# Differences of Quermass- and Dual Quermassintegrals of Blaschke-Minkowski and Radial Blaschke-Minkowski Homomorphisms *

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**Abstract**

In this article, we establish Minkowski, Brunn-Minkowski and Aleksandrov-Fenchel type inequalities for differences of quermass- and dual quermassintegrals of mixed Blaschke-Minkowski and mixed radial Blaschke-Minkowski homomorphisms.

## 1 Introduction and Main Results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$ and let $\mathcal{S}^n$ denote the set of star bodies (compact sets, starshaped with respect to the origin with continuous radial functions) in $\mathbb{R}^n$. Let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the $n$-dimensional volume of a body $K$. For the standard unit ball $B$ in $\mathbb{R}^n$, we write $\omega_n = V(B)$ for its volume. Moreover, $W_i(K)$ and $\widetilde{W}_i(L)$ denote the quermassintegrals of $K \in \mathcal{K}^n$ and the dual quermassintegrals of $L \in \mathcal{S}^n$, respectively.

The notion of projection bodies goes back to Minkowski and was surveyed in Bolker's article [6]. A number of important results regarding this classical notion

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and its generalizations were obtained in recent years (see [1, 3, 5, 7, 9, 11, 16, 26, 28, 36, 39, 40, 53]) and the books (see [12, 25, 41]). Moreover, projection bodies were extended in other ways, e.g., to mixed projection bodies by Lutwak (see [28, 29]). Recently, Lutwak, Yang and Zhang introduced for $p \geq 1$ $L_p$-projection bodies and obtained a series of striking results [32], see also [19, 33, 34, 37, 46-52]. Motivated by well known properties of the projection body operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$, Schuster [42] introduced the notion of Blaschke-Minkowski homomorphisms:

A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a) $\Phi$ is continuous.

(b) $\Phi$ is Blaschke-Minkowski additive, i.e., for all $K, L \in \mathcal{K}^n$,

$$\Phi(K\#L) = \Phi K + \Phi L.$$

(c) $\Phi$ intertwines rotations, i.e., for all $K \in \mathcal{K}^n$ and $\vartheta \in SO(n)$,

$$\Phi(\vartheta K) = \vartheta \Phi K.$$

Here $\Phi K + \Phi L$ denotes the Minkowski sum, see (2.3), of $\Phi K$ and $\Phi L$ and $K\#L$ denotes the Blaschke sum of the convex bodies $K$ and $L$, see (2.6). $SO(n)$ is the group of rotations in $n$ dimensions.

Schuster also obtained the following result which generalizes the notion of mixed projection bodies from [42].

**Theorem 1.A.** *There is a continuous operator*

$$\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n,$$

*symmetric in its arguments such that, for $K_1, \cdots, K_m \in \mathcal{K}^n$ and $\lambda_1, \cdots, \lambda_m \geq 0$,*

$$\Phi(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \cdots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Phi(K_{i_1}, \cdots, K_{i_{n-1}}). \tag{1.1}$$

The operator $\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n$ is called the mixed Blaschke-Minkowski homomorphism induced by $\Phi$. If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$, we write $\Phi_i K$ for $\Phi(K, \cdots, K, B, \cdots, B)$. For $0 \leq i < n$, we write $\Phi_i(K, L)$ for $\Phi(\underbrace{K, \cdots, K}_{n-i-1}, \underbrace{L, \cdots, L}_{i})$. Note that $\Phi_0 K = \Phi K$.

Schuster [42] also established the following Minkowski, Brunn-Minkowski and Aleksandrov-Fenchel type inequalities for mixed Blaschke-Minkowski homomorphisms.

**Theorem 1.B.** *If $K, L \in \mathcal{K}^n$ and $0 \leq i \leq n-1$, $1 \leq j < n-1$, then*

$$W_i(\Phi_j(K, L))^{n-1} \geq W_i(\Phi K)^{n-j-1} W_i(\Phi L)^j \tag{1.2}$$

*with equality if and only if $K$ and $L$ are homothetic.*

**Theorem 1.C.** *If $K, L \in \mathcal{K}^n$ and $0 \le i \le n-1$, $0 \le j < n-2$, then*

$$W_i(\Phi_j(K+L))^{\frac{1}{(n-i)(n-j-1)}} \ge W_i(\Phi_j K)^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j L)^{\frac{1}{(n-i)(n-j-1)}} \tag{1.3}$$

*with equality if and only if K and L are homothetic.*

**Theorem 1.D.** *If $K_1, \cdots, K_n \in \mathcal{K}^n$ and $1 \le r \le n-1$, then*

$$W_i(\Phi(K_1, \cdots, K_{n-1}))^r \ge \prod_{j=1}^{r} W_i(\Phi(\underbrace{K_j, \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1})). \tag{1.4}$$

For generalizations of these results (in particular, of Theorem 1.C) to more general Minkowski valuations, we refer to [2, 8, 38, 45].

Intersection bodies first appeared in a paper by Busemann (see [10]) but were explicitly defined and named only later by Lutwak (see [30]). They turned out to be critical for the solution of the Busemann-Petty problem (see [13-15, 21-23, 54] and [17, 18, 27] for recent $L_p$-generalizations of intersection bodies). In 2006, radial Blaschke-Minkowski homomorphisms were introduced by Schuster [42], the most prominent example being the intersection body operator.

A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

($a_1$) $\Psi$ is continuous.

($b_1$) For all $K, L \in \mathcal{S}^n$,

$$\Psi(K \hat{+} L) = \Psi K \tilde{+} \Psi L.$$

($c_1$) For all $K \in \mathcal{S}^n$ and $\vartheta \in SO(n)$,

$$\Psi(\vartheta K) = \vartheta \Psi K.$$

Here $\Psi K \tilde{+} \Psi L$ is the radial Minkowski sum, see (2.8), of $\Psi K$ and $\Psi L$ and $K \hat{+} L$ denotes the radial Blaschke sum of the star bodies $K$ and $L$, see (2.11).

**Theorem 1.A$_*^{[42]}$.** *There is a continuous operator*

$$\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n,$$

*symmetric in its arguments such that, for $L_1, \cdots, L_m \in \mathcal{S}^n$ and $\lambda_1, \cdots, \lambda_m \ge 0$,*

$$\Psi(\lambda_1 L_1 \tilde{+} \cdots \tilde{+} \lambda_m L_m) = \widetilde{\sum_{i_1, \cdots, i_{n-1}}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Psi(L_{i_1}, \cdots, L_{i_{n-1}}). \tag{1.5}$$

Theorem 1.A$_*$ generalizes the notion of radial Blaschke-Minkowski homomorphisms. We call $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ the mixed radial Blaschke-Minkowski homomorphism induced by $\Psi$. Mixed radial Blaschke-Minkowski homomorphisms were first studied in the articles (see [43, 44]). If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$, we write $\Psi_i K$ for $\Psi(K, \cdots, K, B, \cdots, B)$. For $0 \le i < n$, we write $\Psi_i(K, L)$ for $\Psi(\underbrace{K, \cdots, K}_{n-i-1}, \underbrace{L, \cdots, L}_{i})$. Note that $\Psi_0 K = \Psi K$.

For mixed radial Blaschke-Minkowski homomorphisms, Schuster [42] established the following dual Minkowski, Brunn-Minkowski and Aleksandrov-Fenchel type inequalities.

**Theorem 1.B$_*$.** *If $K, L \in \mathcal{S}^n$ and $0 \le i \le n-1$, $1 \le j < n-1$, then*

$$\widetilde{W}_i(\Psi_j(K,L))^{n-1} \le \widetilde{W}_i(\Psi K)^{n-j-1}\widetilde{W}_i(\Psi L)^j \tag{1.6}$$

*with equality if and only if K and L are dilates.*

**Theorem 1.C$_*$.** *If $K, L \in \mathcal{S}^n$ and $0 \le i \le n-1$, $0 \le j < n-2$, then*

$$\widetilde{W}_i(\Psi_j(K\tilde{+}L))^{\frac{1}{(n-i)(n-j-1)}} \le \widetilde{W}_i(\Psi_j K)^{\frac{1}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j L)^{\frac{1}{(n-i)(n-j-1)}} \tag{1.7}$$

*with equality if and only if K and L are dilates.*

**Theorem 1.D$_*$.** *If $K_1, \cdots, K_n \in \mathcal{S}^n$ and $1 \le r \le n-1$, then*

$$\widetilde{W}_i(\Psi(K_1, \cdots, K_{n-1}))^r \le \prod_{j=1}^{r} \widetilde{W}_i(\Psi(\underbrace{K_j, \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1})) \tag{1.8}$$

*with equality if and only if $K_1, \cdots, K_r$ are dilates.*

The aim of this paper is to establish the following new Minkowski, Brunn-Minkowski and Aleksandrov-Fenchel type inequalities for difference of quermass- and dual quermassintegrals of mixed Blaschke-Minkowski and mixed radial Blaschke-Minkowski homomorphisms. First volume difference inequalities were established by Leng [24]. Since this seminal paper, inequalities for differences of geometric functionals have become the focus of increased attention (see e.g., [35, 56-59]).

**Theorem 1.1.** *Let $K, L \in \mathcal{K}^n$, $D, D' \in \mathcal{S}^n$, $D \subseteq K$, $D' \subseteq L$, and let $D'$ be a dilate of $D$. Then for $0 \le i \le n-1$, $1 \le j < n-1$,*

$$\begin{aligned} &\left[W_i(\Phi_j(K,L)) - \widetilde{W}_i(\Psi_j(D,D'))\right]^{n-1} \\ &\ge \left[W_i(\Phi K) - \widetilde{W}_i(\Psi D)\right]^{n-j-1}\left[W_i(\Phi L) - \widetilde{W}_i(\Psi D')\right]^j \end{aligned}$$

*with equality if and only if K and L are homothetic and $W_i(\Phi K)/\widetilde{W}_i(\Psi D) = W_i(\Phi L)/\widetilde{W}_i(\Psi D')$.*

**Theorem 1.2.** *Let $K, L \in \mathcal{K}^n$, $D, D' \in \mathcal{S}^n$, $D \subseteq K$, $D' \subseteq L$, and let $D'$ be a dilate of $D$. Then for $0 \le i < n$, $0 \le j < n-2$,*

$$\begin{aligned} &\left[W_i(\Phi_j(K+L)) - \widetilde{W}_i(\Psi_j(D\tilde{+}D'))\right]^{\frac{1}{(n-i)(n-j-1)}} \\ &\ge \left[W_i(\Phi_j K) - \widetilde{W}_i(\Psi_j D)\right]^{\frac{1}{(n-i)(n-j-1)}} + \left[W_i(\Phi_j L) - \widetilde{W}_i(\Psi_j D')\right]^{\frac{1}{(n-i)(n-j-1)}} \end{aligned}$$

*with equality if and only if $K$ and $L$ are homothetic and $W_i(\Phi_j K)/\widetilde{W}_i(\Psi_j D) = W_i(\Phi_j L)/\widetilde{W}_i(\Psi_j D')$.*

**Theorem 1.3.** *Let $K_i \in \mathcal{K}^n$, $D_i \in \mathcal{S}^n$ and $D_i \subseteq K_i$ $(i = 1, \cdots, n-1)$. If the bodies $D_j$ $(j = 1, \cdots, r)$ are dilates of each other, then for all $1 \leq r \leq n-1$,*

$$\Big[W_i(\Phi(K_1, \cdots, K_{n-1})) - \widetilde{W}_i(\Psi(D_1, \cdots, D_{n-1}))\Big]^r$$
$$\geq \prod_{j=1}^{r} \Bigg[ W_i(\Phi(\underbrace{K_j, \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1})) - \widetilde{W}_i(\Psi(\underbrace{D_j, \cdots, D_j}_{r}, D_{r+1}, \cdots, D_{n-1})) \Bigg].$$

## 2 Preliminaries

### 2.1 Support functions and radial functions

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, \infty)$, is defined by (see[12, 41])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \ \ x \in \mathbb{R}^n, \tag{2.1}$$

where $x \cdot y$ denotes the standard inner product of $x$ and $y$.

If $K$ is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see[12, 41])

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, \ \ u \in S^{n-1}. \tag{2.2}$$

If $\rho_K$ is continuous and positive, then $K$ will be called a star body. Two star bodies $K$, $L$ are said to be dilates (of one another) if $\rho_K(u) \diagup \rho_L(u)$ is independent of $u \in S^{n-1}$.

### 2.2 Mixed volumes

Here, we collect some basic notions and notations from the Brunn-Minkowski Theory that is needed in the proofs of our main theorems (see, e.g, the books [12] and [41]).

For $K_1, K_2 \in \mathcal{K}^n$ and $\lambda_1, \lambda_2 \geq 0$ (not both zero), the support function of the Minkowski linear combination $\lambda_1 K_1 + \lambda_2 K_2$ is

$$h(\lambda_1 K_1 + \lambda_2 K_2, \cdot) = \lambda_1 h(K_1, \cdot) + \lambda_2 h(K_2, \cdot). \tag{2.3}$$

The volume of a Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$ of convex bodies $K_1, \cdots, K_m$ can be expressed in the form

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \cdots, i_n} V(K_{i_1}, \cdots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}. \tag{2.4}$$

The coefficients $V(K_{i_1},\cdots,K_{i_n})$ are called mixed volumes of $K_{i_1},\cdots,K_{i_n}$. These functions are nonnegative, symmetric and translation invariant. Moreover, they are monotone (with respect to set inclusion), multilinear with respect to Minkowski addition and their diagonal form is ordinary volume, i.e., $V(K,\cdots,K) = V(K)$.

We denote by $V_i(K,L)$ the mixed volume $V(K,\cdots,K,L,\cdots,L)$, where $K$ appears $n-i$ times and $L$ appears $i$ times. For $0\le i\le n-1$, write $W_i(K,L)$ for the mixed volume $V(K,\cdots,K,B,\cdots,B,L)$, where $K$ appears $n-i-1$ times and the unit ball $B$ appears $i$ times. The mixed volume $W_i(K,K)$ will be written as $W_i(K)$ and is called the $i$th quermassintegrals of $K$.

For $K_1,\cdots,K_{n-1}\in\mathcal{K}^n$, $S(K_1,\cdots,K_{n-1},\cdot)$ is the Borel measure on $S^{n-1}$ called the mixed surface area measure of $K_1,\cdots,K_{n-1}$, uniquely determined by the property that for each $K\in\mathcal{K}^n$,

$$V(K,K_1,\cdots,K_{n-1}) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS(K_1,\cdots,K_{n-1},u). \tag{2.5}$$

The measures $S_j(K,\cdot) = S(K,\cdots,K,B,\cdots,B,\cdot)$, where $K$ appears $j$ times and $B$ appears $n-j-1$ times, are called the surface area measures of order $j$ of $K$. If $j=n-1$, then we write $S(K,\cdot)$ for $S_{n-1}(K,\cdot)$. The measure $S(K,\cdot)$ is called the surface area measure of $K$.

If $K_1,K_2\in\mathcal{K}^n$ and $\lambda_1,\lambda_2\ge 0$ (not both zero), then there exists a convex body $\lambda_1\cdot K_1\#\lambda_2\cdot K_2$, such that

$$S(\lambda_1\cdot K_1\#\lambda_2\cdot K_2,\cdot) = \lambda_1 S(K_1,\cdot)+\lambda_2 S(K_2,\cdot). \tag{2.6}$$

This addition and scalar multiplication are called Blaschke addition and scalar multiplication. For $K\in\mathcal{K}^n$ and $\lambda\ge 0$, we have $\lambda\cdot K = \lambda^{\frac{1}{n-1}}K$.

One of the most general and fundamental inequalities for mixed volumes is the Aleksandrov-Fenchel inequality: If $K_1,\cdots,K_n\in\mathcal{K}^n$ and $1\le m\le n$, then

$$V(K_1,\cdots,K_n)^m \ge \prod_{j=1}^{m} V(\underbrace{K_j,\cdots,K_j}_{m},K_{m+1},\cdots,K_n). \tag{2.7}$$

### 2.3 Dual mixed volumes

In the following we summarize some results from the dual Brunn-Minkowski Theory (see [31]). Moreover, as an application of the dual Aleksandrov-Fenchel inequality (see below), we obtain an inequality (Theorem 2.1) for mixed radial Blaschke-Minkowski homomorphisms.

For $L_1,L_2\in\mathcal{S}^n$ and $\lambda_1,\lambda_2\ge 0$ (not both zero), the radial Minkowski linear combination $\lambda_1 L_1\tilde{+}\lambda_2 L_2$ is the star body defined by

$$\rho(\lambda_1 L_1\tilde{+}\lambda_2 L_2,\cdot) = \lambda_1\rho(L_1,\cdot)+\lambda_2\rho(L_2,\cdot). \tag{2.8}$$

The volume of a radial Minkowski linear combination $\lambda_1 L_1\tilde{+}\cdots\tilde{+}\lambda_m L_m$ of star bodies $L_1,\cdots,L_m$ can be expressed as a homogeneous polynomial of degree $n$:

$$V(\lambda_1 L_1\tilde{+}\cdots\tilde{+}\lambda_m L_m) = \sum_{i_1,\cdots,i_n}\widetilde{V}(L_{i_1},\cdots,L_{i_n})\lambda_{i_1}\cdots\lambda_{i_n}.$$

The coefficients $\widetilde{V}(L_{i_1}, \cdots, L_{i_n})$ are called dual mixed volumes of $L_{i_1}, \cdots, L_{i_n}$. They are nonnegative, symmetric and monotone (with respect to set inclusion). They are also multilinear with respect to radial Minkowski addition and $\widetilde{V}(L, \cdots, L) = V(L)$. The following integral representation of dual mixed volumes holds:

$$\widetilde{V}(L_1, \cdots, L_n) = \frac{1}{n} \int_{S^{n-1}} \rho(L_1, u) \cdots \rho(L_n, u) du, \tag{2.9}$$

where $du$ is the spherical Lebesgue measure on $S^{n-1}$. The definitions of $\widetilde{V}_i(K, L)$, $\widetilde{W}_i(K, L)$, etc. are analogous to the ones for mixed volumes in Section 2.2. We note that $\widetilde{W}_i(K)$ is called the *i*th dual quermassintegral of $K \in \mathcal{S}^n$. A slight extension of the notation $\widetilde{V}_i(K, L)$ for $r \in \mathbb{R}$ is

$$\widetilde{V}_r(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho^{n-r}(K, u) \rho^r(L, u) du. \tag{2.10}$$

Obviously, we have $\widetilde{V}_r(L, L) = V(L)$ for every $r \in \mathbb{R}$ and every $L \in \mathcal{S}^n$.

If $\lambda_1, \lambda_2 \geq 0$ (not both zero), then the radial Blaschke linear combination $\lambda_1 \circ L_1 \hat{+} \lambda_2 \circ L_2$ of the star bodies $L_1$ and $L_2$ is the star body whose radial function satisfies

$$\rho^{n-1}(\lambda_1 \circ L_1 \hat{+} \lambda_2 \circ L_2, \cdot) = \lambda_1 \rho^{n-1}(L_1, \cdot) + \lambda_2 \rho^{n-1}(L_2, \cdot). \tag{2.11}$$

This addition and scalar multiplication are called radial Blaschke addition and scalar multiplication. For $L \in \mathcal{S}^n$ and $\lambda \geq 0$, the radial Blaschke and the usual scalar multiplication are related by $\lambda \circ L = \lambda^{\frac{1}{n-1}} L$.

The most general inequality for dual mixed volumes is the dual Aleksandrov-Fenchel inequality: If $L_1, \cdots, L_n \in \mathcal{S}^n$ and $1 \leq m \leq n$, then

$$\widetilde{V}(L_1, \cdots, L_n)^m \leq \prod_{j=1}^{m} \widetilde{V}(\underbrace{L_j, \cdots, L_j}_{m}, L_{m+1}, \cdots, L_n) \tag{2.12}$$

with equality if and only if $L_1, \cdots, L_m$ are dilates.

A special case of inequality (2.12) is the dual Minkowski inequality: If $K, L \in \mathcal{S}^n$, then

$$\widetilde{V}_1(K, L) \leq V(K)^{n-1} V(L) \tag{2.13}$$

with equality if and only if $K$ and $L$ are dilates. A more general version of the dual Minkowski inequality is: If $0 \leq i \leq n-2$, then

$$\widetilde{W}_i(K, L)^{n-i} \leq \widetilde{W}_i(K)^{n-i-1} \widetilde{W}_i(L) \tag{2.14}$$

with equality if and only if $K$ and $L$ are dilates.

We now use the dual Aleksandrov-Fenchel inequality to obtain the following inequality for mixed radial Blaschke-Minkowski homomorphims.

**Theorem 2.1.** *Let $K \in \mathcal{S}^n$ and $0 \leq i < j < n-1$. If $0 \leq m < n$, then*

$$\widetilde{W}_m(\Psi_j K)^{n-i-1} \leq r_{\Psi}^{(n-m)(j-i)} \omega_n^{j-i} \widetilde{W}_m(\Psi_i K)^{n-j-1} \tag{2.15}$$

*with equality if and only if $\Psi_i K$ and $\Psi_j K$ are both balls and dilates of each other.*

Here $r_\Psi$ denotes the radius of the ball $\Psi B$. For the proof of Theorem 2.1, we require the following lemmas.

**Lemma 2.1**[42]**.** *If $K, L \in \mathcal{S}^n$ and $0 \le i, j \le n-2$, then*

$$\widetilde{W}_i(K, \Psi_j L) = \widetilde{W}_j(L, \Psi_i K). \tag{2.16}$$

**Lemma 2.2**[42]**.** *If $K \in \mathcal{S}^n$ and $0 \le i \le n-2$, then*

$$\widetilde{W}_{n-1}(\Psi_i K) = r_\Psi \widetilde{W}_{i+1}(K). \tag{2.17}$$

If we take $L_1 = \cdots = L_{n-j} = L$, $L_{n-j+1} = \cdots = L_n = B$ in (2.12), we directly obtain

**Lemma 2.3.** *If $L \in \mathcal{S}^n$ and $0 \le i < j < n$, then*

$$\widetilde{W}_j(L)^{n-i} \le \omega_n^{j-i} \widetilde{W}_i(L)^{n-j}, \tag{2.18}$$

*with equality if and only if $K$ is a ball.*

*Proof of Theorem 2.1.* The case $m = n-1$ of inequality (2.15) follows from a combination of Lemma 2.2 and Lemma 2.3. Therefore, let $m < n-1$ and $Q \in \mathcal{S}^n$. From (2.16), we have

$$\widetilde{W}_m(Q, \Psi_j K) = \widetilde{W}_j(K, \Psi_m Q). \tag{2.19}$$

Thus from the dual Aleksandrov-Fenchel inequality (2.12), it follows that

$$\begin{aligned}\widetilde{W}_j(K, \Psi_m Q)^{n-i-1} &= \widetilde{V}(\underbrace{K, \cdots, K}_{n-j-1}, \underbrace{B, \cdots, B}_{j-i}, \underbrace{B, \cdots, B}_{i}, \Psi_m Q)^{n-i-1} \\ &\le \widetilde{W}_{n-1}(\Psi_m Q)^{j-i} \widetilde{W}_i(K, \Psi_m Q)^{n-j-1}.\end{aligned} \tag{2.20}$$

From Lemma 2.2 and Lemma 2.3, we deduce

$$\widetilde{W}_{n-1}(\Psi_m Q)^{n-m} = r_\Psi^{n-m} \widetilde{W}_{m+1}(Q)^{n-m} \le r_\Psi^{n-m} \omega_n \widetilde{W}_m(Q)^{n-m-1} \tag{2.21}$$

with equality if and only if $Q$ is a ball. And from Lemma 2.1 and inequality (2.14), we obtain

$$\widetilde{W}_i(K, \Psi_m Q)^{n-m} = \widetilde{W}_m(Q, \Psi_i K)^{n-m} \le \widetilde{W}_m(Q)^{n-m-1} \widetilde{W}_m(\Psi_i K) \tag{2.22}$$

with equality if and only if $Q$ and $\Psi_i K$ are dilates. Therefore, by (2.16) and inequalities (2.20), (2.21) and (2.22), we obtain

$$\widetilde{W}_m(Q, \Psi_j K)^{n-i-1} \le r_\Psi^{j-i} \omega_n^{\frac{j-i}{n-m}} \widetilde{W}_m(Q)^{\frac{(n-i-1)(n-m-1)}{n-m}} \widetilde{W}_m(\Psi_i K)^{\frac{n-j-1}{n-m}}. \tag{2.23}$$

Now take $\Psi_j K$ for $Q$ in (2.23) to get the desired inequality (2.15).

From the equality conditions of inequalities (2.21) and (2.22), we see that equality holds in (2.15) if and only if $\Psi_i K$ and $\Psi_j K$ are both balls that are dilates of each other. ■

The mixed intersection body of $K_1, \cdots, K_{n-1} \in \mathcal{S}^n$, $I(K_1, \cdots, K_{n-1})$, was given by (see [55])

$$\rho(I(K_1, \cdots, K_{n-1}), u) = \tilde{v}(K_1 \cap u^{\perp}, \cdots, K_{n-1} \cap u^{\perp}),$$

where $\tilde{v}$ denotes $(n-1)$-dimensional dual mixed volume and $K \cap u^{\perp}$ denotes the intersection of $K \in \mathcal{S}^n$ with the subspace $u^{\perp}$ that passes through the origin and is orthogonal to $u$. Since the mixed intersection body operator $I : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ is a mixed radial Blaschke-Minkowski homomorphism, we obtain from Theorem 2.1 the following result.

**Corollary 2.1.** *Let $K \in \mathcal{S}^n$ and $0 \le i < j < n-1$. If $0 \le m < n$, then*

$$\widetilde{W}_m(I_j K)^{n-i-1} \le \omega_{n-1}^{(n-m)(j-i)} \omega_n^{j-i} \widetilde{W}_m(I_i K)^{n-j-1}$$

*with equality if and only if $K$ is a ball.*

## 3 Proofs of the Main Results

In this section, we complete the proofs of Theorems 1.1-1.3. For the proof of Theorem 1.1, we require the following lemma.

**Lemma 3.1**[57]. *Let $a, b, c, d > 0$, $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha + \beta = 1$. If $a > b$ and $c > d$, then*

$$a^{\alpha} c^{\beta} - b^{\alpha} d^{\beta} \ge (a-b)^{\alpha} (c-d)^{\beta} \tag{3.1}$$

*with equality if and only if $a/b = c/d$.*

*Proof of Theorem 1.1.* From inequality (1.2), we have for $0 \le i \le n-1$ and $1 \le j < n-1$,

$$W_i(\Phi_j(K, L)) \ge W_i(\Phi K)^{\frac{n-j-1}{n-1}} W_i(\Phi L)^{\frac{j}{n-1}} \tag{3.2}$$

with equality if and only if $K$ and $L$ are homothetic. Since $D'$ and $D$ are dilates, it follows from inequality (1.6) that

$$\widetilde{W}_i(\Psi_j(D, D')) = \widetilde{W}_i(\Psi D)^{\frac{n-j-1}{n-1}} \widetilde{W}_i\left(\Psi D'\right)^{\frac{j}{n-1}}. \tag{3.3}$$

Combining (3.2) with (3.3), and using (3.1) we obtain

$$\begin{aligned} &W_i(\Phi_j(K, L)) - \widetilde{W}_i(\Psi_j(D, D')) \\ &\quad \ge W_i(\Phi K)^{\frac{n-j-1}{n-1}} W_i(\Phi L)^{\frac{j}{n-1}} - \widetilde{W}_i(\Psi D)^{\frac{n-j-1}{n-1}} \widetilde{W}_i\left(\Psi D'\right)^{\frac{j}{n-1}} \\ &\qquad \ge \left[W_i(\Phi K) - \widetilde{W}_i(\Psi D)\right]^{\frac{n-j-1}{n-1}} \left[W_i(\Phi L) - \widetilde{W}_i(\Psi D')\right]^{\frac{j}{n-1}}, \end{aligned}$$

i.e.,

$$[W_i(\Phi_j(K,L)) - \widetilde{W}_i(\Psi_j(D,D'))]^{n-1} \geq \left[W_i(\Phi K) - \widetilde{W}_i(\Psi D)\right]^{n-j-1}\left[W_i(\Phi L) - \widetilde{W}_i(\Psi D')\right]^{j}. \quad (3.4)$$

According to the equality conditions of inequalities (3.1) and (3.2), we see that equality holds in (3.4) if and only if $K$ and $L$ are homothetic and $W_i(\Phi K)/\widetilde{W}_i(\Psi D) = W_i(\Phi L)/\widetilde{W}_i(\Psi D')$. ■

If $K_1, \cdots, K_{n-1} \in \mathcal{K}^n$, then the mixed projection body of $K_1, \cdots, K_{n-1} \in \mathcal{K}^n$, $\Pi(K_1, \cdots, K_{n-1})$, was defined by (see [28])

$$h(\Pi(K_1, \cdots, K_{n-1}), u) = v(K_1^u, \cdots, K_{n-1}^u),$$

where $v$ denotes $(n-1)$-dimensional mixed volume and $K^u$ denotes the image of $K \in \mathcal{K}^n$ under an orthogonal projection onto the subspace $u^\perp$ that passes through the origin and is orthogonal to $u$. Since the mixed projection body operator $\Pi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n$ and the mixed intersection body operator $I : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ are mixed Blaschke-Minkowski and mixed radial Blaschke-Minkowski homomorphisms, respectively, Theorem 1.1 directly yields

**Corollary 3.1.** *Let $K, L \in \mathcal{K}^n$, $D, D' \in \mathcal{S}^n$, $D \subseteq K$, $D' \subseteq L$, and let $D'$ be a dilate of $D$. If $0 \leq i \leq n-1$ and $1 \leq j < n-1$, then*

$$\left[W_i(\Pi_j(K,L)) - \widetilde{W}_i(I_j(D,D'))\right]^{n-1} \geq \left[W_i(\Pi K) - \widetilde{W}_i(ID)\right]^{n-j-1}\left[W_i(\Pi L) - \widetilde{W}_i(ID')\right]^{j} \quad (3.5)$$

*with equality if and only if $K$ and $L$ are homothetic and $W_i(\Pi K)/\widetilde{W}_i(ID) = W_i(\Pi L)/\widetilde{W}_i(ID')$.*

Taking $i = 0$, $j = 1$ in inequality (3.5), inequality (3.5) becomes

**Corollary 3.2.** *If $K, L \in \mathcal{K}^n$, $D, D' \in \mathcal{S}^n$, $D \subseteq K$, $D' \subseteq L$ and $D'$ is a dilate of $D$, then*

$$\left[V(\Pi_1(K,L)) - V(I_1(D,D'))\right]^{n-1} \geq [V(\Pi K) - V(ID)]^{n-2}\left[V(\Pi L) - V(ID')\right],$$

*with equality if and only if $K$ and $L$ are homothetic and $V(\Pi K)/V(ID) = V(\Pi L)/V(ID')$.*

**Lemma 3.2**[4]**.** *Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be two series of positive real numbers and let $p > 1$. If $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$, then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{\frac{1}{p}} \leq \left((a_1+b_1)^p - \sum_{i=2}^n (a_i+b_i)^p\right)^{\frac{1}{p}} \quad (3.6)$$

*with equality if and only if* $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

*Proof of Theorem 1.2.* By inequality (1.3), we have for $0 \le i < n$ and $0 \le j < n-2$,

$$W_i(\Phi_j(K+L)) \ge [W_i(\Phi_j K)^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j L)^{\frac{1}{(n-i)(n-j-1)}}]^{(n-i)(n-j-1)} \quad (3.7)$$

with equality if and only if $K$ and $L$ are homothetic. Since $D$ and $D'$ are dilates, by inequality (1.7), we obtain

$$\widetilde{W}_i(\Psi_j(D\tilde{+}D')) = [\widetilde{W}_i(\Psi_j D)^{\frac{1}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j D')^{\frac{1}{(n-i)(n-j-1)}}]^{(n-i)(n-j-1)}. \quad (3.8)$$

Combining (3.7) and (3.8), it follows from inequality (3.6) that

$$\begin{aligned}
&[W_i(\Phi_j(K+L)) - \widetilde{W}_i(\Psi_j(D\tilde{+}D'))]^{\frac{1}{(n-i)(n-j-1)}} \\
&\quad \ge \{[W_i(\Phi_j K)^{\frac{1}{(n-i)(n-j-1)}} + W_i(\Phi_j L)^{\frac{1}{(n-i)(n-j-1)}}]^{(n-i)(n-j-1)} \\
&\quad - [\widetilde{W}_i(\Psi_j D)^{\frac{1}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j D')^{\frac{1}{(n-i)(n-j-1)}}]^{(n-i)(n-j-1)}\}^{\frac{1}{(n-i)(n-j-1)}} \\
&\quad \ge [W_i(\Phi_j K) - \widetilde{W}_i(\Psi_j D)]^{\frac{1}{(n-i)(n-j-1)}} + [W_i(\Phi_j L) - \widetilde{W}_i(\Psi_j D')]^{\frac{1}{(n-i)(n-j-1)}},
\end{aligned}$$

that is

$$\begin{aligned}
&[W_i(\Phi_j(K+L)) - \widetilde{W}_i(\Psi_j(D\tilde{+}D'))]^{\frac{1}{(n-i)(n-j-1)}} \\
&\quad \ge [W_i(\Phi_j K) - \widetilde{W}_i(\Psi_j D)]^{\frac{1}{(n-i)(n-j-1)}} + [W_i(\Phi_j L) - \widetilde{W}_i(\Psi_j D')]^{\frac{1}{(n-i)(n-j-1)}}. \quad (3.9)
\end{aligned}$$

By the equality conditions of inequalities (3.6) and (3.7), we know that equality holds in (3.9) if and only if $K$ and $L$ are homothetic and $W_i(\Phi_j K)/\widetilde{W}_i(\Psi_j D) = W_i(\Phi_j L)/\widetilde{W}_i(\Psi_j D')$. ■

Taking the mixed projection body operator $\Pi_j$ and the mixed intersection operator $I_j$ as the mixed Blaschke-Minkowski and the mixed radial Blaschke-Minkowski homomorphism in Theorem 1.2, we obtain

**Corollary 3.3.** *If* $K, L \in \mathcal{K}^n$, $D, D' \in \mathcal{S}^n$, $D \subseteq K$ *and* $D' \subseteq L$, *and* $D'$ *is a dilate of* $D$, *then for* $0 \le i < n$ *and* $0 \le j < n-2$,

$$\begin{aligned}
&\Big[W_i(\Pi_j(K+L)) - \widetilde{W}_i(I_j(D\tilde{+}D'))\Big]^{\frac{1}{(n-i)(n-j-1)}} \\
&\quad \ge \Big[W_i(\Pi_j K) - \widetilde{W}_i(I_j D)\Big]^{\frac{1}{(n-i)(n-j-1)}} + \Big[W_i(\Pi_j L) - \widetilde{W}_i(I_j D')\Big]^{\frac{1}{(n-i)(n-j-1)}} \quad (3.10)
\end{aligned}$$

*with equality if and only if* $K$ *and* $L$ *are homothetic and* $W_i(\Pi_j K)/\widetilde{W}_i(I_j D) = W_i(\Pi_j L)/\widetilde{W}_i(I_j D')$.

Taking $i = 0$, $j = 0$ in inequality (3.10), yields

**Corollary 3.4.** *If $K, L \in \mathcal{K}^n$, $D, D' \in \mathcal{S}^n$ $D \subseteq K$ and $D' \subseteq L$, and $D'$ is a dilate of $D$, then*

$$\left[V(\Pi(K+L)) - V(I(D\tilde{+}D'))\right]^{\frac{1}{n(n-1)}} \geq \left[V(\Pi K) - V(ID)\right]^{\frac{1}{n(n-1)}} + \left[V(\Pi L) - V(ID')\right]^{\frac{1}{n(n-1)}}$$

*with equality if and only if $K$ and $L$ are homothetic and $V(\Pi K)/V(ID) = V(\Pi L)/V(ID')$.*

**Lemma 3.3**[20]**.** *If $c_i > 0$, $b_i > 0$, $c_i > b_i$, $i = 1, \cdots, n$, then*

$$\left(\prod_{i=1}^{n}(c_i - b_i)\right)^{\frac{1}{n}} \leq \left(\prod_{i=1}^{n} c_i\right)^{\frac{1}{n}} - \left(\prod_{i=1}^{n} b_i\right)^{\frac{1}{n}} \tag{3.11}$$

*with equality if and only if $c_1/b_1 = c_2/b_2 = \cdots = c_n/b_n$.*

*Proof of Theorem 1.3.* By inequality (1.4), we have for $1 \leq r \leq n-1$,

$$W_i(\Phi(K_1, \cdots, K_{n-1})) \geq \left[\prod_{j=1}^{r} W_i(\Phi(\underbrace{K_j, \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1}))\right]^{\frac{1}{r}}. \tag{3.12}$$

Since $D_j (j = 1, \cdots, r)$ are dilates of each other, it follows from inequality (1.8) that

$$\widetilde{W}_i(\Psi(D_1, \cdots, D_{n-1})) = \left[\prod_{j=1}^{r} \widetilde{W}_i(\Psi(\underbrace{D_j, \cdots, D_j}_{r}, D_{r+1}, \cdots, D_{n-1}))\right]^{\frac{1}{r}}. \tag{3.13}$$

Combining (3.12) and (3.13), and using inequality (3.11), we obtain

$$\begin{aligned}
& W_i(\Phi(K_1, \cdots, K_{n-1})) - \widetilde{W}_i(\Psi(D_1, \cdots, D_{n-1})) \\
&\qquad \geq \left[\prod_{j=1}^{r} W_i(\Phi(\underbrace{K_j, \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1}))\right]^{\frac{1}{r}} - \\
&\qquad\quad \left[\prod_{j=1}^{r} \widetilde{W}_i(\Psi(\underbrace{D_j, \cdots, D_j}_{r}, D_{r+1}, \cdots, D_{n-1}))\right]^{\frac{1}{r}} \\
&\geq \left[\prod_{j=1}^{r}(W_i(\Phi(\underbrace{K_j, \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1})) - \widetilde{W}_i(\Psi(\underbrace{D_j, \cdots, D_j}_{r}, D_{r+1}, \cdots, D_{n-1}))\right]^{\frac{1}{r}}.
\end{aligned}$$

Thus

$$\begin{aligned}
& \left[W_i(\Phi(K_1, \cdots, K_{n-1})) - \widetilde{W}_i(\Psi(D_1, \cdots, D_{n-1}))\right]^{r} \\
&\geq \prod_{j=1}^{r}\left[W_i(\Phi(\underbrace{K_j, \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1})) - \widetilde{W}_i(\Psi(\underbrace{D_j, \cdots, D_j}_{r}, D_{r+1}, \cdots, D_{n-1}))\right].
\end{aligned}$$

From Theorem 1.3 we directly obtain

**Corollary 3.5.** *Let $K_i \in \mathcal{K}^n$, $D_i \in \mathcal{S}^n$ and $D_i \subseteq K_i$ $(i = 1, \cdots, n-1)$. If for $1 \le r \le n-1$ the bodies $D_j$ $(j = 1, \cdots, r)$ are dilates of each other, then*

$$\begin{aligned}
&[V(\Pi(K_1, \cdots, K_{n-1})) - V(I(D_1, \cdots, D_{n-1}))]^r \\
\geq &\prod_{j=1}^{r} \left[ V(\Pi(\underbrace{K_j, \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1})) - V(I(\underbrace{D_j, \cdots, D_j}_{r}, D_{r+1}, \cdots, D_{n-1})) \right].
\end{aligned}$$

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