 \arrow[r, "t"] & A^2 \\
A^3 \arrow[r, "c1"'] & A^3 \arrow[r, "1t"'] \arrow[u, phantom, "(b)"] & A^3 \arrow[r, "c^{-1}1"'] & A^3 \arrow[u, "m1"']
\end{tikzcd}
\quad
\begin{tikzcd}
A^2 \arrow[r, "e1"] \arrow[d, "m"'] \arrow[dr, phantom, "(c)"] & A \arrow[d, "e"] \\
A \arrow[r, "e"'] & I
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[rr, "m1"] \arrow[d, "1m"'] & & A^2 \arrow[d, "m"] \\
A^2 \arrow[r, "t"'] & A^2 \arrow[r, "e1"'] \arrow[u, phantom, "(d)"] & A
\end{tikzcd}$$

Proof. A simplicial map $\mathsf{C} \to \mathsf{M}_1$ is given by the images of the non-degenerate 1-simplex α and of the non-degenerate 2-simplices τ and ε. This means, respectively, a semigroup (A, m), a morphism $t\colon A^2 \to A^2$ in $\mathcal{C}$ making the diagrams (a) and (b) in the claim commute, and a morphism $e\colon A \to I$ making diagram (c) commute. The simplicial map can be defined on the non-degenerate 3-simplex ϕ of C if and only if t obeys the fusion equation in the first diagram of (3.1). It can be defined on the 3-simplex λ if and only if the counitality condition in the second diagram of (3.1) holds. It can be defined on ϱ if and only if diagram (d) of the claim commutes, while for κ we get the same condition encoded in diagram (c). ∎

Remark 3.3. If $t\colon A^2 \to A^2$ is a fusion morphism with counit $e\colon A \to I$ in $\mathcal{C}$, then we have a semigroup $(A, m := e1.t)$ in $\mathcal{C}$ for which all diagrams of Proposition 3.2 commute: (a), (b) and (c) can be found in (3.6), (3.5) and (3.4) of [5], respectively, and (d) follows by the associativity of $m = e1.t$. Hence there is a corresponding simplicial map $\mathsf{C} \to \mathsf{M}_1$.

However, there may be more general simplicial maps $\mathsf{C} \to \mathsf{M}_1$ for which the multiplication of the corresponding semigroup (A, m) is different from the multiplication $e1.t$ coming from the counital fusion morphism (A, t, e).

Let us consider the particular kind of simplicial maps $\mathsf{C} \to \mathsf{M}_1$ for which the multiplication m happens to be *non-degenerate* in the sense that both maps

$$\begin{aligned}
&\mathcal{C}(X, AY) \to \mathcal{C}(AX, AY) \qquad && f \mapsto m1.1f \\
&\mathcal{C}(X, YA) \to \mathcal{C}(XA, YA) \qquad && g \mapsto 1m.g1
\end{aligned}$$

are injective, for any objects X and Y. Since by identities (a) and (d) in Proposition 3.2

$$m.e11.t1 = e1.1m.t1 = e1.t.1m = m.m1,$$

we conclude that in this case $m = e1.t$.

By the associativity of m and commutativity of (d), $m = e1.t$ also follows if $1m\colon A^3 \to A^2$ is an epimorphism.

By the same construction as above, we can associate a simplicial set to the monoidal category $\mathcal{C}^{\mathsf{rev}}$, obtained from $\mathcal{C}$ by reversing the monoidal product and using the same braiding c. The opposite of this simplicial set is called M_2. Explicitly, M_2 also has a single 0-simplex $*$ and the semigroups in $\mathcal{C}$ as 1-simplices. The 2-simplices of a given 2-boundary $\{A_{12}, A_{02}, A_{01}\}$ are now morphisms $\psi\colon A_{01}A_{02} \to A_{01}A_{12}$ in $\mathcal{C}$ making the diagrams

$$\begin{tikzcd}
A_{01}A_{01}A_{02} \arrow[r, "1\psi"] \arrow[d, "m_{01}1"'] & A_{01}A_{01}A_{12} \arrow[d, "m_{01}1"] \\
A_{01}A_{02} \arrow[r, "\psi"'] & A_{01}A_{12}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{01}A_{02}A_{02} \arrow[rr, "1m_{02}"] \arrow[d, "\psi 1"'] & & A_{01}A_{02} \arrow[rr, "\psi"] & & A_{01}A_{12} \\
A_{01}A_{12}A_{02} \arrow[r, "1c"'] & A_{01}A_{02}A_{12} \arrow[rr, "\psi 1"'] & & A_{01}A_{12}A_{12} \arrow[r, "1c^{-1}"'] & A_{01}A_{12}A_{12} \arrow[u, "1m_{12}"']
\end{tikzcd}$$

commute. For a given 3-boundary $\{\psi_{123}, \psi_{023}, \psi_{013}, \psi_{012}\}$ there is precisely one filler if

$$\begin{tikzcd}
A_{01}A_{02}A_{03} \arrow[rr, "1\psi_{023}"] \arrow[d, "\psi_{012}1"'] & & A_{01}A_{02}A_{23} \arrow[rr, "\psi_{012}1"] & & A_{01}A_{12}A_{23} \\
A_{01}A_{12}A_{03} \arrow[r, "1c"'] & A_{01}A_{03}A_{12} \arrow[rr, "\psi_{013}1"'] & & A_{01}A_{13}A_{12} \arrow[r, "1c^{-1}"'] & A_{01}A_{12}A_{13} \arrow[u, "1\psi_{123}"']
\end{tikzcd}$$

commutes — in which case we write $(\psi_{123}, \psi_{023}, \psi_{013}, \psi_{012})$ for the filler — and there is no filler otherwise. All higher simplices are generated coskeletally. The unique degenerate 1-simplex is again the monoidal unit I of $\mathcal{C}$ (regarded as a trivial semigroup), while for a 1-simplex (A, m) the degenerate 2-simplices $s_0(A, m)$ and $s_1(A, m)$ are given by

$$\begin{tikzcd}
& * \arrow[dr, "A"] \arrow[d, phantom, "1"] & \\
* \arrow[ur, "I"] \arrow[rr, "A"'] & {} & *
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& * \arrow[dr, "I"] \arrow[d, phantom, "m"] & \\
* \arrow[ur, "A"] \arrow[rr, "A"'] & {} & *
\end{tikzcd}$$

respectively; note that the roles of s_0 and s_1 have been interchanged. As before, on the higher simplices the degeneracy maps are determined by the uniqueness of the filler for a given boundary.

Since the simplicial set C is isomorphic to its opposite, Proposition 3.2 then characterizes the simplicial maps $\mathsf{C} \to \mathsf{M}_2$ as objects A carrying the compatible structures of a semigroup, and a counital fusion morphism in $\mathcal{C}^{\mathsf{rev}}$; once again, the roles of λ and ρ have been interchanged relative to the case of M_1.

As a further possibility, we can use the above construction to associate a simplicial set M_3 to the monoidal category $\overline{\mathcal{C}}$, obtained from $\mathcal{C}$ by keeping the same monoidal product but replacing the braiding c with c^{-1}, and using the twisted multiplication $m.c^{-1}$ for a 1-simplex (that is, semigroup) (A, m), so that in particular the degenerate 2-simplex $s_0(A, m)$ is given by $m.c^{-1}$. Proposition 3.2 can also be used to describe the simplicial maps $\mathsf{C} \to \mathsf{M}_3$.

Finally, applying the above construction to the braided monoidal category $(\overline{\mathcal{C}})^{\mathsf{rev}} = \overline{\mathcal{C}^{\mathsf{rev}}}$ we obtain a simplicial set M_4.

3.2 Multiplier bimonoids

A *multiplier bimonoid* [3] in a braided monoidal category $\mathcal{C}$ consists of a fusion morphism t_1 in $\mathcal{C}$ and a fusion morphism t_2 in $\mathcal{C}^{\mathsf{rev}}$ with a common counit $e\colon A \to I$ such that the diagrams

$$\begin{tikzcd} A^3 \arrow[r, "t_2 1"] \arrow[d, "1t_1"'] & A^3 \arrow[d, "1t_1"] \\ A^3 \arrow[r, "t_2 1"'] & A^3 \end{tikzcd} \qquad \begin{tikzcd} A^2 \arrow[r, "t_2"] \arrow[d, "t_1"'] & A^2 \arrow[d, "1e"] \\ A^2 \arrow[r, "e1"'] & A \end{tikzcd} \tag{3.3}$$

commute. Thus by Remark 3.3, it can be thought of as a pair of simplicial maps $\mathsf{C} \to \mathsf{M}_1$ and $\mathsf{C} \to \mathsf{M}_2$ subject to compatibility conditions expressing the fact that the underlying semigroups and the counits are equal, and the diagrams in (3.3) commute, with the common diagonal of the second of these given by the multiplication. Guided by this fact, we construct below a simplicial set M_{12} whose simplices are suitably compatible pairs consisting of a simplex in M_1 and a simplex in M_2. We prove that a simplicial map $\mathsf{C} \to \mathsf{M}_{12}$ is the same thing as a multiplier bimonoid in $\mathcal{C}$.

The simplicial set M_{12} has a single 0-simplex $*$ and the semigroups of $\mathcal{C}$ as 1-simplices. The 2-simplices are pairs $(\varphi|\psi)$ consisting of a 2-simplex φ of M_1 and a 2-simplex ψ of M_2 with common boundary $\{A_{12}, A_{02}, A_{01}\}$, such that the diagram

$$\begin{tikzcd} A_{01}A_{02}A_{12} \arrow[r, "1\varphi"] \arrow[d, "\psi 1"'] & A_{01}A_{01}A_{12} \arrow[d, "m_{01}1"] \\ A_{01}A_{12}A_{12} \arrow[r, "1m_{12}"'] & A_{01}A_{12} \end{tikzcd} \tag{3.4}$$

commutes. The 3-simplices are pairs $(\varphi_{123}, \varphi_{023}, \varphi_{013}, \varphi_{012}|\psi_{123}, \psi_{023}, \psi_{013}, \psi_{012})$ consisting of a 3-simplex $(\varphi_{123}, \varphi_{023}, \varphi_{013}, \varphi_{012})$ in M_1 and a 3-simplex $(\psi_{123}, \psi_{023}, \psi_{013}, \psi_{012})$ in M_2 such that $(\phi_{ijk}|\psi_{ijk})$ is a 2-simplex in M_{12} for each ijk, and the diagram

$$\begin{tikzcd} A_{01}A_{03}A_{23} \arrow[r, "1\varphi_{023}"] \arrow[d, "\psi_{013}1"'] & A_{01}A_{02}A_{23} \arrow[d, "\psi_{012}1"] \\ A_{01}A_{13}A_{23} \arrow[r, "1\varphi_{123}"'] & A_{01}A_{12}A_{23} \end{tikzcd} \tag{3.5}$$

commutes. The face and degeneracy maps act on the pairs componentwise, and higher simplices are defined coskeletally.

Remark 3.4. We explained in Remark 3.1 a sense in which the simplicial set M_1 is dictated by the fusion equation; in particular, the associativity of the multiplications in the 1-simplices and the commutativity of the diagrams in the definition of the 2-simplices are required in order for various degenerate 3-simplices to satisfy the fusion equation.

The case of M_{12} is similar: it has the same 1-simplices as M_1, and once we impose commutativity of (3.5) on the 3-simplices, any 2-simplex $(\varphi|\psi)$ must obey (3.4) in order to have a 3-simplex with the boundary of $s_1(\varphi|\psi)$.

Theorem 3.5. *There is a bijection between simplicial maps* $\mathsf{C} \to \mathsf{M}_{12}$ *and multiplier bimonoids in* $\mathcal{C}$.

Proof. Again, a simplicial map $\mathsf{C} \to \mathsf{M}_{12}$ is given by the images (A, m) of the non-degenerate 1-simplex α, $(t_1|t_2)$ of the non-degenerate 2-simplex τ and $(e_1|e_2)$ of the non-degenerate 2-simplex ε. By Proposition 3.2 (t_1, e_1) is a counital fusion morphism in $\mathcal{C}$ obeying conditions (a)-(d); and (t_2, e_2) is a counital fusion morphism in $\mathcal{C}^{\mathsf{rev}}$ obeying symmetric counterparts of conditions (a)-(d). Furthermore, there are compatibility conditions between them: $(t_1|t_2)$ is a 2-simplex of M_{12} if and only if diagram (e) in

$$
\begin{tikzcd}
A^3 \arrow[r, "1t_1"] \arrow[d, "t_21"'] \arrow[dr, phantom, "\text{(e)}"] & A^3 \arrow[d, "m1"] \\
A^3 \arrow[r, "1m"'] & A^2
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "e_1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\text{(f)}"] & I \arrow[d, equal] \\
I \arrow[r, equal] & I
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[r, "1t_1"] \arrow[d, "t_21"'] \arrow[dr, phantom, "\text{(g)}"] & A^3 \arrow[d, "t_21"] \\
A^3 \arrow[r, "1t_1"'] & A^3
\end{tikzcd}
\quad
\begin{tikzcd}
A^2 \arrow[r, equal] \arrow[d, "t_2"'] \arrow[dr, phantom, "\text{(h)}"] & A^2 \arrow[d, "m"] \\
A^2 \arrow[r, "1e_1"'] & A
\end{tikzcd}
\quad
\begin{tikzcd}
A^2 \arrow[r, "t_1"] \arrow[d, equal] \arrow[dr, phantom, "\text{(i)}"] & A^2 \arrow[d, "e_21"] \\
A^2 \arrow[r, "m"'] & A
\end{tikzcd}
$$

commutes and $(e_1|e_2)$ is a 2-simplex of M_{12} if and only if (f) does so. The simplicial map is well-defined on the non-degenerate 3-simplices ϕ, λ, ϱ and κ if and only if the respective diagrams (g), (h), (i) and (f) again commute.

From (f) we infer that the counits e_1 and e_2 are equal so we will denote them simply by e. Then (h) and (i) take the equivalent form in the second diagram of (3.3), with common diagonal m. As observed in Remark 3.3, from this it follows that all identities (a)-(d), as well as their symmetric counterparts, hold true. Diagram (g) is identical to the first diagram of (3.3); this implies (e) upon post-composing by $1e1$.

Summarizing, a simplicial map $\mathsf{C} \to \mathsf{M}_{12}$ is the same thing as a pair of counital fusion morphism (t_1, e) in $\mathcal{C}$ and a counital fusion morphism (t_2, e) in $\mathcal{C}^{\mathsf{rev}}$ (with common counit e) rendering commutative the diagrams of (3.3). ∎

Applying the construction of this section to the braided monoidal category $\overline{\mathcal{C}}$, and using the reversed multiplications $m.c^{-1}$ of the semigroups (A, m), we obtain a simplicial set M_{34}.

3.3 Regular multiplier bimonoids

A *regular multiplier bimonoid* [3] in a braided monoidal category $\mathcal{C}$ is a tuple $(A, t_1, t_2, t_3, t_4, e)$ such that (A, t_1, t_2, e) is a multiplier bimonoid in $\mathcal{C}$ and (A, t_3, t_4, e) is a multiplier bimonoid in $\overline{\mathcal{C}}$, and such that the following diagrams commute, where m stands for the common diagonal of the first diagram.

$$
\begin{tikzcd}
A^2 \arrow[r, "t_1"] \arrow[d, "c"'] & A^2 \arrow[dd, "e1"] \\
A^2 \arrow[d, "t_3"'] & \\
A^2 \arrow[r, "e1"'] & A
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[r, "1t_1"] \arrow[d, "c1"'] & A^3 \arrow[d, "c1"] \\
A^3 \arrow[d, "t_31"'] & A^3 \arrow[d, "1m"] \\
A^3 \arrow[r, "1m"'] & A^2
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[r, "1t_1"] \arrow[dd, "t_41"'] & A^3 \arrow[dd, "t_41"] \\
 & \\
A^3 \arrow[r, "1t_1"'] & A^3
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[r, "t_21"] \arrow[d, "1c"'] & A^3 \arrow[d, "1c"] \\
A^3 \arrow[d, "1t_4"'] & A^3 \arrow[d, "m1"] \\
A^3 \arrow[r, "m1"'] & A^2
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[r, "t_21"] \arrow[dd, "1t_3"'] & A^3 \arrow[dd, "1t_3"] \\
 & \\
A^3 \arrow[r, "t_21"'] & A^3
\end{tikzcd}
\tag{3.6}
$$

We shall classify regular multiplier bimonoids via simplicial maps from C to a simplicial set M which we now describe.

The simplicial set M has a single 0-simplex $*$, and its 1-simplices are the semigroups in the braided monoidal category $\mathcal{C}$. The 2-simplices are pairs $(\varphi|\psi\|\varphi'|\psi')$ consisting of a 2-simplex $(\varphi|\psi)$ of M_{12} and a 2-simplex $(\varphi'|\psi')$ of M_{34} with common boundary (A, B, C), obeying the compatibility conditions

$$
\begin{tikzcd}
CBA \arrow[r, "1\varphi"] \arrow[dd, "\psi' 1"'] & C^2A \arrow[d, "c^{-1}1"] \\
& C^2A \arrow[d, "m1"] \\
CA^2 \arrow[r, "1m"'] & CA
\end{tikzcd}
\quad
\begin{tikzcd}
CBA \arrow[r, "\psi 1"] \arrow[dd, "1\varphi'"'] & CA^2 \arrow[d, "1c^{-1}"] \\
& CA^2 \arrow[d, "1m"] \\
C^2A \arrow[r, "m1"'] & CA
\end{tikzcd}
\quad
\begin{tikzcd}
ABA \arrow[r, "1\varphi"] \arrow[d, "c1"'] & ACA \arrow[d, "c1"] \\
BA^2 \arrow[d, "\varphi' 1"'] & CA^2 \arrow[d, "1m"] \\
CA^2 \arrow[r, "1m"'] & CA
\end{tikzcd}
\quad
\begin{tikzcd}
CBC \arrow[r, "\psi 1"] \arrow[d, "1c"'] & CAC \arrow[d, "1c"] \\
C^2B \arrow[d, "1\psi'"'] & C^2A \arrow[d, "m1"] \\
C^2A \arrow[r, "m1"'] & CA.
\end{tikzcd}
$$

The 3-simplices are pairs

$$(\varphi_{123}, \varphi_{023}, \varphi_{013}, \varphi_{012}|\psi_{123}, \psi_{023}, \psi_{013}, \psi_{012}\|\varphi'_{123}, \varphi'_{023}, \varphi'_{013}, \varphi'_{012}|\psi'_{123}, \psi'_{023}, \psi'_{013}, \psi'_{012})$$

consisting of a 3-simplex $(\varphi_{123}, \varphi_{023}, \varphi_{013}, \varphi_{012}|\psi_{123}, \psi_{023}, \psi_{013}, \psi_{012})$ of M_{12} and a 3-simplex $(\varphi'_{123}, \varphi'_{023}, \varphi'_{013}, \varphi'_{012}|\psi'_{123}, \psi'_{023}, \psi'_{013}, \psi'_{012})$ of M_{34} for which $(\varphi_{ijk}|\psi_{ijk}\|\varphi'_{ijk}|\psi'_{ijk})$ is a 2-simplex in M for every $0 \leq i < j < k \leq 3$ and the diagrams

$$
\begin{tikzcd}
A_{01}A_{03}A_{23} \arrow[r, "1\varphi_{023}"] \arrow[d, "\psi'_{013}1"'] & A_{01}A_{02}A_{23} \arrow[d, "\psi'_{012}1"] \\
A_{01}A_{13}A_{23} \arrow[r, "1\varphi_{123}"'] & A_{01}A_{12}A_{23}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{01}A_{03}A_{23} \arrow[r, "\psi_{013}1"] \arrow[d, "1\varphi'_{023}"'] & A_{01}A_{13}A_{23} \arrow[d, "1\varphi'_{123}"] \\
A_{01}A_{02}A_{23} \arrow[r, "\psi_{012}1"'] & A_{01}A_{12}A_{23}
\end{tikzcd}
\tag{3.7}
$$

commute. The higher simplices are generated coskeletally and the face and the degeneracy maps act on the pairs memberwise.

Theorem 3.6. *There is a bijection between simplicial maps* $\mathsf{C} \to \mathsf{M}$ *and regular multiplier bimonoids in* $\mathcal{C}$.

Proof. For a simplicial map $\mathsf{C} \to \mathsf{M}$, denote the image of the 2-simplex α by (A, m), and denote the images of the 3-simplices τ and ε by $(t_1|t_2\|t_3|t_4)$ and $(e|e\|e'|e')$, respectively. By Theorem 3.5, (t_1, t_2, e) is a multiplier bimonoid in $\mathcal{C}$ for which $m = e1.t_1 = 1e.t_2$, and (t_3, t_4, e') is a multiplier bimonoid in $\mathcal{C}^{\mathsf{rev}}$ for which $m.c^{-1} = e'1.t_3 = 1e'.t_4$. Furthermore, from the requirements that $(t_1|t_2\|t_3|t_4)$ and $(e|e\|e'|e')$ be 2-simplices of M we obtain the following identities.

$$
\begin{tikzcd}
A^3 \arrow[r, "1t_1"] \arrow[dd, "t_4 1"'] \arrow[ddr, phantom, "\text{(j)}"] & A^3 \arrow[d, "c^{-1}1"] \\
& A^3 \arrow[d, "m1"] \\
A^3 \arrow[r, "1m"'] & A^2
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[r, "t_2 1"] \arrow[dd, "1t_3"'] \arrow[ddr, phantom, "\text{(k)}"] & A^3 \arrow[d, "1c^{-1}"] \\
& A^3 \arrow[d, "1m"] \\
A^3 \arrow[r, "m1"'] & A^2
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[r, "1t_1"] \arrow[d, "c1"'] & A^3 \arrow[d, "c1"] \\
A^3 \arrow[r, phantom, "\text{(l)}"] \arrow[d, "t_3 1"'] & A^3 \arrow[d, "1m"] \\
A^3 \arrow[r, "1m"'] & A^2
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[r, "t_2 1"] \arrow[d, "1c"'] & A^3 \arrow[d, "1c"] \\
A^3 \arrow[r, phantom, "\text{(m)}"] \arrow[d, "1t_4"'] & A^3 \arrow[d, "m1"] \\
A^3 \arrow[r, "m1"'] & A^2
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "e"] \arrow[dd, "e'"'] \arrow[ddr, phantom, "\text{(n)}"] & I \arrow[dd, equal] \\
& \\
I \arrow[r, equal] & I.
\end{tikzcd}
$$

The boundaries of the images of the 3-simplices ϕ, λ, ϱ and κ under a simplicial map are determined. The fillers to be their images exist if and only if the diagrams

$$
\begin{tikzcd}
A^3 \arrow[r, "1t_1"] \arrow[d, "t_41"'] \arrow[dr, phantom, "\text{(o)}"] & A^3 \arrow[d, "t_41"] \\
A^3 \arrow[r, "1t_1"'] & A^3
\end{tikzcd}
\quad
\begin{tikzcd}
A^3 \arrow[r, "t_21"] \arrow[d, "1t_3"'] \arrow[dr, phantom, "\text{(p)}"] & A^3 \arrow[d, "1t_3"] \\
A^3 \arrow[r, "t_21"'] & A^3
\end{tikzcd}
\quad
\begin{tikzcd}
A^2 \arrow[r, "t_1"] \arrow[d, "t_2"'] \arrow[dr, "m"] & A^2 \arrow[d, "e'1"] \arrow[dl, phantom, "\text{(q)}"] \\
A \arrow[r, "1e'"'] \arrow[ur, phantom, "\text{(r)}"] & A
\end{tikzcd}
\quad
\begin{tikzcd}
A^2 \arrow[r, "t_3"] \arrow[d, "t_4"'] \arrow[dr, "m.c^{-1}"] & A^2 \arrow[d, "e1"] \arrow[dl, phantom, "\text{(s)}"] \\
A \arrow[r, "1e"'] \arrow[ur, phantom, "\text{(t)}"] & A
\end{tikzcd}
$$

commute (in the case of κ the same condition (n) occurs again).

Conditions (l), (o), (m) and (p) are identical to the last four diagrams of (3.6). In light of (n), conditions (q), (r), (s) and (t) are redundant. Conditions (j) and (k) are also redundant: they follow from (o) and (p), respectively, postcomposing them with $1e1 = 1e'1$. Finally (n) implies the commutativity of the first diagram of (3.6), with common diagonal m. ∎

Remark 3.7. We discussed in Remark 3.1 and Remark 3.4 a sense in which the definitions of 2-simplices in M_1 and M_{12} are dictated by the definitions of the respective 3-simplices.

When it comes to M, however, this property breaks down. While commutativity of the first two diagrams in the definition of 2-simplices in M is needed in order for degenerate 3-simplices to exist in M, commutativity of the other two diagrams is not. Commutativity of these last two diagrams would be needed, though, if we were to modify the definition of 3-simplices so as to require that, in addition to the diagrams of (3.7), also

$$
\begin{tikzcd}
A_{13}A_{03}A_{23} \arrow[r, "1\varphi_{023}"] \arrow[d, "c1"'] & A_{13}A_{02}A_{23} \arrow[r, "c1"] & A_{02}A_{13}A_{23} \arrow[r, "1\varphi_{123}"] & A_{02}A_{12}A_{23} \arrow[d, "\varphi'_{012}1"] \\
A_{03}A_{13}A_{23} \arrow[rr, "\varphi'_{013}1"'] & & A_{01}A_{13}A_{23} \arrow[r, "1\varphi_{123}"'] & A_{01}A_{12}A_{23}
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_{01}A_{03}A_{02} \arrow[r, "\psi_{013}1"] \arrow[d, "1c"'] & A_{01}A_{13}A_{02} \arrow[r, "1c"] & A_{01}A_{02}A_{13} \arrow[r, "\psi_{012}1"] & A_{01}A_{12}A_{13} \arrow[d, "1\psi'_{123}"] \\
A_{01}A_{02}A_{03} \arrow[rr, "1\psi'_{023}"'] & & A_{01}A_{02}A_{23} \arrow[r, "\psi_{012}1"'] & A_{01}A_{12}A_{23}
\end{tikzcd}
$$

commute.

Simplicial maps from C to the resulting simplicial set would now correspond to a stronger notion of regular multiplier bimonoid, in which the second diagram of (3.6) is replaced by the fusion equation in the second diagram of [3, Remark 3.10], and with an analogous change to the fourth diagram of (3.6). Although in *general* this would result in a stronger notion of regular multiplier bimonoid, the difference would disappear in the case where the multiplication is non-degenerate, and it was already anticipated in [3, Remark 3.10] that in the not necessarily non-degenerate case such strengthenings might be needed.

3.4 Multiplier bimonoids which are comonoids

In our paper [4], following some ideas in [8], we associated to any braided monoidal category $\mathcal{C}$ a monoidal category $\mathcal{M}$, and we described a correspondence between certain multiplier bimonoids in $\mathcal{C}$ and certain comonoids in $\mathcal{M}$ [4, Theorem 5.1]. In this section, we explain this correspondence in terms of simplicial maps and the Catalan simplicial set.

A comonoid in the monoidal category $\mathcal{M}$ is the same as a monoid in the monoidal category $\mathcal{M}^{\mathsf{op}}$. We now form the nerve $\mathsf{N}(\mathcal{M}^{\mathsf{op}})$ of the monoidal category $\mathcal{M}^{\mathsf{op}}$, as in Section 2.3; this is not to be confused with the nerve of the underlying category of $\mathcal{M}^{\mathsf{op}}$. As explained in [6], simplicial maps $\mathsf{c}\colon \mathsf{C} \to \mathsf{N}(\mathcal{M}^{\mathsf{op}})$ can be identified with monoids in $\mathcal{M}^{\mathsf{op}}$, and so in turn with comonoids in $\mathcal{M}$.

On the other hand, we have shown that simplicial maps $\mathsf{a}\colon \mathsf{C} \to \mathsf{M}_{12}$ can be identified with multiplier bimonoids in $\mathcal{C}$. In order to compare these, we construct a simplicial set Q which is contained in both M_{12} and $\mathsf{N}(\mathcal{M}^{\mathsf{op}})$. Now a multiplier bimonoid in $\mathcal{C}$ corresponds to a comonoid in $\mathcal{M}$ just when there is a common factorization of the corresponding simplicial maps as in the following diagram.

$$\begin{tikzcd}
\mathsf{C} \arrow[dr, dotted] \arrow[drr, bend left, "\mathsf{a}"] \arrow[ddr, bend right, "\mathsf{c}"'] & & \\
& \mathsf{Q} \arrow[r] \arrow[d] & \mathsf{M}_{12} \\
& \mathsf{N}(\mathcal{M}^{\mathsf{op}}) &
\end{tikzcd}$$

We now turn to the details. To define the monoidal category $\mathcal{M}$, in [4] we fixed a class $\mathcal{Q}$ of regular epimorphisms in $\mathcal{C}$ which is closed under composition and monoidal product, contains the isomorphisms, and is right-cancellative in the sense that if $s\colon A \to B$ and $t.s\colon A \to C$ are in $\mathcal{Q}$, then so is $t\colon B \to C$. Since each $q \in \mathcal{Q}$ is a regular epimorphism, it is the coequalizer of some pair of morphisms. Finally we suppose that this pair may be chosen in such a way that the coequalizer is preserved by taking the monoidal product with any fixed object.

The objects of the associated category $\mathcal{M}$ are those semigroups in $\mathcal{C}$ whose multiplication is non-degenerate and belongs to $\mathcal{Q}$. The morphisms $f\colon A \nrightarrow B$ are pairs $(f_1\colon AB \to B \leftarrow BA\colon f_2)$ of morphisms in $\mathcal{Q}$ such that the first two, equivalently, the last two diagrams in

$$\begin{tikzcd}
A^2B \arrow[r, "1f_1"] \arrow[d, "m1"'] & AB \arrow[d, "f_1"] \\
AB \arrow[r, "f_1"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
BAB \arrow[r, "1f_1"] \arrow[d, "f_21"'] & B^2 \arrow[d, "m"] \\
B^2 \arrow[r, "m"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
BA^2 \arrow[r, "f_21"] \arrow[d, "1m"'] & BA \arrow[d, "f_2"] \\
BA \arrow[r, "f_2"'] & B
\end{tikzcd}$$

commute. The composite $g \bullet f$ of morphisms $f\colon A \nrightarrow B$ and $g\colon B \nrightarrow C$ is defined by universality of the coequalizer in the top row of the following diagrams

$$
\begin{tikzcd}
ABC \arrow[r, "1g_1"] \arrow[d, "f_1 1"'] & AC \arrow[d, "(g\bullet f)_1"] \\
BC \arrow[r, "g_1"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
CBA \arrow[r, "g_2 1"] \arrow[d, "1f_2"'] & CA \arrow[d, "(g\bullet f)_2"] \\
CB \arrow[r, "g_2"'] & C.
\end{tikzcd}
$$

The identity morphism $A \twoheadrightarrow A$ is the pair $(m\colon A^2 \to A \leftarrow A^2\colon m)$.

This category $\mathcal{M}$ is monoidal. The monoidal product of semigroups A and C is

$$(AC)^2 \xrightarrow{1c1} A^2C^2 \xrightarrow{mm} AC$$

and the monoidal product of morphisms $f\colon A \twoheadrightarrow B$ and $g\colon C \twoheadrightarrow D$ is the pair

$$ACBD \xrightarrow{1c1} ABCD \xrightarrow{f_1g_1} BD \xleftarrow{f_2g_2} BADC \xleftarrow{1c1} BDAC.$$

If $\mathcal{C}$ is a closed braided monoidal category with pullbacks, then for any semigroup B with non-degenerate multiplication one can define its multiplier monoid $\mathbb{M}(B)$, see [4]. It is a monoid in $\mathcal{C}$ and a universal object characterized by the property that morphisms $(f_1, f_2)\colon A \twoheadrightarrow B$ in $\mathcal{M}$ correspond bijectively to multiplicative morphisms $f\colon A \to \mathbb{M}(B)$ in $\mathcal{C}$ such that

$$AB \xrightarrow{f1} \mathbb{M}(B)B \xrightarrow{i_1} B \quad \text{and} \quad BA \xrightarrow{1f} B\mathbb{M}(B) \xrightarrow{i_2} B$$

are in $\mathcal{Q}$, where $i\colon \mathbb{M}(B) \to \mathbb{M}(B)$ is the identity morphism in $\mathcal{C}$; regarded as a morphism $(i_1, i_2)\colon \mathbb{M}(B) \twoheadrightarrow B$ in $\mathcal{M}$. In the category of vector spaces $\mathbb{M}(B)$ reduces to the multiplier algebra of B as defined in [7].

Take a multiplier bimonoid (A, t_1, t_2, e) in $\mathcal{C}$ for which

- the underlying semigroup has a non-degenerate multiplication $m := e1.t_1 = 1e.t_2$
- m, e, and the morphisms d_1 and d_2 defined by

  $$A^3 \xrightarrow{c1} A^3 \xrightarrow{1t_1} A^3 \xrightarrow{c^{-1}1} A^3 \xrightarrow{m1} A^2$$
  $$A^3 \xrightarrow{1c} A^3 \xrightarrow{t_21} A^3 \xrightarrow{1c^{-1}} A^3 \xrightarrow{1m} A^2$$

  all belong to $\mathcal{Q}$.

The correspondence in [4, Theorem 5.1] associated a multiplier bimonoid of this type to the comonoid in $\mathcal{M}$ with underlying object (A, m), with comultiplication $A \twoheadrightarrow A^2$ having components d_1 and d_2, and with counit $A \twoheadrightarrow I$ whose components are both e.

The simplicial set M_{12} has a simplicial subset Q as follows. The only 0-simplex $*$ of M_{12} is a 0-simplex also in Q. The 1-simplices of Q are those semigroups (A, m) in $\mathcal{C}$ whose multiplication is non-degenerate and belongs to $\mathcal{Q}$. The 2-simplices of Q are those 2-simplices $(\varphi|\psi)$ of M_{12} whose faces belong to Q and for which the morphisms

$$\widehat{\varphi} : A_{02}A_{01}A_{12} \xrightarrow{1c^{-1}} A_{02}A_{12}A_{01} \xrightarrow{\varphi 1} A_{01}A_{12}A_{01} \xrightarrow{1c} A_{01}A_{01}A_{12} \xrightarrow{m_{01}1} A_{01}A_{12}$$
$$\widehat{\psi} : A_{01}A_{12}A_{02} \xrightarrow{c^{-1}1} A_{12}A_{01}A_{02} \xrightarrow{1\psi} A_{12}A_{01}A_{12} \xrightarrow{c1} A_{01}A_{12}A_{12} \xrightarrow{1m_{12}} A_{01}A_{12}$$

are in $\mathcal{Q}$. The 3-simplices of Q are all those 3-simplices of M_{12} whose faces belong to Q. Clearly a simplicial map from C to Q is the same thing as a multiplier bimonoid in $\mathcal{C}$ having the properties listed above.

The desired simplicial map $\mathsf{Q} \to \mathsf{N}(\mathcal{M}^{\mathsf{op}})$ sends the 0-simplex $*$ to the single 0-simplex of the nerve $\mathsf{N}(\mathcal{M}^{\mathsf{op}})$. It sends the 1-simplex (A, m) in Q to its underlying object A. It sends a 2-simplex $(\varphi|\psi)$ to the morphism $A_{02} \nrightarrow A_{01}A_{12}$ in $\mathcal{M}$ whose components are $\widehat{\varphi}$ and $\widehat{\psi}$. On the higher simplices it is unambiguously defined by the uniqueness of the filler of any boundary in $\mathsf{N}(\mathcal{M}^{\mathsf{op}})$. This assignment is injective by non-degeneracy of the relevant multiplications.

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