Three natural subgroups of the Brauer-Picard group of a Hopf algebra with applications

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Abstract

In this article we construct three explicit natural subgroups of the Brauer-Picard group of the category of representations of a finite-dimensional Hopf algebra. In examples the Brauer Picard group decomposes into an ordered product of these subgroups, somewhat similar to a Bruhat decomposition.

Our construction returns for any Hopf algebra three types of braided autoequivalences and correspondingly three families of invertible bimodule categories. This gives examples of so-called (2-)Morita equivalences and defects in topological field theories. We have a closer look at the case of quantum groups and Nichols algebras and give interesting applications. Finally, we briefly discuss the three families of group-theoretic extensions.

1 Introduction

For a finite tensor category $\mathcal{C}$ the *Brauer-Picard group* $\mathrm{BrPic}(\mathcal{C})$ is defined as the group of equivalence classes of invertible exact $\mathcal{C}$-$\mathcal{C}$-bimodule categories. This group is an important invariant of the tensor category $\mathcal{C}$ and appears at essential places such as group-theoretic extension of $\mathcal{C}$ and as defects in mathematical physics, see applications below. By a result in [ENOM09][DN12] the group is isomorphic to braided autoequivalences of the Drinfeld center $\mathrm{BrPic}(\mathcal{C}) \cong \mathrm{Aut}_{br}(\mathcal{Z}(\mathcal{C}))$; this will be crucial in what follows.

Computing the Brauer-Picard group, even for $\mathcal{C} = \mathrm{Rep}(G)$ or equivalently $\mathcal{C} = \mathrm{Vect}_G$ for a finite group G, is already an interesting and non-trivial task, see

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[ENOM09] [NR14] [FPSV14] [LP15b] [MN16]. The group multiplication is particularly hard to pin down. For $\mathcal{C} = H$-mod with H an arbitrary Hopf algebra, not much is known besides few examples, see [FMM14] [Mom12] [BN14] [ZZ13].

In [LP15b] we have proposed an approach to calculate $\mathrm{BrPic}(\mathcal{C})$ for $\mathcal{C} = H$-mod by defining certain natural subgroups[1] $\mathcal{BV}, \mathcal{EV}$ with intersection $\mathcal{V}$ and a set of elements $\mathcal{R}$, such that the Brauer Picard group may decompose as a Bruhat-alike decomposition

$$\mathrm{BrPic}(\mathcal{C}) = \bigcup_{r \in \mathcal{R}} \mathcal{BV}\,\mathcal{EV}\,r$$

In *cit. loc.* we have proven such a decomposition for the case $H = \mathbb{C}[G]$ for elements fulfilling an additional restriction (laziness). Moreover we checked the decomposition in all available examples by hand. It is unclear at this point if it is true in general.
The intuition arises from

Example (Sec. 4.1.5). *Let $G \cong \mathbb{Z}_p^n$ with p a prime number. Our decomposition reduces to the Bruhat decomposition of $\mathrm{BrPic}(\mathrm{Vect}_G)$, which is the Lie group $\mathrm{O}_{2n}(\mathbb{F}_p)$ over the finite field $\mathbb{F}_p$. In this case $\mathcal{BV}, \mathcal{EV}$ are lower and upper triangular matrices, intersecting in the subgroup $\mathcal{V} = \mathrm{GL}_n(\mathbb{F}_p)$. The partial dualizations are Weyl group elements. More precisely, our result reduces to the Bruhat decomposition of the Lie groups D_n relative to the parabolic subsystem A_{n-1}, so reflections are actually equivalence classes corresponding to $n+1$ cosets of the parabolic Weyl group.*

The present article is devoted to start the discussion of the more general case $\mathcal{C} = H$-mod. We shall not try to prove a decomposition theorem, but focus our attention on establishing and discussing the expected natural subgroups $\mathcal{V}, \mathcal{BV}, \mathcal{EV}, \langle \mathcal{R} \rangle$ of the Brauer Picard group. We will also briefly discuss several interesting applications of our results, in particular when H is the Borel part of a quantum group resp. a Nichols algebra.

In Section 2 we briefly recall the induction functor and the ENOM-functor [ENOM09]

$$\mathrm{Aut}_{\mathrm{mon}}(\mathcal{C}) \to \mathrm{BrPic}(\mathcal{C}) \qquad \mathrm{BrPic}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{br}}(\mathcal{Z}(\mathcal{C}))$$

In view of interesting examples and the applications to defects in mathematical physics and Nichols algebras we state the obvious generalization of these concepts to the groupoid setting, so that arbitrary monoidal equivalences $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$ give rise to invertible $\mathcal{C}$-$\mathcal{D}$-bimodule categories, and these are in bijection to braided equivalences $\mathcal{Z}(\mathcal{C}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{D})$.

In Section 3 we define and derive for each subgroup $\mathcal{V}, \mathcal{BV}, \mathcal{EV}$ and the subset $\mathcal{R}$ explicit expressions for the braided autoequivalence as well as the invertible bimodule categories.

[1] We choose these names $\mathcal{EV}, \mathcal{BV}$ for compatibilities with previous conventions. Be advised that $\mathcal{V}$ does **not** necessarily have complement subgroups $\mathcal{B}, \mathcal{E}$ in $\mathcal{BV}, \mathcal{EV}$ in the most general cases.

On one hand $\mathcal{BV}$ resp. $\mathcal{EV}$ are obtained using induction functors from H-mod resp. H^*-mod. So the bimodule categories in BrPic(H-mod) resp. BrPic(H^*-mod) are given by definition. We then calculate explicitly the images under the ENOM functor using Bigalois objects and finally we describe again the preimage of $\mathcal{EV}$ now in BrPic(H-mod). As linear categories, the bimodule categories in $\mathcal{BV}$ are all equal to $\mathcal{C}$, while the bimodule categories in $\mathcal{EV}$ are representation categories of Bigalois objects, as in [FMM14].

On the other hand the set of elements $\mathcal{R}$ is defined as partial dualizations on the $\mathrm{Aut_{br}}$-side of the functor as obtained by the first author in [BLS15]. There are two types of partial dualization, for every way to decompose $H = K \rtimes A$ into a (semidirect) Radford-biproduct. As linear categories, the bimodule categories in $\mathcal{R}$ are representations of semidirect factors of H (so they may be significantly "smaller", down to Vect) but with a largely nontrivial bimodule category structure $(V.M).W \xrightarrow{\sim} V.(M.W)$.

In Section 4 we discuss examples: Mostly we work out the result for $\mathcal{C} = \mathrm{Vect}_G$, which has been discussed extensively. In particular we discuss how our bimodule categories look in the explicit description of [ENOM09][Dav10]. Then we thoroughly discuss the case where H is the Taft algebra and compare our results with [FMM14].

In Section 5 we discuss applications:

a) First we discuss interesting types of bimodule categories that arise from our constructions for a Nichols algebra $H = \mathfrak{B}(M) \rtimes \mathbb{C}[G]$. This includes for example the quantum group Borel parts $U_q^{\geq}(\mathfrak{g})$ resp. $u_q^{\geq}(\mathfrak{g})$.

   First, due to the Bigalois objects there are interesting elements in $\mathcal{BV}, \mathcal{EV}$ related to different *liftings* of quantum groups, most of which have non-equivalent representation categories $\mathcal{C}, \mathcal{D}, \ldots$ but are connected by invertible bimodule categories.

   Even more interesting are the partial dualizations: We may either dualize on the Cartan part $\mathbb{C}[G]$, then we obtain invertible bimodule categories between different *forms* of $u_q^{\geq}(\mathfrak{g})$ e.g. between the adjoint and the simply-connected form.

   Alternatively we we may dualize on parabolic sub-Nichols algebras, then partial dualization reduces to the usual Weyl group reflection of the quantum group. In this way we get invertible bimodule categories connecting different choices of positive roots, and as linear categories these are representations of coideal subalgebras.

   At last, we remark that the Aut_{br}-side of all these elements, which we have worked out explicitly in the previous sections, give rise to braided autoequivalences of the representation category of the full quantum group.

b) An interesting application to mathematical physics are defects: (Bi-)module categories appear as boundary conditions and defects in $3d$-TQFT, in particular the Brauer-Picard group is the symmetry group of such theories, see [FSV13],[FPSV14].

Our results give three systematic, generic families of examples for such defects. More importantly, they give many examples of invertible bimodule categories between different categories. In a general TQFT the defects separate different regions of space, which can be labeled by different categories. Particularly interesting in this matter are again the concrete examples arising from quantum groups.

c) Finally, a leading motivation for the consideration of the Brauer Picard group is, that group-theoretic extensions of categories are parametrized by group homomorphisms into the Brauer Picard group [ENOM09]. We close this article by briefly discussing, which types of categories arise for our three subgroups.

This includes representations of the folded Nichols algebras over nonabelian groups constructed by the first author in [Len12].

2 Categorical Setup

Let $\mathcal{C},\mathcal{D},\ldots$ be finite tensor categories with base field $k=\mathbb{C}$.

Definition 2.1. *The* Brauer Picard Groupoid BrPic *has as objects tensor categories $\mathcal{C},\mathcal{D},\ldots$ and as morphisms equivalence classes of exact invertible bimodule categories ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ and as composition the relative Deligne tensor product $({}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}})\boxtimes_{\mathcal{D}}({}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}})$.*
The automorphism group of an object $\mathcal{C}$ is the Brauer Picard group $\mathrm{BrPic}(\mathcal{C})$. *Categories $\mathcal{C},\mathcal{D}$ for which there exists an isomorphism ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ are called* (2-) Morita equivalent

Definition 2.2. *The* monoidal equivalence groupoid $\mathrm{Eq}_{\mathrm{mon}}$ *has as objects finite tensor categories $\mathcal{C},\mathcal{D}\ldots$ and as morphism monoidal category equivalences $F:\mathcal{C}\to\mathcal{D}$ and as composition concatenation.*
The braided equivalence groupoid $\mathrm{Eq}_{\mathrm{br}}$ *has as objects braided tensor categories $\mathcal{Z},\mathcal{W}\ldots$ and as morphism braided category equivalences $F:\mathcal{Z}\to\mathcal{W}$. We denote by $\mathrm{Eq}^0_{\mathrm{br}}$ the full subgroupoid consisting of objects that are Drinfeld centers (i.e. Witt class 0).*
The automorphism group of an object $\mathcal{C}$ is the group of monoidal autoequivalences $\mathrm{Eq}_{\mathrm{mon}}(\mathcal{C})=\mathrm{Aut}_{\mathrm{mon}}(\mathcal{C})$ *resp.* braided autoequivalences $\mathrm{Eq}_{\mathrm{br}}(\mathcal{Z})=\mathrm{Aut}_{\mathrm{br}}(\mathcal{Z})$.

In fact we are actually dealing with a bicategory with 1-morphisms invertible bimodule categories and with 2-morphisms bimodule category equivalences, respectively with 1-morphism category equivalences and with 2-morphisms natural transformations.

Lemma 2.3 (Induction Functor)**.** *There is an evident groupoid homomorphism* $\mathrm{Ind}:\mathrm{Eq}_{\mathrm{mon}}\to\mathrm{BrPic}$ *given on objects by the identity and on morphisms ${}_{\mathcal{C}}F_{\mathcal{D}}$ by $F\mapsto{}_{F}\mathcal{D}$ where $\mathcal{D}$ is the trivial right $\mathcal{D}$-module category and the trivial left $\mathcal{D}$-module category precomposed with the monoidal functor F.*
This yields in particular an evident group homomorphism $\mathrm{Aut}_{\mathrm{mon}}(\mathcal{C})\to\mathrm{BrPic}(\mathcal{C})$.

The following theorem is due to [ENOM09]; see [DN12] for the non-semisimple case:

Theorem 2.4 (ENOM functor)**.** *There is an equivalence of groupoids* $\Phi : \mathrm{BrPic} \cong \mathrm{Eq}^0_{\mathrm{br}}$*. It is given on objects by sending* $\mathcal{C} \mapsto \mathcal{Z}(\mathcal{C})$*, on morphisms* ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}} \mapsto F_{\mathcal{M}}$ *it fulfills the following defining property:*

$\mathcal{Z}(\mathcal{C})$ acts on ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ as bimodule category automorphism, where the compatibility constraint $(c.m).d \to c.(m.d)$ is given by the bimodule category structure and the compatibility constraint $c'.(c.m) \to c.(c'.m)$ is given by the half-braiding $\tau_{c,c'}$ of the element $(c,\tau) \in \mathcal{Z}(\mathcal{C})$. Similarly $\mathcal{Z}(\mathcal{D})$ acts on ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ as bimodule category automorphism. The defining property for $\Phi(\mathcal{M}) : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$ is that the module category homomorphisms $c.$ and $.\Phi(\mathcal{M})(c)$ are equivalent, i.e. there is a natural transformations between these two functors that satisfy certain coherence properties with the two module category and the bimodule category structure.

3 Subgroups of BrPic

3.1 Motivation

Why should we hope for a Bruhat-like decomposition of $\mathrm{BrPic}(H\text{-mod})$?

The main motivation for our initial work [LP15b] was the case $H = \mathbb{C}[G]$ for G abelian, as treated in the second authors joint paper [FPSV14]. In particular let $G \cong \mathbb{Z}_p^n$ with p a prime number. Then it is known that $\mathrm{BrPic}(\mathrm{Rep}(G)) = \mathrm{O}_{2n}(\mathbb{F}_p)$ and the choice of generators in *cit. loc.* are upper triangular matrices containing the group of group automorphisms $\mathrm{Aut}(G) = \mathrm{GL}_n(\mathbb{F}_p)$, and additional generators are the so-called *EM-dualities*.

As it turned out in our study, these generators are not arbitrary, but rather naturally defined subgroups, in much more general context, that can be written down without prior knowledge of the full Brauer Picard group and come from different sources:

Two sets of generators can be obtained via different induction functors from various categories $\mathcal{C}'$ with $\mathcal{Z}(\mathcal{C}') \cong \mathcal{Z}(\mathcal{C})$, leading in the example for $\mathcal{C} = \mathrm{Vect}_G$ to upper-triangular matrices $\mathcal{BV} = \mathrm{Aut}(G) \ltimes H^2(G, \mathbb{C}^\times)$, as in [NR14], and for $\mathcal{C} = \mathrm{Rep}(G)$ to lower-triangular matrices $\mathcal{EV}$ intersecting precisely in $\mathcal{V} = \mathrm{Aut}(G)$.

A third set of generators, the so-called EM-dualities $\mathcal{R}$, turned out to be rather general braided autoequivalences called *partial dualizations* in the first authors work [BLS15]. These can be defined whenever a Hopf algebra decomposes into a semidirect product, and a special case are simple reflections of quantum groups.

In [LP15b] we have proved that every element fulfilling an additional condition (laziness) decomposes accordingly into an ordered product in these subgroups, also we have checked the Brauer-Picard group in known cases by hand. The Brauer-Picard group decomposition retains roughly the properties that a Lie group over a ring admits (not an honest Bruhat decomposition), which is what we get e.g. for $G = \mathbb{Z}_k^n$ for k not prime.

A maybe more convincing reason for our approach arose during the work on [LP15b]: Every braided autoequivalence of DH-mod is described through its action on objects plus a monoidal structure i.e. an element in $H^2(\mathrm{D}H^*,\mathbb{C}^\times)$. While the action on objects seems easily accessible (one can look at invertible objects, stabilizer etc.), there is in general very many possibilities. In the lazy case this action if given by precomposing a Hopf algebra automorphism, and the automorphism group reminds on a matrix group, but for more general cases we don't have this luxury.

On the other hand $H^2(\mathrm{D}H^*,\mathbb{C}^\times)$ is rather technical, but it should not surprise us that is is connected to the groups $H^2(H,\mathbb{C}^\times)$, $H^2(H^*,\mathbb{C}^\times)$ and some interaction between H, H^*. So we propose to shift classification effort to the monoidal structure of the functor, rather that its action on objects. In fact for abelian groups (and much more general situations) we have by Schauenburg [Schau02] a *Künneth-type* formula, and this decomposition **does** precisely explain the initially observed decomposition.

Another interesting question is, if one can characterize elements inside one Bruhat-cell: Indeed for $H = \mathbb{C}[G]$ the "big cell" $\mathcal{BVEV}$ has the property that (in the language of [NR14]) it sends the Langrangian subcategory $\mathcal{L}_{1,1}$ to some $\mathcal{L}_{N,\mu}$ with μ nondegenerate. Smaller Bruhat-cells $\mathcal{BVEV}r$ can be characterized by the degree of degeneracy of μ, down to $\mu = 1$ which is a pure reflection. A similar picture seems to emerge in this article for the bimodule categories, where the big cell consists of R-mod for some algebra of same dimension as H, while smaller cells are representations of considerable smaller algebras down to Vect for the longest element in $\mathcal{R}$.

However, these are merely speculative observations. As stated in the introduction, the present paper does not concern itself with the decomposition, but focuses solely on the definition and description of these generic subgroups in the general case:

3.2 $\mathcal{V}$ induced from Hopf automorphisms

This obvious subgroup reappears as the intersection of the two upcoming subgroups.

Lemma 3.1. *Let $v \in \mathrm{Iso}_{\mathrm{Hopf}}(H,L)$ be a Hopf algebra isomorphism, then we have in particular a monoidal equivalence $v:$ L-mod $\to$ H-mod by precomposition. Induction (Lm. 2.3) provides an invertible bimodule category $\mathcal{M} := {}_v(H\text{-mod})$.*
We claim that this element in $\mathrm{BrPic}(L\text{-mod}, H\text{-mod})$ *gives under the ENOM functor rise to the functor in* $\mathrm{Eq_{br}}(\mathrm{D}L\text{-mod}, \mathrm{D}H\text{-mod})$ *given on objects by* $\Phi(\mathrm{Ind}(v)) : Z \mapsto {}_v^{v^{-1}}Z$ *and with trivial monoidal structure. Similarly induction of* $v^{-1} : L^*\text{-mod} \to H^*\text{-mod}$ *provides a module category* ${}_{v^{-1}}(H^*\text{-mod})$ *giving rise to the same element.*
In particular this defines a subgroup $\mathcal{V} \subset \mathrm{BrPic}(H\text{-mod})$ *with* $\mathcal{V} \cong \mathrm{Out_{Hopf}}(H)$.

Proof. To apply the defining property of the ENOM functor it suffices to construct a natural isomorphism between the functors $Z.$ and $.\Phi(\mathrm{Ind}(v))Z$ for $M \in L$-mod.

The half-braiding given by the coaction on Z gives a natural isomorphism of H-modules:

$$_vZ \otimes M \to M \otimes .^{v^{-1}}_{v}Z$$
$$z \otimes m \mapsto v^{-1}(z^{(-1)}).m \otimes z^{(0)}$$

We moreover have to check compatibility with the module category constraints, namely for all $W \in L$-mod the following equality, which requires the coaction choice $^{v^{-1}}Z$:

$$_vZ \otimes ({_vW} \otimes M) \to {_vW} \otimes ({_vZ} \otimes M) \to {_vW} \otimes (M \otimes {^{v^{-1}}_{v}Z}) \overset{=}{\to} ({_vW} \otimes M) \otimes {^{v^{-1}}_{v}Z}$$
$$_vZ \otimes ({_vW} \otimes M) \to ({_vW} \otimes M) \otimes {^{v^{-1}}_{v}Z}$$
$$z \otimes w \otimes m \mapsto v^{-1}(z^{(-1)}).(w \otimes m) \otimes z^{(0)} =$$
$$v(v^{-1}(z^{(-2)})).w \otimes v^{-1}(z^{(-1)}).m \otimes z^{(0)}$$
$$_vZ \otimes ({_vW} \otimes M) \to ({_vW} \otimes M) \otimes {^{v^{-1}}_{v}Z}$$
$$z \otimes w \otimes m \mapsto v^{-1}(z^{(-1)}).(w \otimes m) \otimes z^{(0)} =$$
$$v(v^{-1}(z^{(-2)})).w \otimes v^{-1}(z^{(-1)}).m \otimes z^{(0)}$$

as well as the following equality of morphisms for all $W \in H$-mod:

$$_vZ \otimes (M \otimes W) \overset{=}{\to} ({_vZ} \otimes M) \otimes W \to (M \otimes {^{v^{-1}}_{v}Z}) \otimes W \to (M \otimes W) \otimes {^{v^{-1}}_{v}Z}$$
$$z \otimes m \otimes w \mapsto z \otimes m \otimes w \mapsto v^{-1}(z^{(-1)}).m \otimes z^{(0)} \otimes w \mapsto$$
$$v^{-1}(z^{(-2)}).m \otimes v^{-1}(z^{(-1)}).w \otimes z^{(0)}$$
$$_vZ \otimes (M \otimes {_vW}) \to (M \otimes {_vW}) \otimes {^{v^{-1}}_{v}Z}$$
$$z \otimes m \otimes w \mapsto v^{-1}(z^{(-1)}).(m \otimes w) \otimes z^{(0)} = v^{-1}(z^{(-2)}).m \otimes v^{-1}(z^{(-1)}).w \otimes z^{(0)} \quad \blacksquare$$

We also discuss the connection to a different embedding[2]:

Remark 3.2. *The authors of [COZ97] define for a Hopf algebra H the* Quantum Brauer group $\mathrm{BQ}(k,H)$*, an analogue of the Brauer group. It consist of H-Azumaya H-Yetter-Drinfel'd algebras modulo H-Morita equivalence. In [OZ98] they give a map* $\pi : \mathrm{Aut}(H) \to \mathrm{BQ}(k,H)$ *and determine the kernel. An elements in* $A \in \mathrm{BQ}(k,H)$ *gives rise to a* DH-mod*-module category* A-mod*, i.e. an element in the Picard group. By [DN12] in turn the Picard group maps to the Brauer-Picard group and hence to the group of braided autoequivalences - to be precise Thm. 4.3 states that the image of the Picard group consists precisely of those braided autoequivalences which are trivializable on* H-mod $\subset DH$-mod*. This is by construction exactly our subgroup $\mathcal{BV}$ in the next section.*

We shall briefly sketch, how one can explicitly see the surjection of the subgroup $\mathrm{Aut}(H)$ *to our subgroup $\mathcal{V} \subset \mathcal{BV}$ through all these identifications: We first convince ourselves how the identity* $v = \mathrm{id} \in \mathrm{Aut}(H)$ *maps to the identity: The associated Azumaya algebra $A_{v^{-1}}$ is simply* $\mathrm{End}H$ *where H is an H-Yetter-Drinfeld module with*

[2] Thanks to the referee for asking this question

adjoint H-action and diagonal H-coaction. The module category $\mathcal{M} := A_{v^{-1}}\text{-mod}$ *has (as always) the single simple object H with the above Yetter-Drinfeld structure. Now the implicit construction in [DN12] Sec. 2.9 assigns to* $\mathcal{M}$ *the unique equivalence class of autoequivalences* $\partial_{\mathcal{M}} \in \mathrm{Aut}_{\mathrm{br}}(\mathrm{D}H\text{-mod})$*, such that* $\alpha^- \circ \partial_{\mathcal{M}} = \alpha^+$ *are equal as module category morphisms, where* $\alpha^{\pm}(X)$ *means the module category morphisms given on objects by tensoring by* $X \in \mathrm{D}H\text{-mod}$ *and with module category morphism structure given by the braiding resp. the inverse braiding. Equal here means up to natural equivalence and indeed the double-braiding* $X \otimes M \to M \otimes X \to X \otimes M$ *turns out to be such a natural isomorphism between* $X\otimes$ *and itself that switches* α^+, α^-*. This shows how the Hopf-automorphism* id *indeed implies the braided autoequivalence* $\partial = \mathrm{id}$ *as expected.*

For arbitrary $v \in \mathrm{Aut}(H)$ *the situation is more involved, but fairly similar: The Azumaya algebra is defined as* $A_{v^{-1}} := \mathrm{End}H_{v^{-1}}$ *where* $H_{v^{-1}}$ *has again the diagonal coaction but a altered adjoint action* $h.x = v^{-1}(h^{(2)})xS^{-1}(h^{(1)})$*. This is not a Yetter-Drinfel'd module but fulfills the altered relation*

$$(h.a)^{(0)} \otimes (h.a)^{(1)} = h^{(2)}.a^{(0)} \otimes v^{-1}(h^{(3)})a^{(1)}S^{-1}(h^{(1)})$$

Now if $\partial(X) := {}^{v^{-1}}{}_{v}X$ *is the Yetter-Drinfel'd module with modified action and coaction as in the theorem above, then one can roughly see that the double braiding maps*

$${}^{v^{-1}}{}_{v}X \otimes M \longrightarrow M \otimes {}_{v}X \longrightarrow M \otimes X$$

so the double braiding in this sense gives an isomorphism $\alpha^-(\partial(X)) \to \alpha^+(X)$ *on objects, and as for identity the double braiding intertwines the braiding and negative braiding.*

3.3 $\mathcal{BV}$ induced from H-mod

Another rather obvious source of elements in BrPic is the induction functor from arbitrary monoidal equivalences; this of course contains the previous subgroup. While the bimodule category is given by definition, the image of the ENOM-functor requires some preparation:

Let $F : L\text{-mod} \to H\text{-mod}$ be a monoidal equivalence and let us consider the inverse $F^{-1} : H\text{-mod} \to L\text{-mod}$: We are assuming finite dimension, so F^{-1} is given by $R\Box_{H^*}$ with $R = {}_fH^*_\sigma$ an L^*-H^*-Bigalois object [Sch91], where $\sigma \in Z^2(H^*, \mathbb{C})$ is a Hopf 2-cocycle and $f : {}_\sigma(H^*)_{\sigma^{-1}} \to L^*$ is a Hopf algebra isomorphism from the Doi twist of H^* to L^*. On objects F^{-1} is just composing the coaction with f. E.g., for $H = \mathbb{C}^G$ a dual groupring (but not always for a nonabelian groupring), due to the cocommutativity of $H^* = \mathbb{C}[G]$ any Doi twist is equal to H^* and f is a choice of a group isomorphism $H^* \to L^*$.

Theorem 3.3 ([MO98] Thm 2.7)**.** *Given a* 2*-cocycle* $\sigma \in Z^2(H^*, \mathbb{C})$*, then we have the following category equivalence* $\mathcal{Z}(\mathrm{mod}(H^*)) \to \mathcal{Z}(\mathrm{mod}({}_\sigma(H^*)_{\sigma^{-1}}))$*: Send V to* ${}_\sigma V$ *with the same* H^**-coaction and modified* H^**-action*

$$f._\sigma v = \sigma(f^{(1)}, v^{(-1)})\sigma^{-1}((f^{(2)}.v^{(0)})^{(-1)}, f^{(3)}) \cdot (f^{(2)}.v^{(0)})^{(0)}$$

and monoidal structure of the functor given by σ*.*

We can now state:

Lemma 3.4. *Let $F \in \mathrm{Eq_{mon}}(L\text{-mod}, H\text{-mod})$ and σ, f as above. The induction image of F is by definition the bimodule category $\mathcal{M} := {}_F(H\text{-mod})$.*

We claim that this element in $\mathrm{BrPic}(L\text{-mod}, H\text{-mod})$ gives under the ENOM functor rise to the functor in $\mathrm{Eq_{br}}(DL\text{-mod}, DH\text{-mod})$ given on objects by $\Phi(\mathrm{Ind}(v))$: $Z \mapsto {}^{\sigma^{-1}\circ f}_{f^{-1}}Z$ and with the monoidal structure of F.

Here ${}^{\sigma^{-1}\circ f}_{f^{-1}}Z$ means the L-module has been converted by F to a H-module $F(Z)$ which means precompose the action by f^{-1}. On the other hand the L^*-action is pulled back to an ${}_\sigma(H^*)_{\sigma^{-1}}$-action by f and further to a H^*-action by σ^{-1} with the previous Lemma.

In particular this defines a subgroup $\mathcal{BV} \subset \mathrm{BrPic}(H\text{-mod})$ which is the homomorphic image of the group $\mathrm{Aut_{mon}}(H\text{-mod})$.

It is easy to see that the case $\sigma = 1$ reduces to the elements (and the proof) in $\mathcal{V}$.

Proof. We denote the modified coaction by lower indices $z \mapsto z_{(-1)} \otimes z_{(0)}$. The relevant property of its definition is that $Z \mapsto {}^{\sigma^{-1}\circ f}_{f^{-1}}Z$ is a braided category equivalence which coincides with F on the level of modules. More formally $z_{(-1)} \cdot_F w = z^{(-1)}.w$. Using this property the proof works automatically as in the previous section:

The half-braiding (with modified coaction and action, but unmodified action on M!)

$$_FZ \otimes M \to M \otimes {}^{\sigma^{-1}\circ f}_{f^{-1}}Z$$

$$z \otimes m \mapsto z_{(-1)}.m \otimes z_{(0)}$$

gives clearly a natural isomorphism of H-modules, since we can write it as a braiding of ${}^{\sigma^{-1}\circ f}_{f^{-1}}Z \otimes {}_FM'$ with $M' = {}_{F^{-1}}M$.
Then we check the coherence conditions using the relevant property:

$$\begin{aligned}
&{}_FZ \otimes ({}_FW \otimes M) \to {}_FW \otimes ({}_FZ \otimes M) \to {}_FW \otimes (M \otimes {}^{\sigma^{-1}\circ f}_{f^{-1}}Z) \\
&z \otimes w \otimes m \ \mapsto\ z^{(-1)}.w \otimes z^{(0)} \otimes m \ \mapsto\ z^{(-1)}.w \otimes (z^{(0)})_{(-1)}.m \otimes (z^{(0)})_{(0)} \\
&{}_FZ \otimes ({}_FW \otimes M) \to ({}_FW \otimes M) \otimes {}^{\sigma^{-1}\circ f}_{f^{-1}}Z \\
&z \otimes w \otimes m \ \mapsto\ z_{(-1)}.({}_Fw \otimes m) \otimes z^{(0)} \ =\ (z_{(-1)})^{(1)}.{}_F\ w \otimes (z_{(-1)})^{(2)}.m \otimes z^{(0)}
\end{aligned}$$

as well as the more trivial relation

$$\begin{aligned}
&{}_FZ \otimes (M \otimes W) \stackrel{=}{\to} ({}_FZ \otimes M) \otimes W \to (M \otimes {}^{\sigma^{-1}\circ f}_{f^{-1}}Z) \otimes W \to (M \otimes W) \otimes {}^{\sigma^{-1}\circ f}_{f^{-1}}Z \\
&z \otimes m \otimes w \ \mapsto\ z \otimes m \otimes w \ \mapsto\ z_{(-1)}.m \otimes z_{(0)} \otimes w \ \mapsto\ z_{(-2)}.m \otimes z_{(-1)}.w \otimes z_{(0)} \\
&{}_FZ \otimes (M \otimes {}_FW) \to (M \otimes {}_FW) \otimes {}^{v^{-1}}_{v}Z \\
&z \otimes m \otimes w \ \mapsto\ z_{(-1)}.(m \otimes w) \otimes z_{(0)} \ =\ (z_{(-1)})^{(1)}.m \otimes (z_{(-1)})^{(2)}.w \otimes z^{(0)} \quad \blacksquare
\end{aligned}$$

3.4 $\mathcal{EV}$ induced from H^*-mod

Since $\mathcal{Z}(H\text{-mod}) \cong \mathcal{Z}(H^*\text{-mod})$ we may as well induce up from $\mathrm{Aut}_{\mathrm{mon}}(H^*\text{-mod})$, which is in general not related to $\mathrm{Aut}_{\mathrm{mon}}(H\text{-mod})$ - except the common subgroup $\mathrm{Aut}_{\mathrm{Hopf}}(H) \cong \mathrm{Aut}_{\mathrm{Hopf}}(H^*)$. Here by definition $F \in \mathrm{Aut}_{\mathrm{mon}}(H^*\text{-mod})$ induces the H^*-mod-bimodule category ${}_F(H^*\text{-mod})$ and the image of F under the ENOM functor in $\mathcal{Z}(H\text{-mod}) \cong \mathcal{Z}(H^*\text{-mod})$ is dual to the last section. However, it is not clear what the H-mod-bimodule category associated to F is; this is clarified by:

Lemma 3.5. *Let $F \in \mathrm{Eq}_{\mathrm{mon}}(L^*\text{-mod}, H^*\text{-mod})$ and consider again F^{-1}, which we write as cotensoring with a L-H-Bigalois object $R = {}_fH_\sigma$ with $\sigma \in Z^2(H,\mathbb{C})$ and $f : {}_\sigma H_{\sigma^{-1}} \to L$. We already know that (dually) the induction image of F is by definition the L^*-H^*-bimodule category ${}_F(L^*\text{-mod})$ and this gives under the ENOM functor rise to the functor in $\mathrm{Eq}_{\mathrm{br}}(DL\text{-mod}, DH\text{-mod})$ given on objects by $\Phi(\mathrm{Ind}(F)) : Z \mapsto {}_{\sigma^{-1}\circ f}^{f^{-1}}Z$ and with the monoidal structure of F.*
We claim that this braided equivalence coincides with the image of the ENOM functor of the following invertible exact L-H-bimodule category: Let $\mathcal{M} = R\text{-mod}$ as $\mathbb{C}$-linear category. The left and right coaction

$$R \longrightarrow L \otimes R \qquad R \longrightarrow R \otimes H$$

give by pull-back module category actions of L-mod and H-mod on R-mod.

In particular this defines a subgroup $\mathcal{EV} \subset \mathrm{BrPic}(H\text{-mod})$ which is the homomorphic image of the group $\mathrm{Aut}_{\mathrm{mon}}(H\text{-mod})$.[3]

Proof. Let M be an R-module. To prove our formula for $\Phi(\mathcal{M})$ we need to guess a natural transformation:

$$Z \otimes M \to M \otimes {}_{\sigma^{-1}\circ f}^{f^{-1}}Z$$

$$z \otimes m \mapsto \iota(z_{(-1)}).m \otimes z_{(0)}$$

where we denote the F-modified coaction by lower indices $z_{(-1)} \otimes z_{(0)} \in H \otimes {}^FZ$ and the right-H-colinear cleaving identification map $\iota : H \cong H_\sigma$. To prove that this is indeed a natural transformation we need to check that it is an R-module map (it is clearly natural and bijective), so we act with some $\iota(H) \in R$ and wish to prove:

$$\iota((\iota(h)^{(-1)}.z)_{(-1)}).\iota(h)^{(0)}.m \otimes (\iota(h)^{(-1)}.z)_{(0)}$$
$$\stackrel{?}{=} \iota(h)^{(0)}.\iota(z_{(-1)}).m \otimes \iota(h)^{(1)}._{\sigma^{-1}\circ f}z_{(0)}$$

On the right hand side we use the right H-colinearity of ι, on the left hand side the left L-coaction on R via f. Then we use that by definition $\iota(a)\iota(b) = \sigma(a^{(1)}, b^{(1)})$ $\iota(a^{(2)}b^{(2)})$:

$$\sigma((f(h^{(1)}).z)_{(-2)}, h^{(2)})\ \iota((f(h^{(1)}).z)_{(-1)} \cdot h^{(3)}).m \otimes (f(h^{(1)}).z)_{(0)}$$
$$\stackrel{?}{=} \sigma(h^{(1)}, z_{(-2)})\ \iota(h^{(2)} \cdot z_{(-1)}).m \otimes h^{(3)}._{\sigma^{-1}\circ f}z_{(0)}$$

[3] This subgroup of BrPic has been considered first in a different approach of [FMM14]; here we describe it as induction functor and give its image under the ENOM functor.

To prove this relation is true the main issue is to simplify the expression $(f(h^{(1)}).z)_{(-1)}$ using the Yetter-Drinfeld-condition relation action and coaction, but since we have lower-index (i.e. F-modified coaction) we need to also use the modified action, which we obtain by adding and subtracting an appropriate cocycle. The overall calculation is:

$$
\begin{aligned}
&\sigma((f(h^{(1)}).z)_{(-2)},h^{(2)})\ \iota((f(h^{(1)}).z)_{(-1)}\cdot h^{(3)}).m\otimes (f(h^{(1)}).z)_{(0)}\\
&=\underline{\sigma(h^{(1)},z_{(-2)})\sigma^{-1}(h^{(2)},z_{(-1)})}\ \sigma((f(h^{(1)}).z)_{(-2)},h^{(2)})\ \iota((f(h^{(1)}).z)_{(-1)}\cdot h^{(3)}).\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m\otimes (f(h^{(1)}).z)_{(0)}\\
&=\sigma(h^{(1)},z_{(-1)})\ \iota((h^{(2)}\cdot_{\sigma^{-1}\circ f}z_{(0)})_{(-1)}\cdot h^{(3)}).m\otimes (h^{(2)}\cdot_{\sigma^{-1}\circ f}z_{(0)})_{(0)}\\
&=\sigma(h^{(1)},z_{(-2)})\ \iota(h^{(2)}z_{(-1)}\underline{S(h^{(4)})\cdot h^{(5)}}).m\otimes h^{(3)}\cdot_{\sigma^{-1}\circ f}z_{(0)}
\end{aligned}
$$

Having established a natural transformation we check once again the coherence conditions. We have equalities as follows for all $W\in L$-mod:

$$
\begin{array}{l}
Z\otimes(W\otimes M)\to W\otimes(Z\otimes M)\to W\otimes(M\otimes {}^{f^{-1}}_{\sigma^{-1}\circ f}Z)\xrightarrow{=}(W\otimes M)\otimes {}^{f^{-1}}_{\sigma^{-1}\circ f}Z\\
z\otimes w\otimes m \quad\mapsto\quad z^{(-1)}.w\otimes z^{(0)}\otimes m \quad\mapsto\quad z^{(-2)}.w\otimes\iota(z_{(-1)}).m\otimes z^{(0)}\\
Z\otimes(W\otimes M)\to(W\otimes M)\otimes {}^{f^{-1}}_{\sigma^{-1}\circ f}Z\\
z\otimes w\otimes m \quad\mapsto\quad \iota(z_{(-1)}).(w\otimes m)\otimes z^{(0)} \quad=\quad f(z_{(-2)}).w\otimes\iota(z_{(-1)}).m\otimes z^{(0)}
\end{array}
$$

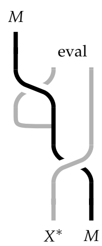
as well as for all $W\in H$-mod:

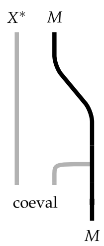
$$
\begin{array}{l}
Z\otimes(M\otimes W)\xrightarrow{=}(Z\otimes M)\otimes W\to(M\otimes {}^{f^{-1}}_{\sigma^{-1}\circ f}Z)\otimes W\to(M\otimes W)\otimes {}^{f^{-1}}_{\sigma^{-1}\circ f}Z\\
z\otimes m\otimes w\mapsto z\otimes m\otimes w\mapsto\iota(z_{(-1)}).m\otimes z_{(0)}\otimes w\mapsto\iota(z_{(-2)}).m\otimes z_{(-1)}.w\otimes z^{(0)}
\end{array}
$$

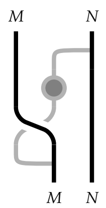
$$
\begin{array}{l}
Z\otimes(M\otimes W)\to(M\otimes W)\otimes {}^{f^{-1}}_{\sigma^{-1}\circ f}Z\\
z\otimes m\otimes w\mapsto\iota(z_{(-1)}).(m\otimes w)\otimes z_{(0)}=\iota(z_{(-1)})^{(0)}.m\otimes\iota(z_{(-1)})^{(1)}.w\otimes z^{(0)}
\end{array}
$$
■

3.5 $\mathcal{R}$ the partial dualizations

We now introduce an additional subset of elements in BrPic which are **not** induced from monoidal equivalences, but constructed from the braided equivalence side of the ENOM functor. We will make thorough use of the second category equivalence Ω_X : DX-mod $\xrightarrow{\sim}$ DX^*-mod [BLS15] Thm. 3.20 for any Hopf algebra X inside a *braided base category* $\mathcal{X}$. The new X^*-action and -coaction on $\Omega(M)$ is as follows:

$\Omega(M,\rho_M,\delta_M) = \Bigg(M,$ 

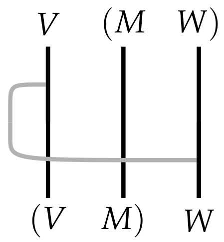
, 

$\Bigg)$, $\quad\Omega_2(M,N) =$ 

.

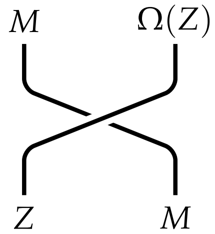
with nontrivial monoidal structure Ω_2 involving the inverse antipode.

Lemma 3.6. *The following* X-mod-X^*-mod *bimodule category fulfills the defining property of the preimage under the ENOM-functor of* Ω*; it is* **not** *necessarily invertible:*
As abelian category $\mathcal{M} = \mathcal{X}$ *with trivial module category structure on either side (forgetting the* X, X^**-module structures) but with nontrivial bimodule category structure* $(V \otimes M) \otimes W \longrightarrow V \otimes (M \otimes W)$ *given by*

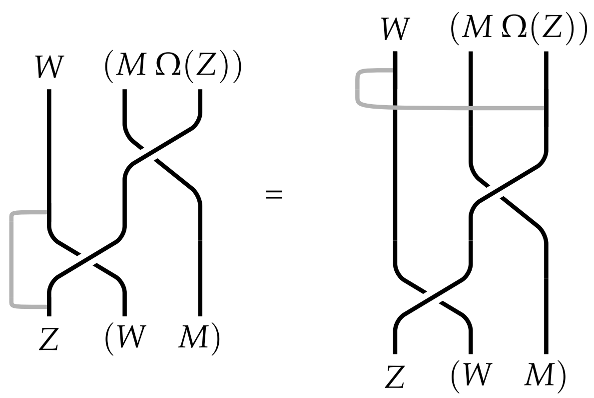


where $V \in X$-mod, $W \in X^*$-mod, $M \in \mathcal{X}$.

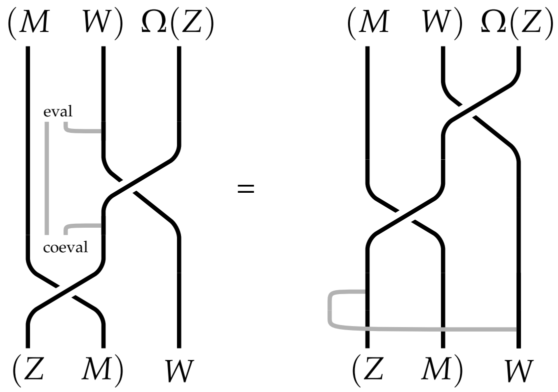
Proof. As natural equivalence $Z \otimes M \longrightarrow M \otimes \Omega(Z)$ we choose the braiding in the category $\mathcal{X}$, where $Z \in DX$-mod inside $\mathcal{X}$ and as objects in $\mathcal{X}$ we have $Z = \Omega(Z)$:



We check the coherence conditions that we have equalities of the following morphisms



as well as of the following morphisms involving the modified action on $\Omega(Z)$:



■

Suppose now we have a projection $\pi : H \to A$ which means we can write $H = K \rtimes A$ where the coinvariants $K = H^{\mathrm{coin}\pi}$ is a Hopf algebra in the braided category DA-mod. Then we can construct two Hopf algebras:

$$r'(H) := K^* \rtimes A \qquad r(H) := \Omega_A(K) \rtimes A^*$$

and category equivalences DH-mod $\to$ D $r(H)$-mod and DH-mod $\to$ D $r'(H)$-mod.

Our previous lemma applied to $\mathcal{X}$ = DA-mod gives a $\mathcal{Z}(H\text{-mod})$-$\mathcal{Z}(r(H)\text{-mod})$-bimodule category $\mathcal{M}' = \mathcal{X}$, which is in general not invertible. But there is an invertible sub-bimodule category stable under the structure maps, namely A-mod (M appears only as undercrossing). This shows for the first part:

Corollary 3.7. *The element* DH-mod $\to$ D $r'(H)$-mod *is the image under the ENOM functor of the module category* $\mathcal{M} = A$-mod *with module structure given by the tensor product* $\otimes_{\mathbb{C}}$ *in* A-mod, *forgetting K- resp. K^*-module structure, and a nontrivial bimodule category structure given by the previous lemma using the pairing between K, K^*.*

Similarly one constructs vice-versa:

Corollary 3.8. *The element* DH-mod $\to$ D $r(H)$-mod *is the image under the ENOM functor of the module category* $\mathcal{M} = (K\text{-mod})_{\Omega_A^{-1}}$ *with module structure given by the tensor product* $\otimes_{\mathbb{C}}$ *in* K-mod, *for the right side after precomposing with* Ω_A^{-1}, *forgetting A- resp. A^*-module structure, and a nontrivial bimodule category structure given by the previous lemma using the pairing between A, A^*.*

Example 3.9. *The extremal case is a full dualization r' with $A = 1$ or equivalently r with $K = 1$. In this case we obtain the (in this case invertible)* H-mod-$H*$-mod-*bimodule category* $\mathcal{M}$ = Vect *from the Lemma with bimodule category structure given by the pairing of H and H^*.*

Very similar formulae construct dually H^*-mod-$r(H)^*$-mod-bimodule categories.

Of particular interest are cases where $r'(H) \cong H$ resp. $r(H) \cong H$ which is the case for self-dual Yetter-Drinfeld Hopf algebra K resp. self-dual Hopf algebra A and Ω-self-dual Yetter-Drinfeld module K. For these cases partial dualizations give rise to elements in BrPic(H-mod).

Remark 3.10. *The bimodule categories should be equivalent to something like* $\mathcal{M} := (K \otimes K^*)_\lambda \rtimes A\text{-mod}$ *resp.* $\mathcal{M} := K \rtimes (A \otimes A^*)_\lambda\text{-mod}$ *for the Bigalois object* $(K \otimes K^*)_\lambda$ *given by the evaluation pairing* $K \otimes K^* \to \mathbb{C}$ *- and with trivial bimodule category structure.*

Remark 3.11. *Partial dualizations can be used to conjugate different forgetful functors* $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ *and hence many different induction functors from* $\mathcal{C}$*. Our approach can be seen as the hope that this exhausts a large amount of different forgetful functors.*

Remark 3.12. *An important fact is that partial dualizations in our (narrow) definition depend on the precise Hopf algebra i.e. is not invariant under monoidal representation category equivalence. This can lead to the effect that* $H\text{-mod} \cong H'\text{-mod}$ *where* H *has a semi direct decomposition while* H' *has not, but still both centers carry the respective partial actualization. This can be either avoided by reformulating the above construction categorically (both categories have a semi direct-product-like decomposition) or by accepting, that partial dualizations can arise from any monoidally equivalent presentation. Compare the group example 4.5 below.*

4 Examples

4.1 Groups

We discuss all module categories and braided equivalences for the case $H = \mathbb{C}^G$ with G a finite group i.e. $H\text{-mod} = \mathrm{Vect}_G$. The module categories can be in this case be check against the explicit description:

Lemma 4.1 ([Dav10] Cor. 3.6.3 [NR14] Prop. 5.2)**.** *Invertible bimodule categories over* Vect_G *are in bijection with pairs* (B, η) *where* $U \subset G \times G^{op}$ *a subgroup and* $\eta \in H^2(B, \mathbb{C}^\times)$ *such that*

- $U(G \times 1) = U(1 \times G^{op}) = G \times G^{op}$
- $U_1 = U \cap (G \times 1)$ *and* $U_2 = U \cap (1 \times G^{op})$ *are abelian.*
- $\eta(h_1, h_2)\eta^{-1}(h_2, h_1)$ *is a nondegenerate pairing* $U_1 \times U_2 \to \mathbb{C}^\times$

In this case $\mathcal{M} = \mathrm{Vect}_{(G \times G^{op})/U}$ *is the* $\mathbb{C}$*-linear category of vector spaces graded by* U*-cosets [O03]. The Lemma holds similarly for invertible* Vect'_G*-*Vect_G*-bimodule categories.*

The braided equivalences of the center can be described very explicitly using the following well-known description:

Lemma 4.2. $\mathcal{Z}(\mathrm{Vect}_G)$ *is semisimple and the simple objects are* $\mathcal{O}_g^\chi$ *where* $[g] \subset G$ *is a conjugacy class and* χ *an irreducible character of the centralizer* $\mathrm{Cent}(g)$

4.1.1 We discuss the group $\mathcal{V}$

Let $v : G' \to G$ be a group isomorphism. The corresponding invertible $\mathrm{Vect}_{G'}$-Vect_G-bimodule category is given ${}_v(\mathrm{Vect}_G)$. This corresponds to the choice $G \cong U \subset G' \times G^{op}$ the graph of v and $U_1 = U_2 = \{1\}, \eta = 1$.

The ENOM functor assigns to this the following category equivalence of the centers:

$$\mathcal{O}_g^\chi \longmapsto \mathcal{O}_{v(g)}^{\chi(v^{-1}(\bullet))}$$

4.1.2 We discuss the group $\mathcal{BV}$

Let $F : \mathrm{Vect}_{G'} \to \mathrm{Vect}_G$ a monoidal equivalence: It is given on objects by a group isomorphism $v : G' \to G$ and the monoidal structure by a 2-cocycle $\mu \in H^2(G', \mathbb{C}^\times)$, which defines a Bigalois object $\mathbb{C}_\sigma[G']$ with left coaction composed with v. Respective, the monoidal equivalence F^{-1} is given by $f = v^{-1}$ and the 2-cocycle $\sigma(g,h) = \mu^{-1}(v^{-1}(g), v^{-1}(h))$. The invertible $\mathrm{Vect}_{G'}$-Vect_G-bimodule category is again given by definition by $\mathcal{M} = {}_F(\mathrm{Vect}_G)$, which corresponds again to the choice $G \cong U \subset G' \times G^{op}$ the graph of v and $U_1 = U_2 = \{1\}$ but now includes nontrivial η.

The ENOM functor assigns to this the following category equivalence of the centers

$$\mathcal{O}_g^\chi \longmapsto \mathcal{O}_{v(g)}^{\chi(v^{-1}(\bullet))\frac{\mu(v^{-1}(\bullet),g)}{\mu(g,v^{-1}(\bullet))}}$$

with nontrivial monoidal structure given by μ on the coaction.

Remark 4.3. *It is informative to also look at the bimodule categories from the dual perspective of the subgroup $\mathcal{EV}$ of* $\mathrm{Rep}(G')$-$\mathrm{Rep}(G)$*-bimodule categories, where we obtain* $\mathcal{M} = {}_v(\mathbb{C}_\sigma[G]\text{-mod})$.

4.1.3 We discuss the group $\mathcal{EV}$

The monoidal equivalences $\mathrm{Rep}(G') \to H$-mod are given by Bigalois objects ${}_f R_{\mathrm{Rep}(G')}$ where H is the Doi twist of $\mathbb{C}[G']$ and $f \in \mathrm{Aut}_{\mathrm{Hopf}}(H)$. By [Dav01] the Galois objects are given by pairs (S, η) where S is a subgroup of G' and $\eta \in Z^2(S, \mathbb{C}^\times)$ nondegenerate; then the Galois object is an induced representation $R = {}_f(\mathbb{C}^{G'} \otimes_S \mathbb{C}_\eta[S])$. The Hopf algebra H, being the Doi twist of $\mathbb{C}[G']$, is fixed up to isomorphism f by the choice S, η. In particular obtaining again a group algebra $H = \mathbb{C}[G]$ is equivalent to S being normal abelian and the cohomology class $[\eta]$ being conjugation invariant. The isomorphism type of G is a certain extension $\hat{S} \mapsto G \to G'/S$ determined by η.
In particular it is sufficient (but not necessary) to achieve $G' \cong G$ that η is conjugation invariant as a 2-cocycle. This additional condition is (see e.g. [LP15b]) equivalent to so-called *laziness*. In particular the extension G is isomorphic to G'

by the trivial isomorphism (identity on G'/S and the nondegenerate form defined by α identifying $S \cong \hat{S}$) and the additional morphism f is actually a Hopf algebra isomorphism induced by a group isomorphism $v : G \to G'$. In this case we may assume $v = \mathrm{id}$ without loss of generality and realize $v \in \mathcal{V}$ as above.

The corresponding invertible $\mathrm{Rep}(G')$-H-mod-bimodule category induced by F has been shown to be R-mod. To link this to the description in Lemma 4.1 we observe that since $\mathbb{C}_\eta[S]$ is by assumption a simple algebra, we have a category equivalence $R\text{-mod} \cong \mathbb{C}^{G'/S}\text{-mod} = \mathrm{Vect}_{G'/S}$. In the lazy case it is easy to check that the following data in the Lemma describes our bimodule category. Identifying $G'/S = G/\hat{S}$, denoting the quotient map by π and identifying $G^{op} \cong G$ via inverse we take

$$U = \{(g', g) \in G' \times G \mid \pi(g') = v(\pi(g)^{-1})\} \qquad (S \times \hat{S}) \to U \to G'/S$$

In particular $U_1 = U \cap (G' \times 1) = S$ and $U_2 = U \cap (1 \times G) = \hat{S}$. There is a diagonal quotient $U \to S \times G'/S$, pulling back the 2-cocycle η gives a 2-cocycle on U which is nondegenerate on $S \times S$, $\hat{S} \times \hat{S}$ and $S \times \hat{S}$ as necessary.

The ENOM-functor assigns to this the category equivalence of the centers obtained above. It can be worked out for a given $\mathcal{O}_g^\chi$ by decomposing the induced representation according to the modified coaction, and the monoidal structure is given by that of F, but there is no convenient group-theoretic formula for this. We work out the following case:

Lemma 4.4. *The formula from Section 3.4 reduces for a lazy* [4] *monoidal equivalence* $\mathrm{Rep}(G) \to \mathrm{Rep}(G)$ *given by* $S, \eta, v = \mathrm{id}$ *as follows on objects* $\mathcal{O}_1^V$*:*
Let the restriction of the irreducible G-representation V to S (abelian, normal) be decomposed according to Clifford theory into irreducible representations $V = \bigoplus_{i=1}^t E_i \otimes V_i$, *where conjugation of G acts transitively on the* 1*-dimensional S-representations* $V_i = \mathbb{C}_{\chi_i}$ *and all the multiplicity spaces E (trivial S-representations) are of same dimension. Use the nondegeneracy of* η *to identify* $\hat{S} \cong S$ *to get a G-conjugacy class* $[s_i]$ *by* $\chi_i(r) = \frac{\eta(r,s_i)}{\eta(s_i,r)} =: \langle r, s_i\rangle$. *Then the centralizer of any* s_i *is the corresponding inertia subgroup* $I_i \subset G$ *fixing* χ_i *and hence acting on* E_i. *Then we claim*

$$\mathcal{O}_1^V \longmapsto \mathcal{O}_{[s_i]}^{E_i \otimes V_i}$$

Proof. Because the lazy case allows without restriction in generality to choose $v = \mathrm{id}$ we have $F = \mathrm{id}$ on objects. Thus as representations $\Phi(\mathrm{Ind}(F))\mathcal{O}_1^V = \mathcal{O}_1^V = V$ and as $\Phi(\mathrm{Ind}(F))\mathcal{O}_g^\chi = {}_{\sigma^{-1}}\mathcal{O}_1^V$. So it remains to determine the σ^{-1}-twisted coaction, which is the σ^{-1}−twisted $\mathbb{C}^G$-action. We need to reformulate also

[4]This "lazy" here is much less critical than in [LP15b], where we classify lazy *braided* autoequivalences of the Drinfeld center. In the present approach it is merely a technical inconvenience that we have good explicit formulae only for (still non-lazy) induction from a lazy *monoidal* autoequivalence of $\mathrm{Rep}(G)$. Does the given group-theoretic formula continue to hold for nonlazy monoidal equivalences?

G-action as $\mathbb{C}^G$-coaction via $v \mapsto \sum_g e_g \otimes g.v$. We decompose $V = \bigoplus_{i=1}^t E_i \otimes V_i$ as asserted and check the twisted action of the projector e_{s_i} for s_i defined as asserted on $v \in E_j \otimes V_j$:

$$e_{s_i} \cdot_\sigma v = \sum_{g,h \in G} \sigma^{-1}(e_{s_i}^{(1)}, e_g) \cdot (h.e_{s_i}^{(2)}.g.v) \cdot \sigma(e_h, e_{s_i}^{(3)})$$

We now use our formula in [LP15a]:

$$\sigma(e_a, e_b) = \frac{\delta_{a,b \in S}}{|S|^2} \sum_{t,t' \in S} \eta(t,t')\langle t, a\rangle\langle t', b\rangle$$

and the fact that v is in grade $1 \in G$ to evaluate our expression. Then we exploit the fact that for a nondegenerate pairing on an abelian group holds $\frac{1}{|S|}\sum_{s's''=s}\langle x, s\rangle\langle s, y\rangle = \delta_{x,y}$ and hence $\frac{1}{|S|}\sum_{s's''=s}\langle x, s'\rangle\langle y, s''\rangle = \delta_{x,y}\langle x, s\rangle$ and that any $r \in S$ acts on v by the 1-dimensional character $\chi_j(r) = \langle r, s_j\rangle$:

$$\begin{aligned}
&= \sum_{g,h \in S, s_i' s_i'' = s_i} \frac{1}{|S|^2} \sum_{t,t' \in S} \eta^{-1}(t,t')\langle t, s_i'\rangle\langle t', g\rangle \cdot (hg.v) \\
&\cdot \frac{1}{|S|^2} \sum_{t'',t''' \in S} \eta(t'',t''')\langle t'', h\rangle\langle t''', s_i''\rangle \\
&= \frac{1}{|S|^2} \sum_{r \in S} \sum_{t,t' \in S} \eta^{-1}(t,t')\langle t, s_i\rangle\langle t', r\rangle \cdot (r.v) \cdot \eta(t',t) \\
&= \frac{1}{|S|^2} \sum_{r \in S} \sum_{t,t' \in S} \langle t', t\rangle\langle t, s_i\rangle\langle t', r\rangle \chi_j(r) \cdot v \\
&= \frac{1}{|S|} \sum_{r \in S} \langle s_i, r\rangle\langle r, s_j\rangle \cdot v = \delta_{i,j} \cdot v
\end{aligned}$$

This shows that $E_j \otimes V_j$ has now a coaction grade s_j as asserted. ∎

Example 4.5. *We also wish to give an example of induction for a non-lazy autoequivalence. Consider* $\mathrm{Sp}_{2n}(\mathbb{F}_2)$ *acting on* $S := \mathbb{Z}_2^{2n}$ *with invariant symplectic form* $\langle \bullet, \bullet \rangle$. *There is a unique nondegenerate cohomology class* $[\eta] \in H^2(S, \mathbb{C}^\times)$ *associated to the symplectic form, which is hence invariant, however no representing* 2*-cocycle is not invariant. It is known ([Dav01] Exm. 7.6) that this relates the semidirect product* $G' = S \rtimes \mathrm{Sp}_{2n}(\mathbb{F}_2)$ *and the nontrivial extension* $G = S.\mathrm{Sp}_{2n}(\mathbb{F}_2)$ *via the (then non-lazy) Bigalois object associated to* S, η.

Of particular interest is the case $n = 1$ *where both groups are isomorphic* $G \cong G' = \mathbb{S}_4$ *but still* v *interchanges the conjugacy classes* $[(12)]$ *and* $[(1234)]$ *(with both* 6 *elements) and is hence no Hopf algebra isomorphism. The non-lazy monoidal autoequivalence* F *of* $\mathbb{S}_4$ *interchanges the two* 3*-dimensional representations* $\chi_3, \chi_3 \otimes \mathrm{sgn}$*and is visible as symmetry in the character table. The induction of this* F *would yield a bimodule category* $\mathcal{M} = R$-mod *which would be described by a* $U \subset \mathbb{S}_4 \times \mathbb{S}_4$ *containing tuples such as* $((12), (1234))$.

4.1.4 We discuss the elements $\mathcal{R}$

We first observe that $\mathbb{C}^G$ seems to have no interesting semidirect decompositions, because of contravariance this would imply a left-split sequence of groups. On the other hand assume $G = N \rtimes Q$, then $H^* = \mathbb{C}[G] = \mathbb{C}[N] \rtimes \mathbb{C}[Q]$. Next we observe that partial dualization $r(\mathbb{C}[G])$ can never return a group ring (except for a direct product, for which it coincides with r'), because the coaction of A on K is trivial, so to be self-dual the action would have to be trivial as well resulting in a direct product.

So we consider partial dualization r' on $H^* = \mathbb{C}[G] = \mathbb{C}[N] \rtimes \mathbb{C}[Q]$ where N is an abelian group and a self-dual Q-module. We have already derived in [LP15b] a formula for the action of r' as a braided equivalence of $\mathcal{Z}(\mathrm{Vect}_G)$ on objects $\mathcal{O}_1^\chi$.

Similar to $\mathcal{EV}$, let the restriction of the irreducible G-representation V to N (abelian, normal) be decomposed according to Clifford theory into irreducible representations $V = \bigoplus_{i=1}^t E_i \otimes V_i$, use the paring to map the 1-dimensional representations $V_i \in N^*$ to a a G-conjugacy class $[s_i] \subset N$. Then the centralizer of any s_i is the corresponding inertia subgroup $I_i \subset G$ fixing V_i and hence acting on E_i. Then we claim

$$\mathcal{O}_1^V \longmapsto \mathcal{O}_{[s_i]}^{E_i}$$

(the only difference is no V_i appears in the centralizer action)

Also the corresponding module category Vect_Q is described in striking similarity to $\mathcal{EV}$ by the same subgroup

$$U = \{(g', g) \in G \times G \mid \pi(g') = \pi(g)^{-1}\} \qquad (N \times \hat{N}) \to U \to Q$$

where $\pi : G \to Q = G/N$. But compared to $\mathcal{EV}$ the 2-cocycle is different: Consider again the diagonal quotient $N \to U \to N \times Q$ and consider the Masumoto spectral sequence

$$\begin{aligned}1 \to H^1(N \times Q, \mathbb{C}^\times) \to H^1(U, \mathbb{C}^\times) \to H^1(N, \mathbb{C}^\times) \to \\ \to H^2(N \times Q, \mathbb{C}^\times) \overset{pullback}{\longrightarrow} H^2(U, \mathbb{C}^\times)_N \overset{form}{\longrightarrow} (N \times Q) \otimes N \to \\ \to H^3(N \times Q, \mathbb{C}^\times) \to H^3(U, \mathbb{C}^\times)\end{aligned}$$

where the subindex N means cohomologically trivial if restricted to the kernel N. For $\mathcal{EV}$ we took the pullback of a 2-cocycle on N, now we should take the preimage of our nondegenerate form on $N \times N$, which becomes trivial in H^3 and is hence in the image.

4.1.5 Example: Elementary abelian groups

For $G = \mathbb{F}_p^n$ a finite vector space we know directly

$$\mathrm{Aut}_{br}(DG\text{-mod}) = O_{2n}(\mathbb{F}_p).$$

For abelian groups, all 2-cocycles over DG are lazy and the results of [BLS15] gives a product decomposition of $\mathrm{BrPic}(\mathrm{Rep}(G))$. The subgroups in question are

- $\mathcal{V} \cong \mathrm{Out}(G) = \mathrm{GL}_n(\mathbb{F}_p)$.
- $\mathcal{BV} = \mathrm{Out}(G) \ltimes (\mathbb{F}_p^n \wedge \mathbb{F}_p^n)$ the latter as an additive group.
- $\mathcal{EV} = \mathrm{Out}(G) \ltimes (\mathbb{F}_p^n \wedge \mathbb{F}_p^n)$ the latter as an additive group.
- The set $\mathcal{R}$ consists of $n+1$ equivalence classes of partial dualizations for each possible dimension d of a direct factor $\mathbb{F}_p^d \cong C \subset G$. Especially the full dualization on $C = G$ conjugates $\mathcal{BV}$ and $\mathcal{EV}$. In this case the proposed decomposition is actually a double coset decomposition, which is a variant of the Bruhat decomposition of $O_{2n}(\mathbb{F}_p)$ of type D_n.
  More precisely, our result reduces to the Bruhat decomposition of the Lie groups D_n relative to the parabolic subsystem A_{n-1}. In particular there are $n+1$ double cosets of the parabolic Weyl group $\mathbb{S}_n$, accounting for the $n+1$ non-isomorphic partial dualizations on subgroups $\mathbb{Z}_p^k$ for $k = 0, \dots, n$.

4.1.6 Examples for nonabelian groups

Let G be a nonabelian simple group, then

- $\mathcal{V} \cong \mathrm{Out}(G)$.
- $\mathcal{BV} = \mathrm{Out}(G) \ltimes H^2(G, \mathbb{C}^\times)$ (the latter as an additive group).
- $\mathcal{EV} = \mathcal{V}$ as there are no nontrivial abelian normal subgroups.
- The set $\mathcal{R}$ is empty as there are no nontrivial semidirect factors.

Let $G = \mathbb{S}_3$, then $\mathrm{BrPic}(G) = \mathbb{Z}_2$ (see already [NR14]), more precisely:

- $\mathcal{V} \cong \mathrm{Out}(G) = 1$.
- $\mathcal{BV} = \mathcal{V} \ltimes H^2(G, \mathbb{C}^\times) = 1$.
- $\mathcal{EV} = \mathcal{V} = 1$ since the only nontrivial abelian normal subgroup is cyclic and has hence no nontrivial cocycles.
- The set $\mathcal{R}$ contains a nontrivial reflection r' on the normal subgroup $\langle(123)\rangle$. As an element in $\mathrm{Aut}_{\mathrm{br}}(D\mathbb{S}_3\text{-mod})$ it permutes the objects as follows[5]:

$$\mathcal{O}_1^{triv},\ \mathcal{O}_1^{sgn},\ \mathcal{O}_1^{ref},\ \mathcal{O}_{(12)}^{+},\ \mathcal{O}_{(12)}^{-},\ \mathcal{O}_{(123)}^{1},\ \mathcal{O}_{(123)}^{\zeta},\ \mathcal{O}_{(123)}^{\zeta^2}$$

$$\longrightarrow\ \mathcal{O}_1^{triv},\ \mathcal{O}_1^{sgn},\ \mathcal{O}_{(123)}^{1},\ \mathcal{O}_{(12)}^{+},\ \mathcal{O}_{(12)}^{-},\ \mathcal{O}_1^{ref},\ \mathcal{O}_{(123)}^{\zeta'},\ \mathcal{O}_{(123)}^{\zeta'^2}$$

[5] $triv, sgn, ref$ the irreducible representations of $\mathbb{S}_3$ and $\pm$ the two 1-dimensional representations of the centralizer $\mathbb{Z}_2$ of (12) and $1, \zeta, \zeta^2$ the three 1-dimensional representations of the centralizer $\mathbb{Z}_3$ of (123). Whether ζ, ζ' are the same roots of unity depends on the right choice of the pairing on $\mathbb{Z}_3$.

As an invertible bimodule category $\mathcal{M}$ is is the abelian category $\mathbb{Z}_2$-mod with highly nontrivial bimodule category constraint.

We remark already at this point, that the associated group-theoretical $\mathbb{Z}_2$-extension of $\mathbb{S}_3$-mod is the fusion category $(\hat{\mathfrak{sl}_2})_4$-mod which decomposes as an abelian category to $\mathbb{S}_3$-mod $\oplus$ $\mathbb{S}_3/\mathbb{Z}_3$-mod with $3+2$ simple objects.

More examples are discussed in [BLS15] Sec. 6.

4.2 Taft algebra

As a example which is not of group type, we now discuss the Taft algebra, for which the description of the Brauer Picard group can be checked against the list of bimodule categories in [FMM14] (although there is unfortunately no description of the Brauer Picard group):

Definition 4.6 (Taft algebra)**.** *Let q be a primitive ℓ-th root of unity prime) and let T_q be the Hopf algebra generated by g, x with relations and coproduct as follows:*

$$g^\ell = 1 \qquad x^\ell = 0 \qquad xg = qgx$$

$$\Delta(g) = g \otimes g \qquad \Delta(x) = g \otimes x + x \otimes 1$$

T_q has dimension ℓ^2 and decomposes into a Radford biproduct product $T_q = K \rtimes A = \mathbb{C}[x] \rtimes \mathbb{C}[\mathbb{Z}_\ell]$ where the A-action and -coaction on K is given by $g.x = qx$ and $\delta(x) = g \otimes x$. It is a self-dual Hopf algebra via the linear forms $g^* : g, x \mapsto q, 0$ and $x^* : g, x \mapsto 1, 1$.

The Taft algebra appears naturally as the Borel part of the small quantum groups $u_{q^{-1/2}}(\mathfrak{sl}_2)^+$. The Drinfel'd double $\mathrm{D}T_q$ is generated by two isomorphic Taft algebras g, x and g^*, x^* with relations

$$x^*g = q^{-1}gx^* \qquad xg^* = q^{-1}g^*x \qquad xx^* - qx^*x = \frac{g - g^*}{q - q^{-1}}$$

It has the full quantum group as quotient by the central element $gg^* - 1$.

We recall some well-known properties of this Hopf algebra:

Fact 4.7.

$$\mathrm{Aut}_{\mathrm{Hopf}}(T_q) \cong \mathbb{C}^\times \qquad \mathrm{Out}_{\mathrm{Hopf}}(H) \cong \mathbb{C}^\times / \langle q \rangle$$

where $c \in \mathbb{C}^\times$ acts by $g, x \mapsto g, cx$. This is because the skew-primitive x is determined uniquely up to scalar and the grouplike g is determined by x; on the other hand the asserted map is a Hopf algebra automorphism. Conjugation by g gives the inner automorphism $c = q$, so $\mathrm{Out}_{\mathrm{Hopf}}(H) \cong \mathbb{C}^\times / \langle q \rangle$

Fact 4.8. *All irreducible T_q-modules are of the form $\mathbb{C}_\chi$ and all indecomposable modules are of the form $\mathbb{C}_\chi[x]/x^d$ for $\chi \in \hat{\mathbb{Z}}_p$ any character of the group ring and $0 < d < \ell$*

Proof. Let V be a finite-dimensional T_q-module. Let v be a g.-Eigenvector to some Eigenvalue $\chi(g)$ defining a character of $\mathbb{Z}_p$. The relation $gxg^{-1} = qx$ shows that $x^k.v$ is a g.-Eigenvector to the Eigenvalue $q^k\chi(g)$ and then at last $x^\ell.v = 0$. Hence the only irreducible representations are 1-dimensional $\mathbb{C}_\chi$ and all indecomposables are $\mathbb{C}_\chi[x]/x^d$ of dimension $0 < d < \ell$. Conversely, each module can be realized as a quotient of the regular representation. ∎

Fact 4.9. *There is a braided subcategory of $\mathcal{Z}(T_q\text{-mod}) = DT_q\text{-mod}$ determined by $gg^* - 1$ acting by zero, which is equivalent to the category of $u_{q^{-1/2}}(\mathfrak{sl}_2)$-mod. We denote the irreducible highest weight module by $V(\lambda)$ for weight $\lambda \in \frac{1}{2}\mathbb{N}_0$.*

We first discuss the group $\mathcal{V} \cong \mathrm{Out}_{\mathrm{Hopf}}(T_q) \cong \mathbb{C}^\times/\langle q\rangle$. The effect of this as a monoidal autoequivalence seems negligible because one easily finds a natural transformation to the trivial autoequivalence by rescaling $x^k v \mapsto c^k \cdot x^k v$. However, a *monoidal* natural transformation will return the trivial autoequivalence with a nontrivial monoidal structure. This can be easily seen for the tensor product of two 2-dimensional indecomposables, which decomposes into 1- and 3-dimensional indecomposables which are rescaled differently; the more general formula for $\mathcal{EV}$ below shows the resulting 2-cocycle systematically for $(a,b) = (c^\ell, 0)$.

The induced bimodule categories are $\mathcal{M}_c^{\mathcal{V}} := {}_c(T_q\text{-mod})$.

The ENOM functor maps this to the braided equivalence of the Drinfel'd center induced by $x, x^*, g, g^-* \mapsto cx, c^{-1}x^*, g, g^*$. Again, this is equivalent to a functor that is trivial on objects but with nontrivial monoidal structure.

To determine the group $\mathcal{EV}$ we need to know the Bigalois objects. This has been done in [Sch00] and can today be understood in the context of nontrivial lifting [M01]:

Lemma 4.10. *The right Galois objects are as follows for any choice $a \in \mathbb{C}^\times, b \in \mathbb{C}$*[6]*:*

$$R_{a,b} = \langle \tilde{g}, \tilde{x}\rangle / (\tilde{g}^\ell = a1,\ \tilde{x}^\ell = b1,\ \tilde{x}\tilde{g} = q\tilde{g}\tilde{x})$$

$$\delta(\tilde{g}) = \tilde{g} \otimes g \qquad \delta(\tilde{x}) = 1 \otimes x + \tilde{x} \otimes g$$

These all become T_q-T_q-Bigalois objects $R_{a,b}$ with the left coaction:

$$\delta(\tilde{g}) = g \otimes \tilde{g} \qquad \delta(\tilde{x}) = 1 \otimes \tilde{x} + x \otimes \tilde{g}$$

So $\mathcal{EV} \cong \mathrm{Bigal}(T_q) \cong \mathbb{C}^\times \ltimes \mathbb{C}$ and the embedding of $\mathcal{V} \cong \mathbb{C}^\times/\langle q\rangle$ goes via $c \mapsto (c^\ell, 0)$.

The induced bimodule categories are $\mathcal{M}_{a,b}^{\mathcal{EV}} := R_{a,b}$-mod. These are the $\mathcal{L}$ in [FMM14]. As a $\mathbb{C}$-linear category this is T_q-mod (for $b = 0$) as discussed in $\mathcal{V}$

[6]The right Galois objects are isomorphic for all values $b \neq 0$, but not as Bigalois objects. There are differently scaled left coactions, but the latter can be rescaled to 1 by a Bigalois isomorphism at the cost of a.

or Vect ($b \neq 0$), since in the latter case there is a unique simple module M of dimension ℓ.

The elements $\mathcal{R}$ for T_q are particularly interesting and will be generalized later:

- Since the Taft algebra is self-dual, we have the full dualization $\star \in \mathrm{Aut}_{\mathrm{br}}(\mathcal{Z}(T_q\text{-mod}))$ (i.e. r for $K = 1$ or equivalently r' for $A = 1$). It decomposes into rr' below.
- For the decomposition $T_q = \mathbb{C}[x] \rtimes \mathbb{C}[\mathbb{Z}_\ell]$ we have $K \cong K^*$ as Yetter-Drinfel'd Hopf algebra, so we have a partial dualization $r' \in \mathrm{Aut}_{\mathrm{br}}(\mathcal{Z}(T_q\text{-mod}))$. It acts on quantum group modules $V(\lambda)$ like a reflection.
- For the decomposition $T_q = \mathbb{C}[x] \rtimes \mathbb{C}[\mathbb{Z}_\ell]$ we also have $A \cong A^*$ and $K \cong \Omega(K)$, so we also have a partial dualization $r \in \mathrm{Aut}_{\mathrm{br}}(\mathcal{Z}(T_q\text{-mod}))$.

5 Applications

5.1 Quantum groups and Nichols algebras

We now discuss some applications of the previously defined general elements if applied to quantum groups.

5.1.1 $\mathrm{Aut}_{\mathrm{br}}$ of nonabelian groups and Nichols algebras

We begin with a little demonstration of the effect of our subgroups of $\mathrm{BrPic}(\mathrm{Rep}(G))$ as subgroups $\mathrm{Aut}_{\mathrm{br}}(H)$ after the ENOM functor. Namely, the Nichols algebra $\mathfrak{B}(M)$ associated to some $M \in \mathrm{D}H$-mod is a fundamental construction with a universal property. It returns e.g. the Borel part of the quantum group $U_q(\mathfrak{g})^+$ over $H = \mathbb{C}[\mathbb{Z}_\ell^{\mathrm{rank}}]$.

Thus it is as a vector space invariant under $\mathrm{Aut}_{\mathrm{br}}(\mathrm{D}H\text{-mod})$. We want to argue that this completely explains certain coincidences in dimension that appeared during the classification of finite-dimensional Nichols algebras over nonabelian groups G. Some of these cases have been known[7] and the only purpose of this section is to collect and unify the argument using our explicit results on $\mathrm{BrPic}(G)$ in the previous section.

Needless to say, this game of changing the realizing group does not reveal much information about the Nichols algebra.

- The following was the first explained coincidence in terms of Doi twist in [Ven12]:
  Let $G = \mathbb{S}_4$ and consider $V = \mathcal{O}_{(12)}^{-+}$ and $V' = \mathcal{O}_{(12)}^{--}$ where $\pm\pm$ indicates the 1-dimensional character of the centralizer $\langle (12), (34) \rangle$. Our results show

[7] SL is indebted to E. Meir for explaining this to him.

that these two objects are interchanged by the braided autoequivalence in $\mathcal{BV}$ induced from the nontrivial 2-cocycle of $\mathbb{S}_4$ (which restricts to a nontrivial class on the centralizer). Both Nichols algebras have dimension 24^2.
More generally let $G = \mathbb{S}_n$ and consider $V = \mathcal{O}_{(12)}^{-+}$ and $V' = \mathcal{O}_{(12)}^{--}$ where the centralizer is $\mathbb{Z}_2 \times \mathbb{S}_{n-2}$. Then again these two objects are interchanged by the unique 2-cocycle inducing up to $\mathcal{BV}$. In case $n = 5$ both Nichols algebras are of finite dimension $4^4 5^2 6^4$.

- Let $G = \mathbb{S}_4$ and consider $V = \mathcal{O}_{(12)}^{--}$ and $V' = \mathcal{O}_{(1234)}^{-1}$. We claim that our results show that these two elements are interchanged by the braided equivalence in $\mathcal{EV}$ induced by the nonlazy monoidal autoequivalence F of $\mathrm{Rep}(\mathbb{S}_4)$ defined by S the Klein-4-group and its unique nondegenerate 2-cocycle. Namely, as objects in $\mathrm{Rep}(\mathbb{S}_4)$ these are $\mathrm{sgn} + \chi_2 + \chi_3 \cdot \mathrm{sgn}$ respectively $\mathrm{sgn} + \chi_2 + \chi_3$ (where $1+\chi_2,\ 1+\chi_3$ are the permutation characters) and F interchanges $[(12)], [(1234)]$ and $\chi_3, \mathrm{sgn}\cdot\chi_3$. Again these Nichols algebras have dimension 24^2.

- On the other hand $V = \mathcal{O}_{(12)}^{-+}$ and $V' = \mathcal{O}_{(1234)}^{-1}$ are directly related by the partial dualization on S, which is due to a relation in BrPic.

- Let $G = \mathbb{Z}_5 \rtimes \mathbb{Z}_5^\times$ and consider $V = \mathcal{O}_{i\rtimes 2}^{-1}$ and $V = \mathcal{O}_{i\rtimes 3}^{-1}$. One can easily see that these two objects are interchanged by an outer automorphism. The respective Nichols algebras have dimension 1280. A similar connection holds between two Nichols algebras over $\mathbb{Z}_7 \rtimes \mathbb{Z}_7^\times$.

- Over the dihedral group $\mathbb{D}_4 = \langle x, y \mid x^2 = y^2 = (xy)^4 = 1\rangle$ with 8 element there are four Nichols algebras of dimension 64, that are all interchanged by the Brauer Picard group, which is $\mathbb{S}_4$ by [NR14].

5.1.2 Braided autoequivalences of quantum groups

Already the well-known fact that $\mathrm{BrPic}(H\text{-mod}) \cong \mathrm{Aut}_{\mathrm{br}}(\mathrm{D}H\text{-mod})$ has interesting implications for $H = \mathfrak{B}(M) \rtimes \mathbb{C}[G]$ as we have already seen in the Taft algebra case:

Quasi-triangular quantum groups $u_q(\mathfrak{g})$ can be obtained[8] as quotients of $\mathrm{D}H$ for suitable Nichols algebras by grouplikes. So the category $\mathrm{D}H$-mod has the category $u_q(\mathfrak{g})$-mod as subcategory. For a given element in $\mathrm{BrPic}(\mathfrak{B}(M) \rtimes \mathbb{C}[G])$ we can ask whether the braided autoequivalence associated by the ENOM functor fixes this subcategory, so we obtain a braided autoequivalence of $u_q(\mathfrak{g})$-mod.

This question seems quite easy to answer (and usually to answer positively) because it involves only knowledge about the action of the grouplikes $\mathbb{C}[G]$ resp. $\mathbb{C}[G \times G]$ in the double: More precisely, a sufficient condition is that the braided autoequivalence preserves the forgetful functor to DG-mod, as is e.g. the case for the interesting elements in $\mathcal{EV}$ we discuss below. More general criteria could be given.

[8] In case q has even order, care has to be taken at this point

5.1.3 Induction images $\mathcal{BV}, \mathcal{EV}$

Since the notion of a Nichols algebra is self-dual, it suffices to restrict to study to $\mathcal{EV}$ (compare to the Taft algebra), so we wish to know the Bigalois objects. This is in general difficult, but there has been significant progress in the context of liftings of Nichols algebras, which we want to briefly comment on:

A long-standing question is to classify algebras L with $\mathrm{gr}(L) = \mathrm{B}(M) \rtimes \mathbb{C}[G]$ and the conjecture stands, that all of these algebras are related by a 2-cocycle Doi twist i.e. there exists a L-H-Bigalois object and hence there is a monoidal equivalence $F : \mathrm{B}(M) \rtimes \mathbb{C}[G]\text{-comod} \to L\text{-comod}$, see [M01][AAIMV13]. In the former paper this has been observed for quantum groups, where the classification of pointed Hopf algebras produces families with free lifting parameters, which turn out to however all be related by 2-cocycle deformations. In the latter paper an impressive program has been presented to systematically determine all different liftings for a given Nichols algebra.

Thus: For a given Nichols algebra, e.g. $u_q^+(\mathfrak{g})$, the BrPic-groupoid contains large (multi-parameter) families of objects L with different liftings, e.g. with deformed relations like $E_{\alpha_i}^{N_i} = \mu_i \in \mathbb{C}$, all of which are connected by elements in $\mathcal{EV}$. Note that this gives bimodule categories between categories H-mod and L-mod that are very different as categories.

Remark 5.1. *From a physical perspective it very interesting to study such defects between different phases labeled H-mod and L-mod, in particular where H is the Borel part of a quantum group and L is a different lifting. Take for example the relation $E_{\alpha_i}^{N_i} = \mu_i$, which resembles closely what one has in finite W-algebras. All different liftings of this type come from different subcategories (sectors) of the Kac-Procesi-DeConcini-Quantum group where $E_{\alpha_i}^{N_i}$ is a central element. The subcategories are enumerated by collections of μ_i that are in bijection to points of the complex Lie group associated to $\mathfrak{g}$. In this view, all these bimodule categories (defects) between different categories can actually be collected to bimodule categories between this new large category.*

Needless to say, these are not the only objects in BrPic, at least not for general Nichols algebras, as the reflections $\mathcal{R}$ in the next two sections show.

5.1.4 Partial dualization on the Cartan part

We want to now more thoroughly treat partial dualization on the Cartan part of a quantum Borel part of a quantum group $U_q(\mathfrak{g})$ and find relations to the L-dual of the respective Lie group, at least in the simply-laced case. We assume that the TFT side of our construction is actually related to T-duality; this could explain why an L-dual appears, see [DE14]:

Let $H = U_q(\mathfrak{g})^{\geq} = U_q(\mathfrak{g})^{+} \rtimes \mathbb{C}[\Lambda]$ where $\Lambda = \mathbb{Z}^{\mathrm{rank}}$ is a lattice (resp. a quotient at roots of unity) sitting between root- and weight-lattice of $\mathfrak{g}$ i.e. $\Lambda_W \supset$

$\Lambda \supset \Lambda_R$. The embedding and the scalar product determine the Yetter-Drinfel'd structure of $U_q(\mathfrak{g})^+$ via

$$K_\lambda.E_\alpha = q^{(\lambda,\alpha)}E_\alpha \qquad \delta(E_\alpha) \mapsto K_\alpha \otimes E_\alpha$$

The choice of Λ is parametrized by a subgroup of $\Lambda_W/\Lambda_R = \pi_1$ which determines the fundamental group of the respective complex Lie group, which parametrizes different topological coverings. Correspondingly the usual choice $\Lambda = \Lambda_R$ is the adjoint form and $\Lambda = \Lambda_W$ is often called the simply-connected form.

Lemma 5.2. *For $\mathfrak{g}$ simply laced partial dualization r on the Cartan part $\mathbb{C}[\Lambda]$ interchanges $U_q(\mathfrak{g})^+ \rtimes \mathbb{C}[\Lambda]$ and $U_q(\mathfrak{g})^+ \rtimes \mathbb{C}[\Lambda^\vee]$; e.g. it interchanges adjoint and simply-connected form. In particular for small quantum groups at an ℓ-th root of unity it interchanges $u_q(\mathfrak{g})^+ \rtimes \mathbb{C}[\Lambda/\ell\Lambda^\vee]$ and $u_q(\mathfrak{g})^+ \rtimes \mathbb{C}[\Lambda^\vee/\ell\Lambda]$*

Proof. In the case of the Taft algebra this has been checked explicitly in our paper [BLS15]. Take the obvious group pairing $\Lambda \times \Lambda^\vee \to \mathbb{C}^\times$ given by $\lambda \otimes \mu \mapsto q^{(\lambda,\mu)}$. It gives in particular rise to a nondegenerate group pairing:

$$\Lambda/\ell\Lambda^\vee \times \Lambda^\vee/\ell\Lambda \to \mathbb{C}^\times$$

We need to convince ourselves that this dualization interchanges action and coaction. But this is clearly true

$$K_\lambda ._r E_\alpha := \langle K_\lambda, K_\alpha\rangle E_\alpha = q^{(\lambda,\alpha)}\ E_\alpha \qquad \blacksquare$$

Corollary 5.3. *Partial dualization as discussed above gives rise to the braided category equivalence between the Drinfel'd centers of $u_q(\mathfrak{g})^+ \rtimes \mathbb{C}[\Lambda/\ell\Lambda^\vee]$ and $u_q(\mathfrak{g})^+ \rtimes \mathbb{C}[\Lambda^\vee/\ell\Lambda]$. It restricts to a braided category equivalence between the respective quantum groups $u_q(\mathfrak{g})$ associated to Λ and $\Lambda^\vee$.*

Corollary 5.4. *Partial dualization as discussed above is the image under the ENOM functor of the module category $\mathcal{M}_r := u_q(\mathfrak{g})^+$-mod, which is as $\mathbb{C}$-linear category the Nichols algebra representation category and has a nontrivial bimodule category structure defined by $q^{(\lambda,\mu)}$ for $\lambda \in \Lambda$ and $\mu \in \Lambda^\vee$*

5.1.5 Partial dualization and Weyl reflection

We now turn our attention to reflections of the Nichols algebra in the original sense: Let $M = \bigoplus_i M_i$ a decomposition of the object M into simple objects, then α_i is a simple root for the Nichols algebra $\mathrm{B}(M)$ in the sense of [AHS10]. For example for the semisimple complex finite-dimensional Lie algebra $\mathfrak{g}$ we have $U_q(\mathfrak{g})^+ = \mathfrak{B}(M)$, resp. $U_q(\mathfrak{g})^+ = \mathfrak{B}(M)$ for roots of unity, for a choice of a $\mathbb{Z}^{\mathrm{rank}}$-Yetter-Drinfeld module $M = \bigoplus_i E_{\alpha_i}\mathbb{C}$ where α_i a simple root in the usual sense.

Then the reflection of this Nichols algebra is the special case of a partial dualization r with respect to the projection, see [HS13][BLS15]

$$\pi_i : \mathfrak{B}(M) \to \mathfrak{B}(M_i) \qquad \mathfrak{B}(M) = \mathfrak{B}(M)^{\mathrm{coin}\pi} \rtimes \mathfrak{B}(M_i)$$

For semisimple Lie algebras there is an algebra isomorphism $r(\mathfrak{B}(M)) \cong \mathfrak{B}(M)$, namely Lusztig's reflection automorphism T_{w_i} for the simple reflection w_i, but for general Nichols algebras these two algebras can be non-isomorphic. Nevertheless our results *(in cit. loc.)* show in all cases a category equivalence

$$\mathcal{Z}(\mathfrak{B}(M)\text{-mod}) \cong \mathcal{Z}(r(\mathfrak{B}(M))\text{-mod})$$

In particular for the Lie algebra case this restricts to a braided equivalence $T_{w_i} : U_q(\mathfrak{g})\text{-mod} \to U_q(\mathfrak{g})\text{-mod}$ and more general for every Weyl group element $w \in W$.

We now discuss the $\mathfrak{B}(M)$-mod-$r(\mathfrak{B}(M))$-mod-bimodule categories associated to these partial dualizations. This is interesting already in the Lie algebra case: Our results in Section 3.5 show that the preimage of there is a bimodule category

$$\mathcal{M}_{w_i} := \mathfrak{B}(M)^{\mathrm{coin}\pi}\text{-mod}$$

With left resp. right categorical action by $\mathfrak{B}(M)$-mod resp. $r(\mathfrak{B}(M))$-mod, forgetting $\mathfrak{B}(M)$ resp. $\mathfrak{B}(M^*)$-action, and a nontrivial bimodule category constraint $(V \otimes M) \otimes W \cong V \otimes (M \otimes W)$ given by the evaluation map $\mathfrak{B}(M) \otimes \mathfrak{B}(M^*) \to \mathbb{C}$.

Remark 5.5. *Iterating this procedure yields for every Weyl group element $w \in W$ a bimodule category*

$$\mathcal{M}_w := U^+[w]\text{-mod}$$

It is worth mentioning that these are precisely the homogeneous coideal subalgebras of $U^+(\mathfrak{g})$; so it would be interesting to consider (and recognize in our ansatz) bimodule categories for all coideal subalgebras C, which are classified by [HK11] to be character shifts $C = (\mathrm{id} \otimes \chi)\Delta U^+[w]$.

5.2 Defects in 3D topological field theories

An oriented $(3,2,1)$-extended TQFT is a symmetric monoidal weak 2-functor:

$$Z : \mathrm{Bord}^{or}_{3,2,1} \to \mathrm{2Vect}$$

where $\mathrm{Bord}^{or}_{3,2,1}$ is the symmetric monoidal bicategory of oriented 3-cobordisms and 2Vect the symmetric monoidal bicategory of Kapranov-Voevodsky 2-vector spaces, thus objects of 2Vect are k-linear, abelian, semisimple categories, morphisms are k-linear functors and 2-morphisms are natural transformations. (See [KV94], [Mo11] and the Appendix of [BDSV15] for more details on 2Vect and other targets).
Oriented $(3,2,1)$-extended TQFTs are classified by anomaly free modular tensor categories (by Thm. 2 in [BDSV15]), where a functor Z corresponds to the

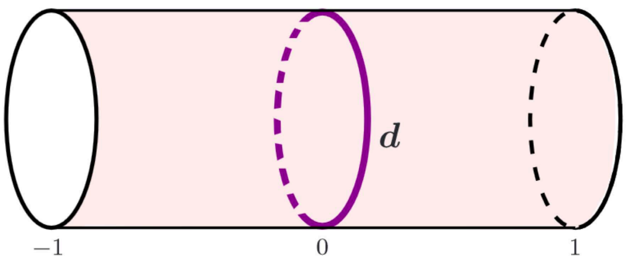
anomaly free modular tensor category $Z(S^1)$, which we also refer to as the category of bulk Wilson lines. For general modular tensor categories, such theories are called Reshetikhin-Turaev type theories. In the case the modular tensor category is $Z(S^1) = \mathcal{Z}(\mathcal{C})$, the Drinfeld center of some fusion category $\mathcal{C}$, such theories are called Turaev-Viro type theories. One can use the Reshetikhin-Turaev construction [RT91], which is essentially based on surgery on 3-manifolds along links, to define a Reshetikhin-Turaev type theory explicitly.

A special case are *Dijkgraaf-Witten theories* with $Z(S^1) = \mathcal{Z}(\mathrm{Vect}_G^\omega)$ where Vect_G^ω is the category of G-graded vector spaces for some finite group G and nontrivial associativity constraints determined by 3-cocycles $\omega \in Z^3(G, k^\times)$. If $\omega = 1$ the Dijkgraaf-Witten theory is called untwisted. Dijkgraaf-Witten theories can be realized explicitly by linearizing the category of principal G-bundles on a manifolds i.e.

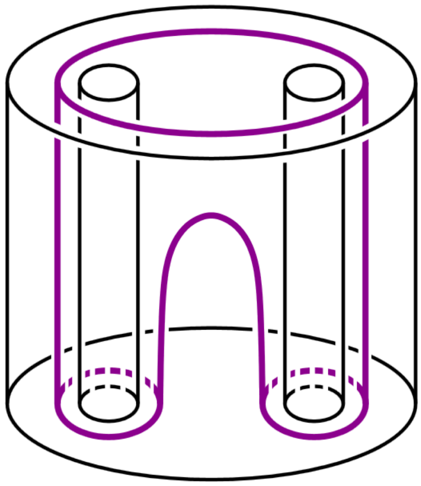
$$Z(\Sigma) := \mathrm{Fun}(\mathrm{Bun}_G(\Sigma), \mathrm{Vect}) \qquad \mathrm{Bun}_G(\Sigma) \cong \mathrm{Hom}(\pi_1(\Sigma), G)$$

and Z(M) by so-called pull-push-construction, that sums over all possible continuations of bundles on Σ to M, see e.g. [FPSV14].

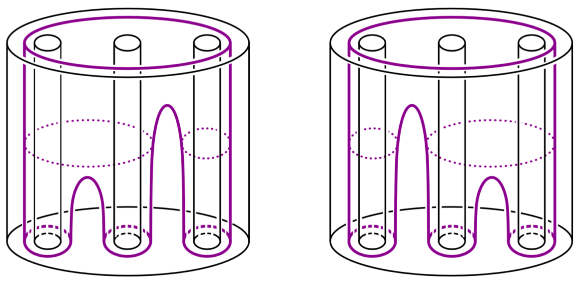
We now consider additional data on the manifold, namely *surface defects*: These are codimension 1 submanifolds. Suppose for example $\Sigma_{\text{Transm}} = S^1 \times [-1, 1]$ and a middle circle belonging to a defect d, then the two bounding circles get assigned some $Z(S^1 \times \{-1\}) = \mathcal{Z}(\mathcal{C})$ and $Z(S^1 \times \{1\}) = \mathcal{Z}(\mathcal{D})$ and the defect a bimodule category $Z(S^1 \times \{0\}) = {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$. On the other hand the TFT assigns to this situation a (due to the defect possibly nontrivial) morphism $\mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$:



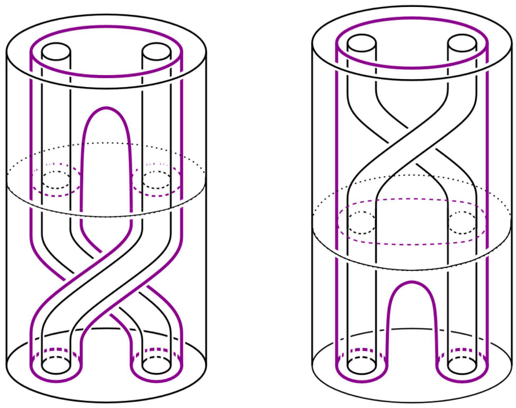
This becomes a monoidal functor with the monoidal structure given by $Z(M_{pants\ defects})$ for the following 3-manifold with defect: (the cylinder has been flattened to a annulus)



The coherence condition is checked by noticing that the following two manifolds are diffeomorphic:



and the following two diffeomorphic manifolds show the functor $Z(\Sigma_{\mathrm{Transm}})$ is braided:



For details we refer to [FPSV14]. We repeat their very interesting question linking this natural functor from the TFT construction to the ENOM functor, which they solve in the case Vect_G for G abelian by explicit calculation using the bundle construction:

Question 5.6. *Does the assignment of the functor $Z(\Sigma_{Transm}) : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$ to an exact invertible bimodule category ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ coincide with ENOM functor?*

The results of the present article give many new families of examples for such situations. The final hope is, that there are three types of defects and every defect can be written as a product. This would also open the possibility of checking the previous question explicitly for the given subgroups.

The TFT approach is also a reason for insisting in the formulation of exact invertible $\mathcal{C}$-$\mathcal{D}$-bimodule categories with $\mathcal{C} \neq \mathcal{D}$: As we saw, for quantum groups many of the interesting examples appear between different categories - an effect that is present (but rare) for group examples, see Example 4.5. From a physics perspective, it is very natural to assign different categories to different "phases regions" i.e. connected regions separated by defects.

5.3 Outlook: Group-theoretic extensions

By [ENOM09] group-theoretic extensions

$$\mathcal{D} = \bigoplus_{t \in \Sigma} \mathcal{D}_t \qquad \mathcal{D}_1 = \mathcal{C}$$

of the category H-mod by Vect_Σ are associated to homomorphisms $\psi : \Sigma \to \mathrm{BrPic}(\mathcal{C})$ (plus additional coherence data we omit here) with $\mathcal{D}_t = \psi(t)$ a $\mathcal{C}$-$\mathcal{C}$-bimodule category.

We finally sketch briefly what the result is for $\mathcal{C} = H$-mod when ψ lands in our three subgroups $\mathcal{BV}, \mathcal{EV}, \langle \mathcal{R} \rangle$ in $\mathrm{BrPic}(H\text{-mod})$. The idea is that there are essentially three types of generic group-theoretic extensions associated to the three subgroups:

Let $\psi : \Sigma \to \mathcal{BV} = \mathrm{Ind}(\mathrm{Aut}_{\mathrm{mon}}(H\text{-mod}))$. This is the trivial case considered by several authors: All the bimodule categories are $\mathcal{D}_t = {}_{F_t}H$-mod so $\mathcal{D} = \mathrm{Vect}_\Sigma \boxtimes \mathcal{C}$, while $\Sigma \to \mathrm{Aut}_{\mathrm{mon}}(\text{H-mod})$ gives a categorical action and accordingly is the tensor product defined.

Example 5.7. *Take the case $\mathcal{V}$ i.e. let $v \in \mathrm{Aut}_{\mathrm{Hopf}}(H)$ of order n and let $\Sigma = \langle v \rangle$ and ψ just the identity. Then the associated category is*

$$\mathcal{D} = \mathrm{Vect}_\Sigma \boxtimes H\text{-mod} = \bigoplus_{i=0}^{n-1} H\text{-mod}$$

with a tensor product $X_i \otimes Y_j = (X \otimes v^i(Y))_{i+j}$. Hence $\mathcal{D}$ should be the representations of the cosmash product Hopf algebra $\mathbb{Z}_n \ltimes H$, with $\mathbb{Z}_n$-coaction on H given by v, which is as an algebra just $\mathbb{Z}_n \otimes H$.

Let $\psi : \Sigma \to \mathcal{EV} = \mathrm{Ind}(\mathrm{Aut}_{\mathrm{mon}}(H^*\text{-mod})) = \mathrm{Bigal}(H)$. Then $\mathcal{D} = \tilde{H}$-mod where the new Hopf algebra is as an algebra

$$\tilde{H} = \bigoplus_{t \in \Sigma} R_t$$

with Bigalois objects R_t. This type of extensions has been considered in the first authors work [Len12], in particular in its application to construct new Nichols algebras.

Example 5.8. *Let $\Sigma^* \to G \to \Gamma$ a central extension of groups, then associated one has a 2-cocycle in $Z^2(\Gamma, \Sigma^*)$ and hence a homomorphism $\phi : \Sigma \to Z^2(\Gamma, \mathbb{C}^\times)$. We viewing the target as the subgroup of $\mathcal{BV}$ for $H = \mathbb{C}[\Gamma]$. Then our construction returns bimodule categories $\mathcal{D}_t = R_t$-mod for Bigalois objects being twisted group rings $R_t = \mathbb{C}_{\phi(t)}[\Gamma]$ and overall we get*

$$\tilde{H} = \mathbb{C}[G] \qquad \mathcal{D} = \mathrm{Rep}(G) = \bigoplus_{t \in \Sigma} \mathbb{C}_{\phi(t)}[\Gamma]\text{-mod}$$

For example $\mathbb{C}[\mathbb{D}_4] = \mathbb{C}[\mathbb{Z}_2^2] \oplus \mathbb{C}_\sigma[\mathbb{Z}_2^2]$ and $\mathcal{D} = \mathrm{Rep}(\mathbb{D}_4)$ is a $\mathbb{Z}_2$-extension of $\mathrm{Rep}(\mathbb{Z}_2^2)$.

Example 5.9 ([Len12])**.** *Let $\phi : \Sigma \to Z^2(\Gamma, \mathbb{C}^\times)$ as above and $\mathfrak{B}(M)$ a Nichols algebra over Γ, and assume we are given a so-called twisted symmetry action of Σ on $\mathfrak{B}(M)$. Then this data gives rise to a homomorphism*

$$R_t : \Sigma \to \mathrm{Bigal}(\mathfrak{B}(M) \rtimes \mathbb{C}[\Gamma])$$

and our construction returns

$$\tilde{H} = \mathfrak{B}(\tilde{M}) \rtimes \mathbb{C}[G] \qquad \mathcal{D} = \tilde{H}\text{-mod} = \bigoplus_{t \in \Sigma} R_t\text{-mod}$$

where $\mathfrak{B}(\tilde{M})$ is a Nichols algebra over the centrally extended group G.

Let $\Sigma = \mathbb{Z}_2$ and $\psi(g) = r$ a partial dualization on a semidirect product decomposition $H = K \rtimes A$ with K-self-dual. Then again we obtain a group-theoretical extension

$$\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_r = H\text{-mod} \times A\text{-mod}$$

Example 5.10 ([ENOM09] Sec. 9.2)**.** *Let $A = 1$, e.g. for $H = K = \mathbb{C}[G]$ with G abelian. Then $\mathcal{D}$ is a Tambara-Yamigami category with $\mathcal{D}_1 = \mathcal{C}$ pointed and $\mathcal{D}_r$ consisting of a unique simple object.*

Question 5.11. *What are the category extension associated to $H = U_q(\mathfrak{g})^+$ and the homomorphism $\phi : W \to \mathrm{BrPic}(H)$ with W the Weyl group generated by all reflections r_i? (or say a cyclic subgroup generated by a single element w)*

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