Module Maps and Invariant Subsets of Banach Modules of Locally Compact Groups

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Abstract

For a locally compact group G, Lau [6] and Ghaffari [3] provided many results about G-invariant subsets of G-modules, and the relationship between G-module maps, $L^1(G)$-module maps and $M(G)$- module maps. In both papers their results were specified for one module action. In this paper we extend many of their results to arbitrary Banach G-modules and G-module maps.

1 Introduction

Let G always denote a locally compact group with a Haar measure λ and modular function Δ. Let $L^p(G)$, $1 \leq p \leq \infty$, be the Banach space of λ-measurable functions $f : G \to \mathbb{C}$, such that $\|f\|_p < \infty$ and when $s \in G$, we let δ_s denote the Dirac measure at s. In addition, we let $C_0(G)$ denote the set of all continuous functions $f : X \to \mathbb{C}$ vanishing at ∞ and denote the set of all complex regular Borel measures on G by $M(G) \cong C_0(G)^*$. Define the convolution product between two measures μ, ν in $M(G)$ by

$$\langle \mu * \nu, f \rangle = \iint f(xy)d\mu(x)d\nu(y) \qquad (f \in C_0(G)).$$

The group algebra $(L^1(G), *)$, where for $f, g \in L^1(G)$,

$$(f * g)(x) = \int f(y)g(y^{-1}x)d\lambda(y),$$

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is a closed ideal of the measure algebra via $f \mapsto \mu_f$ where $\langle \mu_f, \phi \rangle = \int \phi(s) f(s) ds$ whenever $\phi \in C_0(G)$.

Our definitions of a Banach G-module and of a Banach $\mathcal{A}$-module for a Banach algebra $\mathcal{A}$, follow [5]; also see [7]. Definitions of dual modules and the relationship between Banach G-modules, Banach $M(G)$-modules and Banach $L^1(G)$-modules are found in [5] and [7].

The main purpose of this paper is to generalize many of the results from Lau and Ghaffari's papers [6] and [3] respectively. Both Lau and Ghaffari's results were about G-invariant subsets of G-modules and the relation between G-module maps, $L^1(G)$-module maps and $M(G)$-module maps. Lau's results were specified for the G-module action $s \cdot f = \delta_s * f$ where $s \in G, f \in L^p(G), 1 \leq p < \infty$, and Ghaffari's results were specified for the G-module action $s \cdot f(t) = \delta_s \star f(t) = \Delta(s)^{\frac{1}{p}} f(s^{-1}ts)$ whenever $s \in G, f \in L^p(G), 1 \leq p < \infty$. In this paper we will obtain many of the results proved for specific actions in [6] and [3] for arbitrary Banach G-modules and dual G-modules. We also correct an inaccurate statement found in [3]. The ideas of our proofs combine those of Lau, Ghaffari and our own.

This paper will be a part of the author's M.Sc. thesis, written under the supervision of Ross Stokke. The author would like to express her deep gratitude to Dr. Stokke for his support, encouragement and helpful academic advice.

2 G-Module Maps Between Left Banach G-Modules

Let X be a left Banach G-module. Recall that X is a unital left Banach $M(G)$-module and a neo-unital left Banach $L^1(G)$-module with respect to the weak integral

$$\mu \cdot x = \int_G s \cdot x d\mu(s) \qquad (x \in X, \mu \in M(G)).$$

Definition 2.1. *Let X and Y be left Banach G-modules, and let $T : X \to Y$ be a (bounded, linear) operator. Then we will say that T is a G-module map if*

$$T(s \cdot x) = s \cdot Tx \qquad (s \in G, x \in X).$$

Similarly, we can define right G-module maps, (right) $M(G)$-module maps and (right) $L^1(G)$-module maps.

The following two theorems generalize [3, Theorem 2.1].

Theorem 2.2. *Let X and Y be left Banach G-modules and $T : X \to Y$ a bounded linear map. Then the following statements are equivalent:*

(i) T is a G-module map.

(ii) T is an $M(G)$-module map.

(iii) T is an $L^1(G)$-module map.

Proof. $(i) \Rightarrow (ii)$ Let X and Y be left Banach G-modules and suppose that $T : X \to Y$ is a bounded linear G-module map and $T^* : Y^* \to X^*$ is its adjoint operator. For $\mu \in M(G), x \in X$ and $\phi \in Y^*$,

$$\langle T(\mu \cdot x), \phi \rangle = \int \langle s \cdot x, T^* \phi \rangle d\mu(s) = \int \langle T(s \cdot x), \phi \rangle d\mu(s) = \int \langle s \cdot Tx, \phi \rangle d\mu(s) = \langle \mu \cdot Tx, \phi \rangle.$$

Since Y^* separates points of Y, $T(\mu \cdot x) = \mu \cdot Tx$.

$(ii) \Rightarrow (iii)$ This is obvious since $L^1(G) \subset M(G)$.

$(iii) \Rightarrow (i)$ Suppose that $T : X \to Y$ is a bounded linear $L^1(G)$-module map. Let $s \in G, x \in X$, and let $(e_\alpha)_\alpha$ be a bounded approximate identity (BAI) for $L^1(G)$. Then $(e_\alpha)_\alpha$ is a BAI for both the neo-unital $L^1(G)$-modules X and Y, so

$$s \cdot Tx = \delta_s \cdot Tx = \lim \underbrace{(\delta_s * e_\alpha)}_{\in L^1(G)} \cdot Tx = \lim T((\delta_s * e_\alpha) \cdot x) = T(\delta_s \cdot x) = T(s \cdot x). \quad \blacksquare$$

Note that because X^* and Y^* are not necessarily right *Banach* G-modules (e.g. $L^1(G)^* = L^\infty(G), s \cdot f = \delta_s * f$), the next result is not immediately contained in the right module version of Theorem 2.2.

Theorem 2.3. *Let X and Y be left Banach G-modules and suppose that $T : Y^* \to X^*$ is linear, bounded and $w^* - w^*$ continuous. Then the following statements are equivalent:*

(i) T is a right G-module map.

(ii) T is a right $M(G)$-module map.

(iii) T is a right $L^1(G)$-module map.

Proof. As $T : Y^* \to X^*$ is linear, bounded and $w^* - w^*$ continuous, T is the adjoint operator of some $L : X \to Y$.
$(i) \Rightarrow (ii)$ Suppose that T is a right G-module map. Then for $s \in G, x \in X$ and $\phi \in Y^*$,

$$\langle \phi, L(s \cdot x) \rangle = \langle T(\phi \cdot s), x \rangle = \langle \phi, s \cdot (Lx) \rangle,$$

So L is a G-module map. Hence by Theorem 2.2, L is an $M(G)$-module map and therefore, for $x \in X, \phi \in Y^*$, and $\mu \in M(G)$,

$$\langle x, T(\phi \cdot \mu) \rangle = \langle L(\mu \cdot x), \phi \rangle = \langle x, T\phi \cdot \mu \rangle.$$

Hence, T is $M(G)$-module map. This proves $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ is obvious. That $(iii) \Rightarrow (i)$ follows the argument used to prove $(i) \Rightarrow (ii)$. $\blacksquare$

The following corollary [4, Theorem 35.5] is an immediate consequence of Theorem 2.2 applied to the Banach G-module action $s \cdot h = \delta_s * h$ of G on $L^1(G)$.

Corollary 2.4. *Let G be a locally compact group and let $T : L^1(G) \to L^1(G)$ be a bounded linear operator. Then the following are equivalent:*

(i) $T(\delta_s * h) = \delta_s * Th$ *whenever* $s \in G$, *and* $h \in L^1(G)$.

(ii) $T(\mu * h) = \mu * Th$ *whenever* $\mu \in M(G)$, *and* $h \in L^1(G)$.

(iii) $T(f * h) = f * Th$ *whenever* $f, h \in L^1(G)$.

Let X be a left Banach G-module and define $UC(X^*)$ by the following:

$$UC(X^*) := \{\phi \in X^* : s \mapsto \phi \cdot s : G \to (X^*, \|\cdot\|) \text{ is continuous}\}.$$

It is easy to see that $UC(X^*)$ is a closed linear subspace of X^*. The next observation can be found in a forthcoming paper by Y. Choi, E. Samei and R. Stokke. Related results are found in the author's thesis.

Lemma 2.5. *Let X be a left Banach G-module. Then $UC(X^*)$ is a right Banach G-submodule of X^*. Moreover, as we already noted, X^* itself is not necessarily a right Banach G-module. If we let $\phi \bullet \mu$ and $\phi \cdot \mu$ respectively denote the corresponding $M(G)$-module action on $UC(X^*)$, and dual $M(G)$-module action on X^* restricted to $UC(X^*)$, then $\phi \bullet \mu = \phi \cdot \mu$. Hence the notation $\phi \cdot \mu$ is unambiguous and $UC(X^*) = X^* \cdot L^1(G)$.*

Now by the above Lemma, we can obtain the following corollary, which includes [3, Theorem 2.4], as an immediate corollary to the right module version of Theorem 2.2.

Corollary 2.6. *Let X and Y be left Banach G-modules, and let $T : UC(X^*) \to UC(Y^*)$ be a bounded linear operator. Then the following statements are equivalent:*

(i) $T(\phi \cdot s) = T\phi \cdot s$ *where* $\phi \in UC(X^*), s \in G$.

(ii) $T(\phi \cdot \mu) = T\phi \cdot \mu$ *where* $\phi \in UC(X^*), \mu \in M(G)$.

(iii) $T(\phi \cdot f) = T\phi \cdot f$ *where* $\phi \in UC(X^*), f \in L^1(G)$.

3 Closed Convex G-Invariant Subsets of Left Banach G-modules

Definition 3.1. *Let X be a left Banach G-module. If C is a convex subset of X, then C is called G-invariant if $s \cdot x \in C$ whenever $s \in G, x \in C$. Similarly, we can define $L^1(G)$-invariant, and $M(G)$-invariant convex sets.*

We denote the probability measures in $M(G)$ by $M(G)_1^+$ and let $L^1(G)_1^+ = M(G)_1^+ \cap L^1(G)$. The following theorem includes [6, Theorem 4.1(a)]. Note that by [1, V. Corollary 1.5], a convex subset of a Banach space is closed if and only if it is weakly closed.

Theorem 3.2. *Let X be a left Banach G-module and C a closed convex subset of X. Then the following are equivalent:*

(i) C is G-invariant.

(ii) C is $M(G)_1^+$-invariant.

(iii) C is $L^1(G)_1^+$-invariant.

Proof. $(i) \Rightarrow (ii)$ Let $\mu \in M(G)_1^+, x \in C$ and suppose that C is G-invariant. Suppose $\mu \cdot x \notin C$. Then by the Hahn-Banach Separation Theorem, there is $x^* \in X^*, \gamma \in \mathbb{R}$, and $\epsilon > 0$, such that

$$Re\langle x^*, c\rangle \leq \gamma < \gamma + \epsilon \leq Re\langle x^*, \mu \cdot x\rangle \qquad (c \in C),$$

so

$$Re\langle x^*, s \cdot x\rangle \leq \gamma < \gamma + \epsilon \leq Re\langle x^*, \mu \cdot x\rangle \qquad (s \in G).$$

But

$$\begin{aligned} Re\langle x^*, \mu \cdot x\rangle = Re \int \langle x^*, s \cdot x\rangle d\mu(s) = \int Re\langle x^*, s \cdot x\rangle d\mu(s) \\ \leq \int \gamma d\mu(s) = \gamma\mu(G) = \gamma, \end{aligned}$$

a contradiction. Hence, $\mu \cdot x \in C$.

$(ii) \Rightarrow (iii)$ This is obvious.

$(iii) \Rightarrow (i)$ Let $s \in G, x \in C$ and suppose that C is $L^1(G)_1^+$-invariant. Let $(e_\alpha)_\alpha \subseteq L^1(G)_1^+$ be a BAI for $L^1(G)$. Then $e_\alpha \cdot (s \cdot x) = e_\alpha \cdot (\delta_s \cdot x) = \underbrace{(e_\alpha * \delta_s)}_{\in L^1(G)_1^+} \cdot x \in C$. Since $s \cdot x \in X$, and X is a neo-unital Banach $L^1(G)$-module, $e_\alpha \cdot (s \cdot x) \to s \cdot x$, so $s \cdot x \in C$ because C is closed. ∎

The following theorem includes [3, Theorem 2.5] and [6, Theorem 4.1(b)]. The proof is similar to that of Theorem 3.2.

Theorem 3.3. *Let X be a left Banach G-module, L a w^*-closed convex subset of X^*. Then the following are equivalent:*

(i) L is G-invariant.

(ii) L is $M(G)_1^+$-invariant.

(iii) L is $L^1(G)_1^+$-invariant.

If A is a subset of X, co(A) denotes the convex hull of A. The next corollary includes [6, Corollary 4.2].

Corollary 3.4. *Let X be a left Banach G-module, $x \in X$ and $\phi \in X^*$. Then the following statements hold:*

(i) $\overline{co}\{s \cdot x : s \in G\} = \overline{\{f \cdot x : f \in L^1(G)_1^+\}} = \overline{\{\mu \cdot x : \mu \in M(G)_1^+\}}$.

(ii) $\overline{co}^{w^*}\{\phi \cdot s : s \in G\} = \overline{\{\phi \cdot f : f \in L^1(G)_1^+\}}^{w^*} = \overline{\{\phi \cdot \mu : \mu \in M(G)_1^+\}}^{w^*}$.

Proof. We establish (i); the proof of (ii) is similar. Let $x \in X$, $C_1 = \overline{co}\{s \cdot x : s \in G\}$, $C_2 = \overline{\{\mu \cdot x : \mu \in M(G)_1^+\}}$, and $C_3 = \overline{\{f \cdot x : f \in L^1(G)_1^+\}}$. Obviously $co\{s \cdot x : s \in G\}$ is G-invariant, so continuity of $y \mapsto s \cdot y : X \to X$ $(s \in G)$ gives G-invariance of C_1. Hence by Theorem 3.2, C_1 is $M(G)_1^+$-invariant and $L^1(G)_1^+$-invariant. Since $x = e \cdot x \in C_1$, $\{\mu \cdot x : \mu \in M(G)_1^+\} \subseteq C_1$. As C_1 is closed, $C_2 = \overline{\{\mu \cdot x : \mu \in M(G)_1^+\}} \subseteq C_1$, and clearly $C_3 \subseteq C_2$. Now let $(e_\alpha)_\alpha \subseteq L^1(G)_1^+$ be a BAI for $L^1(G)$. Then $e_\alpha \cdot x \to x$, so $x \in C_3$. But C_3 is closed, convex and $L^1(G)_1^+$-invariant so by Theorem 3.2, C_3 is G-invariant. Hence $C_1 \subseteq C_3$. ■

4 G-Module Maps Between Closed Convex G-Invariant Subsets of Left Banach G-Modules

Let X, Y be normed spaces, and C, D convex subsets of X, Y respectively. Recall that a map $f : C \to D$ is called affine if for all $x, y \in C$ and $\alpha \in [0, 1]$,

$$f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha) f(y).$$

Definition 4.1. *Let τ be the locally convex topology on $M(G)$ generated by the collection of seminorms $\{P_f : f \in CB(G)\}$, such that*

$$P_f(\mu) = |\langle \mu, f\rangle| = |\int f d\mu| \qquad (\mu \in M(G)).$$

So $\mu_\alpha \xrightarrow{\tau} \mu$ means that whenever $f \in CB(G)$, $\int f d\mu_\alpha \to \int f d\mu$.

The following theorem contains [6, Theorem 5.1].

Theorem 4.2. *Let X, Y be left Banach G-modules, and B, C be closed G-invariant convex subsets of X and Y respectively. If $T : B \to C$ is continuous and affine, then the following are equivalent:*

(i) $T(s \cdot x) = s \cdot Tx$ *whenever* $s \in G, x \in B$.

(ii) $T(\mu \cdot x) = \mu \cdot Tx$ *whenever* $\mu \in M(G)_1^+, x \in B$.

(iii) $T(f \cdot x) = f \cdot Tx$ *whenever* $f \in L^1(G)_1^+, x \in B$.

Proof. Note that by Theorem 3.2, B and C are $M(G)_1^+$-invariant and $L^1(G)_1^+$-invariant.
$(i) \Rightarrow (ii)$ Let $x \in B$ and suppose $T(s \cdot x) = s \cdot Tx$ $(s \in G)$. Let $\mu \in M(G)_1^+$, $(\mu_\alpha) = (\sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{s_i^\alpha}) \subseteq co\{\delta_s : s \in G\}$ be a net converging to μ in $\tau-$topology; see

[6, Lemma 3.1]. Let $\phi \in X^*$. Then noting that $f \in CB(G)$ where $f(s) = \langle \phi, s \cdot x \rangle$ $(s \in G)$, we obtain

$$\langle \phi, \mu_\alpha \cdot x \rangle = \int \langle \phi, s \cdot x \rangle d\mu_\alpha(s) \to \int \langle \phi, s \cdot x \rangle d\mu(s) = \langle \phi, \mu \cdot x \rangle.$$

Therefore, $\mu_\alpha \cdot x \to \mu \cdot x$ weakly in X; also $\mu_\alpha \cdot Tx \to \mu \cdot Tx$ weakly in Y. By [2, Remark 2], T is continuous when A and B have their respective weak topologies. This, and our assumption(i), give

$$\begin{aligned} T(\mu \cdot x) = w - \lim T(\mu_\alpha \cdot x) &= w - \lim T(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha (s_i^\alpha \cdot x)) \\ &= w - \lim(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha (s_i^\alpha \cdot Tx)) \\ &= w - \lim \mu_\alpha \cdot Tx = \mu \cdot Tx. \end{aligned}$$

$(ii) \Rightarrow (iii)$ This is obvious.

$(iii) \Rightarrow (i)$ Let $s \in G, x \in B$ and suppose that $T(f \cdot x) = f \cdot Tx$ for every $f \in L^1(G)_1^+$. Letting $(e_\alpha)_\alpha \subseteq L^1(G)_1^+$ be a BAI for $L^1(G)$, $(e_\alpha)_\alpha$ is a BAI for both X and Y, so

$$s \cdot Tx = \delta_s \cdot Tx = \lim(\underbrace{(\delta_s * e_\alpha)}_{L^1(G)_1^+} \cdot Tx) = \lim T((\delta_s * e_\alpha) \cdot x) = T(\delta_s \cdot x) = T(s \cdot x),$$

as needed. ∎

The next theorem contains [3, Theorem 2.6] and [6, Theorem 5.2]. The proof is similar to the proof of Theorem 4.2.

Theorem 4.3. *Let X, Y be left Banach G-modules and let L, K be w^*-closed G-invariant convex subsets of X^* and Y^* respectively. If $T : L \to K$ is $w^* - w^*$ continuous and affine, then the following are equivalent:*

(i) $T(\phi \cdot s) = T\phi \cdot s$ *whenever* $s \in G, \phi \in L$.

(ii) $T(\phi \cdot \mu) = T\phi \cdot \mu$ *whenever* $\mu \in M(G)_1^+, \phi \in L$.

(iii) $T(\phi \cdot f) = T\phi \cdot f$ *whenever* $f \in L^1(G)_1^+, \phi \in L$.

In the next two theorems $L^1(G)$ is viewed as a left Banach G-module via $s \cdot f = \delta_s * f$. We first observe that [6, Theorem 5.3] can be generalized as follows:

Theorem 4.4. *Let G be a locally compact non-compact group. Let B be a non-empty closed convex left G-invariant subset of $L^1(G)$, and C a non-empty weakly compact closed convex left G-invariant subset of a left Banach G-module X. If $T : C \to B$ is a continuous affine G-module map, then $T(f) = 0$ for every $f \in C$.*

Proof. By [2, Remark 2], T is affine continuous when C and B have their respective weak topologies, so $T(C)$ is a weakly compact convex left G-invariant subset of $L^1(G)$. Hence by [6, Theorem 4.6], $T(C) = \{0\}$. ∎

Also [6, Theorem 5.5], can be made more general:

Theorem 4.5. *Let G be any locally compact group, let C be a weakly compact closed bounded left G-invariant subset of a left Banach G-module X. Also let $T : L^1(G)_1^+ \to C$ be a continuous affine map. Then the following are equivalent:*

(i) T is a G-module map.

(ii) There is $x \in C$, such that $T(u) = u \cdot x$ whenever $u \in L^1(G)_1^+$.

Proof. $(i) \Rightarrow (ii)$ Let $s \in G$ and suppose $T(s \cdot u) = s \cdot Tu$ whenever $u \in L^1(G)_1^+$. Observe that $L^1(G)_1^+$ is a G-invariant weakly closed convex subset of $L^1(G)$, so by Theorem 4.2, we have

$$T(f * u) = f \cdot Tu \qquad (f, u \in L^1(G)_1^+).$$

Suppose $(u_\alpha)_\alpha \subseteq L^1(G)_1^+$ is a BAI for $L^1(G)$. Since $T(u_\alpha) \in C$ for each α and C is weakly compact, there is an $x \in C$ such that by passing to a subnet if necessary, $T(u_\alpha) \to x$ in the weak topology. Hence, for $\phi \in X^*$,

$$\langle u \cdot x, \phi \rangle = \lim \langle Tu_\alpha, \phi \cdot u \rangle = \lim \langle u \cdot Tu_\alpha, \phi \rangle = \lim \langle T \underbrace{(u * u_\alpha)}_{\in L^1(G)_1^+}, \phi \rangle = \langle Tu, \phi \rangle.$$

$(ii) \Rightarrow (i)$ Let $s \in G$ and suppose there is an $x \in C$, such that $T(u) = u \cdot x$ for all $u \in L^1(G)_1^+$. Then

$$T(s \cdot u) = (s \cdot u) \cdot x = s \cdot (u \cdot x) = s \cdot Tu.$$ ∎

Ghaffari's action of $L^1(G)$ on $L^p(G)$ $(1 \leq p < \infty)$ in his paper [3] is

$$f \star h(t) = f \cdot h(t) = \int \Delta(s)^{\frac{1}{p}} h(s^{-1}ts) f(s) ds \qquad (f \in L^1(G), h \in L^p(G), s, t \in G);$$

the corresponding G-module action is $s \star h(t) = \Delta(s)^{\frac{1}{p}} h(s^{-1}ts)$. In that paper it is stated that $(L^1(G), \star)$ is a Banach algebra. Unfortunately $\star$ is not always associative on $L^1(G)$.

Theorem 4.6. *Let G be any non-abelian discrete group. Then $\star$ is not associative on $L^1(G) = \ell^1(G)$.*

Proof. Observe that $\delta_x \star \delta_y = \delta_{xyx^{-1}}$ whenever $x, y \in G$. Suppose G is a non-abelian discrete group, and choose $s, t, r \in G$, such that $tr \neq rt$. If $\star$ is associative on $\ell^1(G)$, then

$$\delta_{rst(rs)^{-1}} = \delta_r \star (\delta_s \star \delta_t) = (\delta_r \star \delta_s) \star \delta_t = \delta_{(rsr^{-1})t(rsr^{-1})^{-1}}.$$

So $rsts^{-1}r^{-1} = rsr^{-1}trs^{-1}r^{-1}$ and hence $t = r^{-1}tr$; therefore $rt = tr$, a contradiction. Hence $\star$ is not associative. ∎

Proposition 4.7. *Let G be a locally compact group. If G is abelian, then $\star$ is associative on $L^1(G)$.*

Proof. Let G be an abelian locally compact group, and let $f, g \in L^1(G)$. Then

$$f \star g(t) \;=\; \int \underbrace{\Delta(s)}_{=1} g(\underbrace{s^{-1}ts}_{=t}) f(s)ds = \int g(t)f(s)ds = \left(\int f(s)ds\right) g(t).$$

Now let $h \in L^1(G)$. Then

$$\begin{aligned}(f \star g) \star h(t) &= \left(\int (f \star g)(s)ds\right) h(t) \\ &= \left(\int \left(\int f(r)dr\right) g(s)ds\right) h(t) \\ &= \int f(r)dr \left(\int g(s)dsh(t)\right) = \int f(r)dr(g \star h(t)) \\ &= f \star (g \star h)(t).\end{aligned}$$

■

Corollary 4.8. *Let G be a discrete group. Then $\star$ is associative on $\ell^1(G)$ if and only if G is abelian.*

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