From finite groups to finite-dimensional Hopf algebras

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Introduction

Finite-dimensional Hopf algebras H have an elusive nature. On one hand they resemble finite groups in many aspects, while on the other, elementary facts about finite groups are either hard to translate or even false. In fact, Kaplansky's famous conjectures from 1975[**?**], are all attempts to generalize results from groups to Hopf algebras. Some of his conjectures are still open.

Even the naive translation of normal subgroups to normal Hopf subalgebras is problematic. There exist semisimple Hopf algebras of dimension 36 that have no nontrivial normal Hopf subalgebras [**?**], yet Burnside's theorem on solvable groups assures the existence of normal subgroups for groups of order 36. A more suitable translation of subgroups are left coideal subalgebras. When N is a subgroup of a group G then kN can be considered as a left coideal subalgebra of the Hopf algebra kG. Normal left coideal subalgebras correspond to normal subgroups in this way.

Explicitly, a Hopf algebra H is an algebra over a field k endowed with a counital coassociative coalgebra structure map $\Delta : H \to H \otimes H$, compatible with the algebra structure of H, an augmentation algebra map $\varepsilon : H \to k$ and an antipode $S : H \to H$. A left coideal subalgebra N of H is a subalgebra of H such that $\Delta(N) \subset H \otimes N$. There exists also an appropriate notion of normality.

Just as normal subgroups for groups, normal left coideal subalgebras give rise to Hopf quotients. Explicitly, if N is a normal left coideal subalgebra of H then the Hopf quotient related to N is H/HN^+ where $N^+ = N \cap \ker \varepsilon$ and $HN^+ = N^+H$ is then a Hopf ideal. On the level of groups, Hopf quotients and group algebras

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of quotient groups coincide. For if N is a normal subgroup of G then the group algebra $k(G/N)$ is precisely the Hopf quotient $kG/(kG)(kN)^+$.

However, while group quotients are always of the form G/N, N a normal subgroup, Hopf quotients arise from normal left coideal subalgebras which are not necessarily Hopf subalgebras. This is the cause for a major difference between general Hopf algebras and group algebras. We avoid this problem by using integrals.

The existence of integrals for finite dimensional Hopf algebras is one of the first major achievements in their structure theory, due to Larson and Sweedler from the late 60's [12]. Integrals play the role of the averaging elements $\sum_{g\in G} g$ for finite groups G. Since the 90's the existence of integrals for left coideal subalgebras was established as well (e.g [11, **?**, 13, 14]). Using integrals for the Hopf algebra H, for its dual Hopf algebra H^* and for left coideal subalgebras of both, enable us to generalize results from groups to Hopf algebras.

In this survey we describe conjugacy classes, character tables, commutators, nilpotency and solvability for semisimple Hopf algebras. At each stage we show how our general approach coincides with the classical notions for groups.

1 Preliminaries and notations

Throughout H is a d-dimensional semisimple Hopf algebra over an algebraically closed field k of characteristic 0 and H^* is its dual which is a semisimple Hopf algebra as well. In several of the following we assume that $k = \mathbb{C}$. Let $\varepsilon : H \to k$ denote the augmentation map. We use the Sweedler notation for the coalgebra structure

$$\Delta(h) = \sum h_1 \otimes h_2 \in H \otimes H$$

An element $h \in H$ is cocommutative if $\sum h_1 \otimes h_2 = \sum h_2 \otimes h_1$. for all $h \in H$.

We denote by S and s the antipodes of H and H^* respectively. It is known that when H is semisimple then $S^2 = \mathrm{Id}$.

We denote by Λ and λ the integrals of H and H^* respectively. Recall, $h\Lambda = \Lambda h = \langle \varepsilon, h\rangle \Lambda$ for all $h \in H$ and similarly $p\lambda = \lambda p = \langle p, 1\rangle \lambda$ for all $p \in H^*$. We choose Λ to be idempotent and λ to satisfy $\langle \lambda, \Lambda\rangle = 1$.

The Hopf algebra H^* becomes a right and left H-module by the *hit* actions $\leftharpoonup$ and $\rightharpoonup$ defined for all $a \in H$, $p \in H^*$,

$$\langle p \leftharpoonup a, a'\rangle = \langle p, aa'\rangle \qquad \langle a \rightharpoonup p, a'\rangle = \langle p, a'a\rangle$$

H becomes a left and right H^* module analogously.

It is known that Λ is a free generator for H as a left and a right H^*-module. That is, $H^* \rightharpoonup \Lambda = \Lambda \leftharpoonup H^* = H$.

Denote by $\dot{}_{ad}$ the left adjoint action of H on itself, that is, for all $a, h \in H$,

$$h_{\dot{ad}} a = \sum h_1 a S(h_2).$$

A left coideal subalgebra of H is a subalgebra N so that $\Delta(N) \subset H \otimes N$. It is called **normal** if it is stable under the left adjoint action of H.

Every left coideal subalgebra of H is semisimple as an algebra and equipped with a 1-dimensional ideal of integrals, we denote by Λ_N the unique idempotent integral of N.

The most basic example of a semisimple Hopf algebra is $H = kG,\ G$ a finite group, we keep this assumption throughout.

Example 1.1. Recall, kG is a Hopf algebra with $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$ and $\langle \varepsilon, g\rangle = 1$ for all $g \in G$. Clearly kG is a cocommutative Hopf algebra. We have,

$$\Lambda = \frac{1}{|G|} \sum_{g \in G} g$$

is an idempotent integral for kG. The dual Hopf algebra kG^* has a linear basis $\{p_g\}$ where $\langle p_g, g'\rangle = \delta_{g,g'}$, that is, a dual basis for the basis $\{g\}_{g\in G}$ of kG. We have $\lambda = |G| p_1$.

It can be checked that

$$\Delta(p_g) = \sum_{t \in G} p_t \otimes p_{t^{-1}g}.$$

The semisimple Hopf algebra kG^* is commutative and the elements $\{p_g\}$ form a set of primitive idempotents for it. We have,

$$p_g \rightharpoonup g' = \delta_{g,g'} g \qquad g \rightharpoonup p_{g'} = p_{g'g^{-1}}$$

and thus

$$p_g \rightharpoonup \Lambda = \frac{1}{|G|} g \qquad g \rightharpoonup \lambda = |G| p_{g^{-1}}.$$

2 Conjugacy classes, Class sums and Character tables

The representation theories of semisimple Hopf algebras and of finite groups are closely related in the sense that they both give rise to fusion categories (that is, k-linear semisimple rigid tensor categories with finitely many simple objects and finite dimensional spaces of morphisms). These categories are tensor categories due to the fact that the tensor product of two representations is a representation as well via the coproduct structure. As a result their characters form an algebra.

Explicitly, for a finite-dimensional left H-module V denote its structure map $H \to \mathrm{End}_k(V)$ by ρ_V. Then the character χ_V of V is defined by

$$\langle \chi_V, h\rangle = \mathrm{Trace}(\rho_V(h))$$

for all $h \in H$. Let $d = \dim H$ and $\{k = V_0, \dots V_{n-1}\}$ be a complete set of non-isomorphic irreducible H-modules. We set $\mathrm{Irr}(H) = \{\chi_i\}$, the corresponding characters where $\chi_i = \chi_{V_i}$ and $\langle \chi_1, 1\rangle = d_i$, the dimension of the irreducible module V_i. Then we have

$$\chi_V \chi_W = \chi_{V \otimes W},$$

where the product on the left coincide with the product in the Hopf algebra H^*. We have also that λ is the character of the left (right) regular representation of H, that is,

$$\lambda = \sum_{i=0}^{n-1} d_i \chi_i, \qquad \langle \lambda, 1 \rangle = \sum d_i^2 = d.$$

Denote by $R(H) \subset H^*$ the k-span of $\mathrm{Irr}(H)$. It is in fact an algebra which coincides with the algebra of cocommutative elements of H^*. By ([**?**, 15]) it is a semisimple algebra. Let $\{\frac{1}{d}\lambda = F_0, \ldots F_{m-1}\}$ be a complete set of central primitive idempotents of $R(H)$, and let $\{f_0, \ldots, f_{m-1}\}$ be primitive orthogonal idempotents in $R(H)$ so that $f_i F_j = \delta_{ij} f_i$.

In what follows we give the initial definitions that allow us to generalize results from group theory to the structure theory of Hopf algebras. We define a **conjugacy class** as follows:

$$\mathfrak{C}_i = \Lambda \leftharpoonup F_i H^* \tag{1}$$

We generalize also the notions of a **Class sum** and of a representative of a conjugacy class:

$$C_i = \Lambda \leftharpoonup dF_i \qquad \eta_i = \frac{C_i}{\dim(f_i H^*)}. \tag{2}$$

We refer to η_i as a **normalized class sum**. By results in [**?**], the elements C_i are central in H.

When $H = kG$, G a finite group, the definition of conjugacy classes, class sums and representatives reduce to the usual definitions. To see this we continue with Example 1.1

Example 2.1. Let $H = kG$, G a finite group. Since kG^* is a commutative algebra, so is $R(H)$. It can be checked that if $\mathfrak{C}_i$ is the ith conjugacy class of G, then the element $F_i \in kG^*$,

$$F_i = \sum_{g \in \mathfrak{C}_i} p_g$$

is cocommutative, hence belongs to $R(H)$. In fact $\{F_i\}$ is the set of (central) primitive idempotents of $R(H)$. In particular $F_i = f_i$ in our notations.

The classical notions of conjugacy classes $\mathfrak{C}_i$ and corresponding class sums C_i are given respectively by:

$$\mathfrak{C}_i = \{x^{-1} g_i x,\ x \in G\} \quad \text{and} \quad C_i = \sum_{g \in \mathfrak{C}_i} g,$$

where g_i is an arbitrary representative of $\mathfrak{C}_i$.

By using the notations and the formulas in Example 1.1, they can be realized exactly as in (1) and (2). That is, since $\Lambda = \frac{1}{|G|} \sum_{g \in G} g$, $|G| = \dim(H) = d$, we obtain

$$Sp_k\{x^{-1} g_i x,\ x \in G\} = \Lambda \leftharpoonup F_i H^* \quad \text{and} \quad \sum_{g \in \mathfrak{C}_i} g = \Lambda \leftharpoonup dF_i.$$

Note also that

$$\dim(F_i H^*) = |\mathfrak{C}_i|.$$

For general Hopf algebras we can not choose a representative g_i from the conjugacy class $\mathfrak{C}_i$. We replace it by an "average" - the normalized class sum η_i defined in (2). For groups it boils down to

$$\eta_i = \frac{1}{|\mathfrak{C}_i|} \sum_{g \in \mathfrak{C}_i} g.$$

The following result describes, in a certain sense, the "essence" of conjugacy classes. Given a Hopf algebra H, its **Drinfel'd double** $D(H)$ defined in [9] is a Hopf algebra structure defined on $H^* \otimes H$, the tensor product of H and its dual H^*. From a categorical point of view, the category of representations of $D(H)$ is the center of the category of H-representations. We have shown in [6] that:

Proposition 2.2. *Each conjugacy class $\mathfrak{C}_i$ is an irreducible left $D(H)$-module and moreover,*

$$H \cong \oplus_{i=0}^{m-1} \mathfrak{C}_i^{\oplus m_i}$$

as $D(H)$-modules, where $m_i = \dim f_i R(H)$.

We are ready now to define the character table for Hopf algebras. Recall, the (i,j)-th entry in the character table for G is $\langle \chi_i, g_j \rangle$ where g_j is an element of $\mathfrak{C}_j$. Since this value does not depend on the representative, we can replace g_j by the 'average' representative η_j. We thus define a generalized character table for H as follows:

Definition 2.3. The generalized character table (ξ_{ij}) of a semisimple Hopf algebra H over $\mathbb{C}$ is given by:

$$\xi_{ij} = \langle \chi_i, \eta_j \rangle,$$

for all irreducible characters χ_i and generalized class sums η_j of H.

When $R(H)$ is commutative then $\{F_i\}$ forms a basis of $R(H)$ and $f_i = F_i$ for all i. In this case, $H \cong \oplus_{i=0}^{m-1} \mathfrak{C}_i$ and the character table is a square matrix. Moreover, many other properties of character tables for groups can be generalized in this case.

The common family of Hopf algebras for which $R(H)$ is a commutative algebra consists of **quasitriangular** Hopf algebras introduced by Drinfel'd[9] in the context of quantum groups. They can be described as quotients of Drinfel'd doubles. Alternatively, they can be described as Hopf algebras whose category of representations is also braided. Group algebras are always quasitriangular. **Factorizable** Hopf algebras are special kind of quasitriangular Hopf algebras. Drinfel'd doubles are the basic examples of factorizable Hopf algebras.

In [4] we listed properties of groups related to their character tables and showed how they can be generalized to Hopf algebras for which $R(H)$ is a commutative algebra. We summarize:

Theorem 2.4. *Let H be a semisimple Hopf algebra over $\mathbb{C}$ and assume $R(H)$ is commutative. Then the following hold.*

1. *The entries of the character table ξ_{ij} are algebraic integers.*

2. *In each row i, $\xi_{i0} = \langle \chi_i, 1 \rangle$ is maximal amongst all absolute values of the entries in this row. That is, $|\xi_{ij}| \leq \langle \chi_i, 1 \rangle$ for all $0 \leq i \leq n-1$.*

3. *For groups, the product of two class sums is an integral sum of class sums. The structure constants of this product are obtained from the character table.*

   For Hopf algebras, if C_i, C_j are any conjugacy sums, then

$$C_i C_j = \sum_t c_{ijt} C_t$$

   where for any t,

$$c_{ijt} = \frac{\dim(H^* F_i)(\dim(H^* F_j)}{d} \sum_k \frac{\xi_{ki}\xi_{kj}\overline{\xi_{kt}}}{d_k}.$$

   If H is quasitriangular then we have integral coefficients $\{c_{ijt}\}$ up to a factor of d^{-3}. If H is factorizable then these coefficients are integral up to a factor of d^{-1}.

4. *For groups, different columns are orthogonal, while the square of the norm of each column j equals $\frac{|G|}{|\mathfrak{C}_j)|}$. Different rows satisfy generalized orthogonality relations.*

   The same result for Hopf algebras takes the following form.

$$(a) \qquad \sum_j \dim(F_j H^*) \xi_{nj} \overline{\xi_{mj}} = \delta_{mn} d.$$

$$(b) \qquad \sum_m \xi_{mi} \overline{\xi_{mj}} = \delta_{ij} \frac{d}{\dim(H^* F_i)}$$

   As a result, the size of each conjugacy class can be determined from the norm of the corresponding column.

5. *For groups, the kernel of the irreducible representation V_i is the union of all conjugacy classes $\mathfrak{C}_j$ for which $\langle \chi_i, \eta_j \rangle = \langle \chi_i, 1 \rangle = \dim V_i$.*

   For Hopf algebras we use the term **Left kernel** *defined in [1] that generalizes kernels of group representations to left kernels of H-representations. While kernels of group representations are normal subgroups, left kernels of H-representations are normal left coideal subalgebras. We have:*

   Let V_i be an irreducible representation of H, then

$$\mathrm{Lker} V_i = \bigoplus_{j \in J} \mathfrak{C}_j, \text{ where } J = \{ j \mid \langle \chi_i, \eta_j \rangle = \langle \chi_i, 1 \rangle \}.$$

6. *For groups, a collection of conjugacy classes forms a normal subgroup if and only if it is an intersection of kernels of some irreducible representations. As a result, the order of all the normal subgroups and the inclusion relations among them can be determined from their character tables.*

   Similar result holds for normal left coideal subalgebras. We have,

   Let $L = \bigoplus_{j \in J} C_j$*, for some set* $J \subset \{0, \dots, n-1\}$*. Then* L *is a subalgebra (necessarily normal left coideal) of* H *if and only if there exists* $I \subset \{0, \dots, n-1\}$ *so that* $J = J_I$*, where*

   $$J_I = \{j \mid \chi_i F_j = \langle \chi_i, 1 \rangle F_j \text{ for all } i \in I\}.$$

   As a result, the dimensions of all the normal left coideal subalgebras of H*, and the inclusion relations among them can be determined from its character table.*

When $R(H)$ is not commutative, generalizing results from groups is more complicated. The character tables are no longer square matrices and the central idempotents F_i in $R(H)$ are sums of primitive idempotents f_{i_j}. As a result, class sums C_i do not belong to a unique conjugacy class, but to a sum of isomorphic conjugacy classes. Explicitly, set

$$\mathfrak{C}^i = \Lambda \leftharpoonup F_i H^* = \bigoplus_j \mathfrak{C}_{i_j}.$$

If we choose arbitrarily $f_i = f_{i_j}$, for some j, then we have,

$$C_i \in \mathfrak{C}^i \cong \mathfrak{C}_i^{\oplus m_i},$$

where $m_i = \dim f_i R(H)$.

In [5] we extended as possible the results in Theorem 2.4 to the general case where $R(H)$ is not necessarily commutative. We summarize:

Theorem 2.5. *Let* H *be a* d*-dimensional semisimple Hopf algebra over* $\mathbb{C}$*, then*

- *Integrability: All entries of the character table are algebraic integers.*
- *Absolute maximality:* (i) *Each entry of the i-th row of the character table satisfies:*

  $$|\langle \chi_i, \eta_j \rangle| \leq m_j \langle \chi_i, 1 \rangle.$$

  (ii) *Equality holds if and only if right multiplication by* χ_i*,* r_{χ_i} *acts on* $f_j R(H)$ *as* $\alpha_i \operatorname{Id}_{f_j R(H)}$*, where* $|\alpha_i| = \langle \chi_i, 1 \rangle$*.*

  (iii) $\langle \chi_i, \eta_j \rangle = m_j \langle \chi_i, 1 \rangle$ *if and only if* r_{χ_i} *acts on* $f_j R(H)$ *as* $\langle \chi_i, 1 \rangle \operatorname{Id}_{f_j R(H)}$.
- *Orthogonality of columns:*

  $$\sum_k \xi_{ki} \overline{\xi_{kj}} = \delta_{ij} \frac{d \dim(f_i R(H))}{\dim(f_i H^*)}.$$

- *Analogues of characterizations of kernels. Let $\{F_j\}$ be the set of central primitive idempotents of $R(H)$ and let $\{V_i\}$ be the set of irreducible representations of H. Then the following are equivalent:*

  (i) $F_j\chi_i = \langle \chi_i, 1\rangle F_j$

  (ii) $\langle \chi_i, \eta_j\rangle = m_j \langle \chi_i, 1\rangle$.

  (iii) $\mathfrak{C}^j \subset \mathrm{LKer}_{V_i}$.

3 Commutators, nilpotency and solvability

The commutator subalgebra H' of a semisimple Hopf algebra H was first defined in [1]. It is a normal left coideal subalgebra of H for which H/HH'^+ is commutative and is minimal with respect to this property.

It is not hard to see that

$$H' = \{h \in H \mid \sigma \rightharpoonup h = h \ \forall \sigma \in G(H^*)\}. \tag{3}$$

Here $G(H^*)$ denotes the group of group-like elements of H^*, that is, the 1-dimensional (and thus multiplicative) characters on H.

Generalizing from groups we described H' in terms of Hopf algebraic commutators [6]. Let H be any Hopf algebra over k. For $a, b \in H$, define their commutator $\{a, b\}$ as:

$$\{a, b\} = \sum a_1 b_1 S a_2 S b_2. \tag{4}$$

Define

$$Com = \mathrm{span}_k\{\{a, b\} \mid a, b \in H\}. \tag{5}$$

Then Com is a left coideal of H. In [6] We showed:

Proposition 3.1. *Let H be a semisimple Hopf algebra then the commutator subalgebra H' of H is the algebra generated by Com.*

Example 3.2. Let G be a finite group and G' its commutator subgroup. Then G/G' is an abelian group and G' is minimal with respect to this property. Moreover, all 1-dimensional characters of G, are trivial (that is, act as 1) when restricted to G'. This is the analogue of the commutator subalgebra H' of H and of (3).

Since $\Delta(g) = g \otimes g$, we find that the definition of the commutator given in (4), boils down to the group commutator $[a, b] = aba^{-1}b^{-1}$ for all $a, b \in G$. The commutator group G' is the group generated by all commutators and is normal in G. Proposition 3.1 is a generalization of this fact.

Another concept which appears in the literature for group algebras is the commutator space of kG

$$K = Sp_k\{xy - yx,\ x, y \in kG\}.$$

The ideal generated by K is in fact $kG(kG')^+$, which is the kernel of the algebra map $kG \to kG'$. Moreover, it is contained in the kernel of any algebra map $kG \to k\overline{G}$ when $\overline{G}$ is commutative. For general Hopf algebras, the corresponding Hopf ideal is HH'^+.

In [6] we introduced a family of elements in H' denoted by z_n, $n > 1$, which arise from the idempotent integral of H. This family consists of powers of the S-fixed central invertible element z_2,

$$z_2 = \{\Lambda, \Lambda'\} = \sum \Lambda_1 \Lambda'_1 S\Lambda_2 S\Lambda'_2$$

where Λ, Λ' are two copies of the idempotent integral of H. We refer to z_2 as an extensive commutator.

The extensive commutator z_2 has a very nice form which is the key for determining commutativity. We showed

Theorem 3.3. *Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic* 0*, then H is commutative if and only if $z_2 \in k$.*

Let $\{E_i\}$ be the set of central orthogonal idempotents of H. Then

$$z_2 = \{\Lambda, \Lambda'\} = \sum_i \frac{1}{d_i^2} E_i \in Z(H).$$

The element z_2 generates H' in the following sense. H' is the algebra generated by the left coideal $z_2 \leftharpoonup H^*$.

What is the role played by the extensive commutator z_2 inside kG?

Example 3.4. Recall for groups $\Lambda = \frac{1}{|G|}\sum_{g\in G} g$. Hence

$$\{\Lambda, \Lambda'\} = \frac{1}{|G|^2}\sum_{a,b\in G} aba^{-1}b^{-1} = \frac{1}{|G|^2}\sum_{g\in G} f(g)g,$$

where f is the Frobenius counting function. That is, $f(g)$ counts the number of times g can be obtained as a commutator $aba^{-1}b^{-1}$. Clearly, if $\sum_{a,b\in G} aba^{-1}b^{-1} \in k$, then all commutators are trivial which means that G is abelian. This fact is expressed in the first statement of Theorem 3.3.

The group elements $\{g\}$ that appear in $\{\Lambda, \Lambda\}$ with a nonzero coefficient, (that is $f(g) \neq 0$), are precisely those that generate G'. This fact is expressed in the last part of Theorem 3.3 since $z_2 \leftharpoonup p_g = f(g)g$ for all $g \in G$.

The fact that $d_i||G|$ is a famous property of groups. Frobenius has shown that the counting function f has the form

$$f = \sum_i \frac{|G|}{d_i}\chi_i,$$

meaning that f is a character on G. It can be seen that f and the extensive commutator z_2 are related by:

$$f = \lambda \leftharpoonup dz_2.$$

For general Hopf algebras, Kaplansky's 6th conjecture states that the dimension of any irreducible H-module divides $\dim H$. This conjecture is still open. However, in [7] we refer to the function $f_{com} = \lambda \leftharpoonup dz_2$ and to some other counting functions as distribution functions.

Nilpotent groups can be defined in terms of upper central series of normal subgroups,

$$1 = Z_0(G) \subset Z_1(G) \subset Z_2(G) \subset \cdots \subset Z_t(G) = G,$$

where $Z_1(G) = Z(G)$, the center of the group G, and

$$Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)).$$

Equivalently, it is defined in terms of lower central series of commutators,

$$G \supset G_1 \supset G_2 \supset \cdots \supset 1,$$

where $G_1 = [G, G], G_i = [G, G_{i-1}]$, the ith commutator subgroup.

Hopf commutators on one hand and integrals of certain left coideal subalgebras enable us to define nilpotent Hopf algebras as a natural generalization of nilpotent groups. The generalization of upper central series is not trivial though. While the set $\{Z_i(G)\}$ are all normal subgroups of G, it makes sense to consider quotients of successive subgroups, this is no longer the case for Hopf algebras. Though we can replace normal subgroups with normal left coideal subalgebras, we do not have a meaningful replacement for quotients.

We overcome this problem by using integrals for left coideal subalgebras. Observe that $Z_{i+1}(G)/Z_i(G)$ is the image of $Z_{i+1}(G)$ under the group projection $\pi_i : G \to G/Z_i(G)$. In [8], we showed that if $\pi_i : H \to H/HZ_i^+$ is a Hopf algebra projection, and Λ_{Z_i} is the integral of Z_i, then $\pi_{i|H\Lambda_{Z_i}}$ is an algebra (and H-module) isomorphism. In particular, if $Z_{i+1} \supset Z_i$, then we have an algebra isomorphism

$$\pi_i(Z_{i+1}) \equiv Z_{i+1}\Lambda_{Z_i}.$$

This observation enables us to replace quotients with subalgebras and thus suggest a definition for an upper central series for Hopf algebras as follows:

Definition 3.5. An ascending (upper) series for H is a series of normal left coideal subalgebras

$$k = Z_0 \subset Z_1 \subset \cdots \subset Z_t \subset$$

so that

$$Z_{i+1}\Lambda_{Z_i} = Z(H\Lambda_{Z_i}),$$

for all $1 \leq i \leq t$, where Λ_{Z_i} is the integral of Z_i and $Z(H\Lambda_{Z_i})$ is the center of $H\Lambda_{Z_i}$.

The lower series for H are described via Hopf commutators as follows.

Definition 3.6. A descending lower series for H is a series of iterated commutators,

$$H = N_0 \supset N_1 \supset \cdots \supset N_t \supset,$$

so that $N_0 = H,\ N_i =$ the normal left coideal subalgebra generated by $\{N_{i-1}, \Lambda\}$.

As for groups, when H is nilpotent the first series ends with H while the second one ends with k. They are also interrelated. Based on [7, Th. 3.7] and [8, Prop. 3.8] we have,

Theorem 3.7. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic* 0, *let $\{Z_i\}$ be an ascending central series and $\{N_i\}$ be a descending central series. Then*

$$N_t = k1 \iff Z_t = H.$$

In this case H is **nilpotent** *and we have: $N_{t-i} \subseteq Z_i$ for all $0 \leq i \leq t$.*

Iterated commutators of the extensive commutator z_2 yield another criterion for nilpotency of H. Define an operator $T : Z(H) \longrightarrow Z(H)$ by

$$T(z) = \{z, \Lambda\}.$$

We have:

Proposition 3.8. *(i) The matrix of T with respect to the basis $\{\frac{E_0}{d_0^2}, \ldots, \frac{E_{n-1}}{d_{n-1}^2}\}$ is A, where*

$$A_{ij} = \frac{\langle \chi_i s(\chi_i) s(\chi_j), \Lambda \rangle}{d_j}, \quad 0 \leq i, j \leq n-1. \tag{6}$$

(ii) A has non-negative rational entries and the first column of A has all entries equal 1.

(iii) The first row of A^m is $(1, 0, \ldots, 0)$ for all $m \geq 0$.

(iv) In the first column of A^m we have:

$$(A^m)_{i0} = \sum_j (A^{m-1})_{ij},$$

For all $m > 0$, $0 \leq i \leq n-1$. In particular, The first column of A^m consists of positive rational numbers.

We refer to A as the **Commutator matrix** of H. Observe that this matrix depends only on the structure constants of the ring of characters on H. Eigenvalues of A are related to nilpotancy of H.

Theorem 3.9. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic* 0 *and assume $\chi_i s(\chi_i) \in Z(R(H))$ for each irreducible character χ_i. Then H is nilpotent if and only if its commutator matrix A has eigenvalues $\{1, 0\}$ where the algebraic multiplicity of* 1 *is* 1.

The condition $\chi_i s(\chi_i) \in Z(R(H))$ is satisfied by an abundance of semisimple H. In particular by all quasitriangular Hopf algebras, but also by $H = kG^*$, even though $R(H) = kG$ is not necessarily commutative.

We next define an important family of central iterated commutators .

$$\gamma_0 = \Lambda,\ \gamma_1 = T(\Lambda) = \{\Lambda, \Lambda\},\ \ldots,\ \gamma_m = T^m(\Lambda) = \{\gamma_{m-1}, \Lambda\}. \tag{7}$$

We can describe the central iterated commutators γ_n in terms of the coefficients of A. In particular, they are all positive rational numbers.

Proposition 3.10. *Let γ_m be defined as in* (7) *and the matrix A be defined as in Proposition 3.8. Then*

$$\gamma_m = \sum_i (A^m)_{i0} \frac{E_i}{d_i^2} = \sum_i \left(\sum_j (A^{m-1})_{ij} \right) \frac{E_i}{d_i^2}$$

for all $m \geq 1$. Moreover, the coefficient of each E_i in γ_m is a non-zero rational number, in particular γ_m is invertible.

Proposition 3.10 yields another criterion for nilpotency of H.

Theorem 3.11. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic* 0*, and let γ_t be defined as in* (7)*. Assume $\chi_i s(\chi_i) \in Z(R(H))$ for each irreducible character χ_i. Then H is nilpotent if and only if $\gamma_m = 1$ for some $m \in \mathbb{Z}^+$.*

Its index of nilpotency is the least integer m so that $\gamma_m = 1$.

The last concept we wish to discuss is solvability. On the level of category theory there exists a notion of solvability [10] and it is customary to define solvable Hopf algebras H as those for which $Rep(H)$ is a solvable category. However, this non-intrinsic definition is unsatisfactory as it contradicts our intuition from group theory. Commutative or nilpotent Hopf algebras are not always solvable in this sense [10, Prop. 4.5(ii),Remark 4.6(i)]. We suggested in [8] an intrinsic definition of solvability which is consistent with solvability for finite groups and as desired, commutative or nilpotent Hopf algebras are indeed solvable.

A solvable group has a subnormal series of subgroups. That is,

$$1 = G_0 \subset G_1 \subset \cdots \subset G,$$

so that G_i is normal in G_{i+1}. G is solvable if it has a subnormal series so that G_{i+1}/G_i is an abelian group. Generalizing this definition to Hopf algebras requires an appropriate translation for subnormal series in addition to abelian quotients. This is done using integrals as follows:

Definition 3.12. Let H be a semisimple Hopf algebra. A subnormal series of left coideal subalgebras of H

$$k = N_0 \subset N_1 \subset \cdots \subset N_t = H$$

is a series satisfying:

$$\Lambda_{N_i} \in Z(N_{i+1})$$

for all $0 \leq i \leq t-1$, where Λ_{N_i} is the integral of N_i. H is solvable if for all $a, b \in N_{i+1}$,

$$(a_{\dot{ad}} b)\Lambda_{N_i} = \langle \varepsilon, a \rangle b \Lambda_{N_i}.$$

Example 3.13. For finite groups, the definition of solvable Hopf algebras boils down to the usual definition of solvability for groups.

To see this, let $G_1 \subset G$ and $\Lambda_{G_1} = \frac{1}{|G_1|}\sum_{x\in G_1} x$. If Λ_{G_1} is central in G, then for $g \in G$ we have,

$$\frac{1}{|G_1|}\sum_{x\in G_1} xg = \Lambda_{G_1}g = g\Lambda_{G_1} = \frac{1}{|G_1|}\sum_{x\in G_1} gx.$$

It follows that for $x \in G_1,\ xg = gy$ for some $y \in G_1$. Thus G_1 is normal in G.

We claim now that G/G_1 is abelian if and only if $(a_{\dot{ad}}b)\Lambda_{G_1} = b\Lambda_{G_1}$, for all $a, b \in G$. Indeed, assume $\overline{a}\overline{b} = \overline{b}\overline{a}$ for all $\overline{a}, \overline{b} \in G/G_1$. Then $aba^{-1} = by$ for some $y \in G_1$. This implies that

$$aba^{-1}\sum_{x\in G_1} x = by\sum_{x\in G_1} x = b\sum_{x\in G_1} x.$$

Conversely, if $aba^{-1}\sum_{x\in G_1} x = b\sum_{x\in G_1} x$ then $aba^{-1} = by$ for some $y \in G_1$. This shows our claim.

We showed in [8] that many properties of solvable groups can be generalized to solvable Hopf algebras. In particular, any nilpotent Hopf algebra is solvable. We proved also an analogue of Burnside's p^aq^b theorem for semisimple quasitriangular Hopf algebras.

Theorem 3.14. *Let H be a quasitriangular semisimple Hopf algebra of dimension p^aq^b over a field k of characteristic 0, p, q primes. Then H is solvable.*

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