The Nörlund operator on ℓ^2 generated by the sequence of positive integers is hyponormal

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Abstract

First it is shown that the Nörlund matrix associated with the sequence of positive integers is a coposinormal operator on ℓ^2. This fact then turns out to be useful for showing that this operator is also posinormal and hyponormal. In contrast with the analogous weighted mean matrix result **[6]**, the proof of hyponormality is accomplished without resorting to determinants or Sylvester's criterion.

1 Introduction

Lat $a :\equiv \{a_n\}$ denote a sequence of nonnegative numbers with $a_0 > 0$, and take $A_n :\equiv \sum_{j=0}^{n} a_j > 0$. The Norlund matrix $M_a :\equiv [m_{ij}]_{i,j\geq 0}$ is defined by

$$m_{ij} = \begin{cases} a_{i-j}/A_i & for \quad 0 \leq j \leq i \\ 0 & for \quad j > i. \end{cases}$$

Over the last few decades, these operators, acting on various Banach spaces, have been studied by various authors in a number of papers, including **[1]**, **[2]**, **[3]**, **[7]**, and **[8]**.

If $\mathcal{B}(H)$ denotes the set of all bounded linear operators on a Hilbert space H, then the operator $A \in \mathcal{B}(H)$ is *hyponormal* if

$$< (A^*A - AA^*)f, f > \ \geq \ 0$$

Received by the editors in June 2015 - In revised form in June 2015.
Communicated by G. Godefroy.
2010 *Mathematics Subject Classification* : Primary 47B20.
Key words and phrases : Nörlund matrix, posinormal operator, hyponormal operator.

for all $f \in H$. The aim of this note is to show that the Nörlund matrix associated with the sequence of positive integers is a hyponormal operator on ℓ^2. This example is not easily seen to be hyponormal directly from the definition, but, fortunately, posinormality turns out to be a useful tool for achieving our goal. The operator $A \in \mathcal{B}(H)$ is said to be *posinormal* (see [**5**]) if

$$AA^* = A^*PA$$

for some positive operator $P \in \mathcal{B}(H)$, called the *interrupter*, and A is *coposinormal* if A^* is posinormal. The following proposition (see [**4**]) contains some key facts about posinormal operators.

Proposition 1.1. *If $A \in \mathcal{B}(H)$, then the following are equivalent.*

(a) There exists a nonnegative $P \in \mathcal{B}(H)$ such that $AA^* = A^*PA$ (i.e., A is posinormal).

(b) There exists a nonnegative $P \in \mathcal{B}(H)$ such that $AA^* \leq A^*PA$.

(c) There exists a nonnegative $\alpha \in \mathbb{R}$ such that $AA^* \leq \alpha^2 A^*A$.

(d) $Ran(A) \subseteq Ran(A^*)$, where $Ran(A) = \{g \in H : g = Af$ for some $f \in H\}$.

(e) There exists a $B \in \mathcal{B}(H)$ such that $A = A^*B$.

Moreover, each of the above assertions implies the following one.

(f) $Ker(A) \subseteq Ker(A^*)$, where $Ker(A) = \{f \in H : Af = 0\}$.

Furthermore, if $Ran(A)$ is closed, then these six assertions are all equivalent.

Proof. See [**4**]. ∎

Note that A is hyponormal if part (c) of the proposition is satisfied for $\alpha = 1$, so hyponormal operators are necessarily posinormal, although they need not be coposinormal – e.g., the unilateral shift $U \in \mathcal{B}(\ell^2)$ is hyponormal but not coposinormal. However, for the Nörlund operator M_a on ℓ^2 determined by the sequence of positive integers, coposinormality can be demonstrated more readily than posinormality can, and that demonstration will become our first step toward the goal of proving hyponormality for this operator.

2 Main Result

Under consideration here will be the Nörlund matrix $M :\equiv M_a$ associated with the sequence $a :\equiv \{a_n\}$ given by

$$a_n = n + 1$$

for all n, so the entries m_{ij} of M are given by

$$m_{ij} = \begin{cases} \frac{2(i+1-j)}{(i+1)(i+2)} & for \quad 0 \leq j \leq i \\ 0 & for \quad j > i. \end{cases}$$

An alternative equivalent formulation of part (e) of Proposition 1.1 provides the starting place for our computations.

(e′) There exists a $B \in \mathcal{B}(H)$ such that $BA = A^*$.

For the operator M currently under consideration, it is a more straightforward computation initially to produce a candidate for an operator B such that $BM^* = M$, which gives coposinormality, rather than trying to immediately produce a candidate satisfying $BM = M^*$, as required for posinormality. An additional benefit of this approach – determining coposinormality – will come in the form of a corollary presented at the end of this section.

Lemma 2.1. *If M is the Nörlund matrix determined by the sequence of positive integers and $B :\equiv [b_{ij}]_{i,j\geq 0}$ is the matrix defined by*

$$b_{ij} = \begin{cases} \frac{2(i+1-3j)}{(i+1)(i+2)} & if \quad j \leq i+1 \\ 1 & if \quad j = i+2 \\ 0 & if \quad j > i+2 \end{cases},$$

then

(a) B is a bounded linear operator on ℓ^2,

(b) $BM^* = M$,

(c) $BB^* = diag\{\frac{(n+3)(n+4)}{(n+1)(n+2)} : n \geq 0\}$,

(d) B is injective, and

(e) B is invertible.

Proof.

(a) First note that for $j \leq i$, $b_{ij} \in [-4/(i+2), 2/(i+2)]$ for all i, so the lower triangular part T of matrix B is a bounded operator on ℓ^2. If U denotes the unilateral shift and W is the weighted shift with weight sequence $\{4/(n+2) : n \geq 0\}$, then

$$B = T - W^* + (U^2)^*,$$

so $B \in \mathcal{B}(\ell^2)$.

(b) For $j \leq i+1$, the (i,j)-entry in the product matrix BM^* is given by

$$\sum_{k=0}^{j} \frac{2(i+1-3k)}{(i+1)(i+2)} \cdot \frac{2(j+1-k)}{(j+1)(j+2)} = \frac{2(i+1-j)}{(i+1)(i+2)}.$$

For $j \geq i+2$, the (i,j)-entry of BM^* is given by

$$\sum_{k=0}^{i+1} \frac{2(i+1-3k)}{(i+1)(i+2)} \cdot \frac{2(j+1-k)}{(j+1)(j+2)} + 1 \cdot \frac{2[j+1-(i+2)]}{(j+1)(j+2)} = 0.$$

(c) For $i = j$, the (i,j)-entry in the product matrix BB^* is given by

$$\sum_{k=0}^{i+1} \left\{ \frac{2(i+1-3k)}{(i+1)(i+2)} \right\}^2 + 1 = \frac{(i+3)(i+4)}{(i+1)(i+2)}.$$

For $j \geq i+1$, the (i,j)-entry of BB^* is given by

$$\sum_{k=0}^{i+1} \frac{2(i+1-3k)}{(i+1)(i+2)} \cdot \frac{2(j+1-3k)}{(j+1)(j+2)} + 1 \cdot \frac{2[j+1-3(i+2)]}{(j+1)(j+2)} = 0.$$

For $i \geq j+1$, the computation is similar to the one immediately above, so it is left to the reader.

(d) Suppose $x :\equiv < x_0, x_1, x_2, x_3, ..., x_n, ... >^T \in Ker(B)$. An induction argument shows that

$$x_{n+2} = (n+2)x_1 - (n+1)x_0$$

for all $n \geq 0$. Therefore, $Ker(B)$ consists of linear combinations of the form

$$x_0 < 1,\ 0, -1, -2, -3, -4,\ ...\ , -(n+1),\ .... >^T$$

$$+\ \ x_1 < 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ ...\ ,\ n+2,\ .... >^T,$$

which can belong to ℓ^2 only when $x_0 = x_1 = 0$. Therefore $Ker(B) = \{0\}$, so B is injective.

(e) Using part (c), we find that

$$B(B^* diag\{(n+1)(n+2)/(n+3)(n+4) : n \geq 0\}) = I,$$

so B is right-invertible. From part (d), we know that B is also left-invertible. Therefore B is invertible, and

$$B^{-1} = B^* diag\{(n+1)(n+2)/(n+3)(n+4) : n \geq 0\}. \qquad \blacksquare$$

Before continuing on, we note that the range of M contains all the e_n's from the standard orthonormal basis for ℓ^2 since

$$M[((n+1)(n+2)/2)(e_n - 2e_{n+1} + e_{n+2})] = e_n.$$

Theorem 2.2. *The Nörlund matrix M associated with the sequence $a_n = n+1$ is a coposinormal, posinormal, and hyponormal operator on ℓ^2.*

Proof. By Lemma 2.1 (b), M is coposinormal. It then follows from part (e) of the lemma that M is also posinormal and

$$M^* = B^{-1}M = B^*DM$$

where $D :\equiv diag\{(n+1)(n+2)/(n+3)(n+4) : n \geq 0\}$. Therefore,

$$MM^* = (M^*DB)(B^*DM) = M^*DM.$$

Consequently,

$$\langle (M^*M - MM^*)f, f \rangle = \langle (M^*M - M^*DM)f, f \rangle = \langle (I-D)Mf, Mf \rangle \geq 0$$

for all $f \in \ell^2$, which means that M is hyponormal. $\blacksquare$

Corollary 2.3. *If M is the Nörlund operator on ℓ^2 determined by the sequence $a_n = n+1$, then both M and M^* are injective and have dense range with*

$$Ran(M) = Ran(M^*).$$

Proof. Since M is both posinormal and coposinormal, it follows from Proposition 1.1 that $Ker(M) = Ker(M^*)$ and $Ran(M) = Ran(M^*)$. It is easy to see that $Ker(M) = \{0\}$. Consequently, both M and M^* are one-to-one, and both have dense range. ∎

3 Concluding remarks

Will the procedure above be successful with other operators? In trying to apply Proposition 1.1 (e′) to the adjoint of the Nörlund operator $M_{odd} \in \mathcal{B}(\ell^2)$ associated with the sequence of odd positive integers

$$a_n = 2n+1,$$

one finds that the first row of the matrix B is required to be

$$< 1, -3, 4, -4, 4, -4, 4, -4, 4..., -4, 4, ... >,$$

so $B \notin \mathcal{B}(\ell^2)$, which means that M_{odd} cannot be coposinormal. Similarly, if the same procedure is applied to the adjoint of the Nörlund operator $M_{squares} \in \mathcal{B}(\ell^2)$ associated with the sequence of squares of the positive integers

$$a_n = (n+1)^2,$$

the first row of B is then required to be

$$< 1, -4, 7, -8, 8, -8, 8, -8, 8, ..., -8, 8, ... >,$$

so once again $B \notin \mathcal{B}(\ell^2)$, meaning that $M_{squares}$ cannot be coposinormal either. Of course, this does not settle the question of hyponormality for these two operators, but it does make it clear that a different approach would be needed.

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