Certain meromorphic functions sharing a nonconstant polynomial with their linear polynomials*

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Abstract

Let B be the class of meromorphic functions f such that the set sing (f^{-1}) is bounded, where sing (f^{-1}) is the set of critical and asymptotic values of f. Suppose that f has at most finitely many poles in the complex plane, and that $L(f) - P$ and $f - P$ share 0 CM, where $L[f] = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f$, where k is a positive integer and $a_0, a_1, \cdots, a_{k-1}$ are complex numbers, P is a nonconstant polynomial. Then, the hyper-order of f is nonnegative integer or ∞. Applying this result, we obtain some uniqueness results for transcendental meromorphic functions having the same fixed points with their linear differential polynomials, where the meromorphic functions belong to B and have at most finitely many poles in the complex plane. The results in this paper are concerning a conjecture of Brück [5]. An example is provided to show that the results in this paper are best possible.

1 Introduction and Main Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in Nevanlinna theory of meromorphic functions as explained in [13, 17, 33, 34]. It will be convenient

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to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \to \infty$ and $r \notin E$.

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g share the value a CM, provided that $f-a$ and $g-a$ have the same zeros and each common zero of $f-a$ and $g-a$ has the same multiplicity related to f and g. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros and each common zero of $f-a$ and $g-a$ is counted only once. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM (see [34]). Let $\varphi \not\equiv \infty$ be a meromorphic function such that $T(r,\varphi) = S(r,f)$. Then we say that φ is a small function of f. A point $b \in \mathbb{C}$ is said to be an asymptotic value of f if there exists a curve $\Gamma \subset \mathbb{C}$ tending to ∞ such that $f(z) \to b$ as $z \to \infty$ along Γ. A value b is called a critical value of f if there exists $z_0 \in \mathbb{C}$ such that $f'(z_0) = 0$ and $f(z_0) = b$. Throughout this paper, we denote by sing (f^{-1}) the set of critical and asymptotic values of f, and denote by B the class of meromorphic functions f, where f is such that the set sing (f^{-1}) is bounded. This class has been considered extensively in iteration theory, see, e.g. [3, 11, 18]. In this paper, we also need the following definition:

Definition 1.1. For a nonconstant entire function f, the lower order $\mu(f)$, the order $\rho(f)$, the lower hyper-order $\mu_2(f)$, the hyper-order $\rho_2(f)$ are defined as

$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

$$\mu_2(f) = \liminf_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}$$

and

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}$$

respectively, where and in what follows, $M(r,f) = \max_{|z|=r} |f(z)|$.

In 1977, Rubel and Yang [27] proved that if an entire function f shares two distinct finite complex numbers CM with its derivative f', then $f = f'$. What can be said about the relationship between an entire f and its first order derivative f', if f shares one finite value a CM with f' ? In 1996, Brück [5] made the conjecture that if f is a nonconstant entire function satisfying $\rho_2(f) < \infty$, where $\rho_2(f)$ is not a positive integer, and if f and f' share one finite complex number a CM, then $f - a = c(f' - a)$ for some constant $c \neq 0$. For the case that $a = 0$, the above conjecture had been proved by Brück [5]. Brück [5] also proved the above conjecture is true, provided that $a \neq 0$ and $N(r, 1/f') = S(r,f)$, where f is an entire function. Later on, Gundersen and Yang [12], Chen and Shon [8] proved that the above conjecture is true, provided that $\rho(f) < \infty$ and $\rho_2(f) < 1/2$ respectively, where f is an entire function. In 2005, Al-Khaladi [1] showed that the conjecture remains true for a nonconstant meromorphic function f such that $N(r, 1/f') = S(r, f)$. In this direction, some other research works have been obtained, see, e. g., Banerjee and Bhattacharjee [2], Chang and Zhu [6], Chang and Fang[7], Heittokangas, Korhonen, Laine, Rieppo and Zhang [14], Lahiri and Sarkar [16], Li and Gao[19, 20], Li and Yi [21-26], Wang [29], Wang and Laine [30], Wang and Li [31], Xiao and Li [32], Zhang [35], Zhang and Yang [36-37]. But the conjecture remains open by now.

First we recall the following result due to Gundersen and Yang:

Theorem A ([12, Theorem 1]). Let f be a nonconstant entire function of finite order, and let $a \neq 0$ be a finite complex number. If f and f' share a CM, then $f' - a = c(f - a)$ for some nonzero constant c.

Wang [29] obtained the following result to improve Theorem A:

Theorem B ([29, Theorem 1]). Let f be a nonconstant entire function of finite order, let P be a polynomial with degree $p \geq 1$, and let k be a positive integer. If $f - P$ and $f^{(k)} - P$ share 0 CM, then $f^{(k)} - P = c(f - P)$ for some complex number $c \neq 0$.

Consider the following linear differential polynomial related to f.

$$L[f] = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f, \tag{1.1}$$

where k is a positive integer and $a_0, a_1, \cdots, a_{k-1}$ are complex numbers.

One may ask, what can be said about the relations between a meromorphic function f and $L[f]$, if $f - P$ and $L[f] - P$ share 0 CM, where f belongs to the class B and has at most finitely many poles in the complex plane, $L[f]$ is defined as in (1.1), $k \geq 1$ is a positive integer and P is a nonconstant polynomial. In this direction, we first prove the following results:

Theorem 1.1. Suppose that $f \in B$ is a nonconstant entire function and P is a nonconstant polynomial such that $f - P$ and $L[f] - P$ share 0 CM, where $L[f]$ is defined as in (1.1), at least one of $a_0, a_1, \cdots, a_{k-1}$ is not zero, $k \geq 1$ is a positive integer. Then, there exists an entire function α_1 such that $L[f] - P = (f - P)e^{\alpha_1}$ and $\mu_2(f) = \rho_2(f) = \rho(e^{\alpha_1})$.

Theorem 1.2. Let $f \in B$ be a transcendental meromorphic function such that f has at least one pole but at most finitely many poles $\omega_1, \omega_2, \cdots, \omega_{n-1}$ and ω_n, where $n \geq 1$ is a positive integer, and let P be a nonconstant polynomial. Suppose that $f - P$ and $L[f] - P$ share 0 CM, where $L[f]$ is defined as in (1.1), at least one of $a_0, a_1, \cdots, a_{k-1}$ is not zero, $k \geq 1$ is a positive integer. Then, there exists an entire function α_2 such that $(L[f] - P)P_1 = (f - P)e^{\alpha_2}$ and $\rho_2(f) = \rho(e^{\alpha_2})$, where $P_1 = (z - \omega_1)^k (z - \omega_2)^k \cdots (z - \omega_n)^k$.

From Theorem 1.1 we get the following corollary:

Corollary 1.1. Let $f \in B$ be a transcendental entire function. Suppose that $f - z$ and $L[f] - z$ share 0 CM, where $L[f]$ is defined as in (1.1), at least one of $a_0, a_1, \cdots, a_{k-1}$ is not zero. If $\rho_2(f)$ is not a positive integer and $\rho_2(f) < \infty$, then $L[f] - z = c(f - z)$ for some nonzero constant c.

In 1995, Yi and Yang posed the following question.

Question 1.1 ([34, p.398]). Let f be a nonconstant meromorphic function, and let a be a finite nonzero complex constant. If f, $f^{(n)}$ and $f^{(m)}$ share the value

a CM, where n and m $(n < m)$ are distinct positive integers not all even or odd, then can we get the result $f = f^{(n)}$?

Gundersen and Yang [12] proved the following result to deal with Question 1.1:

Theorem C ([12, Theorem 2]). Let f be a nonconstant entire function of finite order, let $a \neq 0$ be a complex number, and let k be a positive integer. If a is shared by f, $f^{(k)}$ and $f^{(k+1)}$ IM, and shared by $f^{(k)}$ and $f^{(k+1)}$ CM, then $f = f'$.

We will prove the following result, which is an analogue of Theorem C concerning meromorphic functions having the same fixed points with their certain differential polynomials, where the meromorphic functions belongs to the class B and have at most finitely many poles:

Theorem 1.3. Let f be a transcendental meromorphic function such that f has at most finitely many poles and such that $\rho_2(f) < \infty$, where $\rho_2(f)$ is not a positive integer. Suppose 0 is shared by $f - z$, $L[f] - z$ and $L'[f] - z$ IM, and shared by $L[f] - z$ and $L'[f] - z$ CM, where

$$L[f] = f' + a_0 f \tag{1.2}$$

and

$$L'[f] = f'' + a_0 f', \tag{1.3}$$

in which a_0 is a finite value. If $L[f] \in B$, then $a_0 = 0$ and that f is a transcendental entire function such that f is given as $f = ce^z$, where $c \neq 0$ is some constant.

From Theorem 1.3 we get the following result:

Corollary 1.2. Let f be a transcendental meromorphic function such that f has at most finitely many poles and such that $\rho_2(f) < \infty$, where $\rho_2(f)$ is not a positive integer. Suppose 0 is shared by $f - z$, $f' - z$ and $f'' - z$ IM, and shared by $f' - z$ and $f'' - z$ CM. If $f' \in B$, then f is a transcendental entire function such that $f(z) = de^z$, where $d \neq 0$ is some constant.

2 Some Lemmas

In order to prove our theorems, we need the following preliminary results. Lemma 2.1 plays an important role in proving the main results of this paper. Lemma 2.2 is the lemma of the logarithmic derivative. Lemma 2.5 is a result of the Wiman-Valiron theory, which together with Lemma 2.4, Lemma 2.6 and Lemma 2.7 is a powerful tool for hyper-order and order considerations of entire solutions of linear differential equations.

Lemma 2.1 ([4, Lemma 2]) Let g be a transcendental meromorphic function, and suppose that $g(0) \neq \infty$ and that the set of finite critical and asymptotic values of g is bounded. Then there exists $R > 0$ such that

$$|g'(z)| \geq \frac{|g(z)|}{2\pi|z|} \log \frac{|g(z)|}{R},$$

as $|z|$ is large.

Lemma 2.2 ([17, Corollary 2.3.4]) Let f be a transcendental meromorphic function and $k \geq 1$ be an integer. Then $m(r, f^{(k)}/f) = O(\log(rT(r,f))$, outside of a possible exceptional set E of finite linear measure, and if f is of finite order of growth, then $m(r, f^{(k)}/f) = O(\log r)$.

Lemma 2.3 ([28]). Let f be a meromorphic function and k a positive integer. If f is a solution of the differential equation $a_0 f^{(k)} + a_1 f^{(k-1)} + \cdots + a_k f = 0$, where $a_0, a_1, \cdots, a_k$ are complex numbers with $a_0 \neq 0$, then $T(r,f) = O(r)$. Moreover, if f is transcendental, then $r = O(T(r,f))$.

Let $f = \sum\limits_{n=0}^{\infty} a_n z^n$ be an entire function. Next we define by $\mu(r) = \max\{|a_n| r^n : n = 0, 1, 2, \cdots\}$ the maximum term of f, and define by $\nu(r,f) = \max\{m : \mu(r) = |a_m| r^m\}$ the central index of f (see[15, pp. 33-35]).

Lemma 2.4 ([25, Lemma 2.2]). Let f be an entire function of infinite order with the lower-order $\mu(f)$ and the hyper-order $\mu_2(f)$. Then

$$\mu(f) = \liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log r}$$

and

$$\mu_2(f) = \liminf_{r\to\infty} \frac{\log\log \nu(r,f)}{\log r}.$$

Lemma 2.5 (Wiman-Valiron, [15, pp. 187-199]). Let $g(z)$ be a transcendental entire function, and let $0 < \delta < \frac{1}{4}$. Then, exists a set $E \subset \mathbb{R}^+$ of finite logarithmic measure, i.e., $\int_E dt/t < +\infty$, such that for all z with $|z| = r \notin E$ and

$$|g(z)| > M(r,g)\nu(r,g)^{-\frac{1}{4}+\delta},$$

one has

$$g^{(m)}(z) = \left(\frac{\nu(r,g)}{z}\right)^m \{1 + o(1)\} g(z),$$

where $m \geq 0$ is an integer.

Lemma 2.6 ([9, Lemma 2] or [10, Lemma 4]). If f is a transcendental entire function of hyper order $\rho_2(f)$, then

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\mathrm{loglog}\nu(r,f)}{\log r}.$$

Lemma 2.7 ([15, Satz 4.5]). If f is a nonconstant entire function of order $\rho(f)$, then

$$\rho(f) = \limsup_{r\longrightarrow\infty} \frac{\log \nu(r,f)}{\log r}.$$

Lemma 2.8 ([13, Theorem 3.2]). Let $f(z)$ be a nonconstant meromorphic function, and let $\Psi(z)$ be a nonconstant meromorphic function such that $\Psi(z) = \sum\limits_{j=0}^{k} a_j(z) f^{(j)}(z)$, where $a_1(z), a_2(z), \cdots, a_k(z)$ are small function of $f(z)$. Then

$$T(r,f) \leq \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right) + S(r,f),$$

where $N_0\left(r,\frac{1}{\psi'}\right)$ denotes the counting function of those zeros of ψ', which are not zeros of $\psi-1$.

Lemma 2.9 ([24, Lemma 2.7]). Suppose that α and β are nonconstant entire functions, and that a_1, a_2, b_1 and b_2 are meromorphic functions satisfying $T(r,a_1)+T(r,a_2)=S(r,e^{\alpha})$, $T(r,b_1)+T(r,b_2)=S(r,e^{\beta})$ and $a_1a_2b_1b_2\not\equiv 0$. If $a_1e^{\alpha}-a_2$ and $b_1e^{\beta}-b_2$ share 0 IM, then $a_1b_2e^{\alpha}=a_2b_1e^{\beta}$ or $a_1b_1e^{\alpha+\beta}=a_2b_2$.

Lemma 2.10 ([24, Lemma 2.8]). Suppose that R_1 and R_2 are rational functions, and that a_1 and a_2 are two constants satisfying $0<|a_1|\le|a_2|$ and $a_1\neq a_2$. Then there exists a constant $A>1$ such that

$$AT(r,e^{a_1z})\le T(r,R_1e^{a_1z}+R_2e^{a_2z})+O(\log r).$$

Lemma 2.11 ([24, Lemma 2.5]). Let f_j $(j=1,2,\cdots,n)$ be nonconstant meromorphic functions satisfying $N\left(r,\frac{1}{f_j}\right)+\overline{N}\left(r,f_j\right)=S(r,f_j)\quad(j=1,2,\cdots,n)$, and let $F=a+\sum\limits_{j=1}^{n}f_j$, where a is a meromorphic function such that $a\not\equiv 0$. If F is not a constant, and $T(r,a)=S(r,F)$, then $T(r,F)=N\left(r,\frac{1}{F}\right)+S(r,F)$.

3 Proof of Theorems

Proof of Theorem 1.1. By the condition that $f-P$ and $L[f]-P$ share 0 CM we have

$$\frac{L[f(z)]-P(z)}{f(z)-P(z)}=e^{\alpha_1(z)},\tag{3.1}$$

where $\alpha_1(z)$ is an entire function. If $f(z)$ is a polynomial, from (3.1) we see that $e^{\alpha_1(z)}$ is a nonzero constant, and so $\rho_2(f)=\rho(e^{\alpha_1})=0$, which reveals the conclusion of Theorem 1.1. We next suppose that f is a transcendental entire function.

Suppose that $|z|$ is large and that $|f(z)-P(z)|\le 1$. Then, $|P(z)|$ and $|f(z)|$ are large. Combining this with Lemma 2.1 and the condition $f\in B$, we know that there exists a sufficiently large positive number R such that

$$|zf'(z)|\ge\frac{|f(z)|}{2\pi}\log\frac{|f(z)|}{R}\ge\frac{1}{4\pi}|P(z)|\log|P(z)|,\tag{3.2}$$

as $|z|$ is large. Therefore, from (3.2) we get

$$|z(f'(z)-P'(z))|\ge\frac{1}{4\pi}|P(z)|\left(\log|P(z)|-4\pi\left|\frac{zP'(z)}{P(z)}\right|\right),\tag{3.3}$$

as $|z|$ is large. From (3.3) and Lemma 2.2 we get

$$\begin{aligned}m\left(r,\frac{1}{f-P}\right)&\le m\left(\frac{z(f'-P')}{f-P}\cdot\frac{1}{z(f'-P')}\right)\\&\le m\left(r,\frac{z(f'-P')}{f-P}\right)+m\left(r,\frac{1}{z(f'-P')}\right)\\&\le O(\log r+\log T(r,f)),\end{aligned}$$

as $|z| = r \notin E$ is large, where $E \subset \mathbb{R}^+$ is some subset with linear measure $mesE < \infty$. This together with (3.1) and Lemma 2.2 gives

$$\begin{aligned} m(r, e^{\alpha_1}) &= m\left(r, \frac{L[f(z) - P(z)] + L[P(z)] - P(z)}{f(z) - P(z)}\right) \\ &\leq m\left(r, \frac{L[f(z) - P(z)]}{f(z) - P(z)}\right) + m\left(r, \frac{L[P(z)] - P(z)}{f(z) - P(z)}\right) + O(1) \\ &\leq O(\log r + \log T(r, f)), \end{aligned}$$

as $|z| = r \notin E$ is large. Combining this with the standard reasoning of removing exceptional set (see [17, Lemma 1.1.2]), we can find that there exists some sufficiently large positive number r_0 such that

$$m(r, e^{\alpha_1}) \leq O(\log r + \log T(2r, f) + \log 2), \tag{3.4}$$

as $r \geq r_0$. From (3.4) and Definition 1.1 we get

$$\rho(e^{\alpha_1}) = \mu(e^{\alpha_1}) \leq \mu_2(f) \leq \rho_2(f). \tag{3.5}$$

We consider the following two cases:

Case 1. Suppose that

$$\mu(f) < \infty. \tag{3.6}$$

Then, from (3.6) and Definition 1.1 we get $\mu_2(f) = 0$, this together with (3.5) implies that e^{α_1} is a constant. Therefore, from (3.1) we can get

$$f^{(\gamma_P+k+1)} + a_{k-1}f^{(\gamma_P+k)} + \cdots + a_1 f^{(\gamma_P+2)} + (a_0 - e^{\alpha_1})f^{(\gamma_P+1)} = 0, \tag{3.7}$$

where γ_P is the degree of P. From (3.7) and Lemma 2.3 we deduce $\rho(f) \leq 1$. Hence $\mu_2(f) = \rho_2(f) = \rho(e^{\alpha_1}) = 0$, and so the conclusion of Theorem 1.1 holds.

Case 2. Suppose that

$$\mu(f) = \infty. \tag{3.8}$$

From (3.8) and Lemma 2.4 we get

$$\liminf_{r \longrightarrow \infty} \frac{\log \nu(r, f)}{\log r} > 1. \tag{3.9}$$

Noting that f is a transcendental entire function, we have

$$M(r, f) \longrightarrow \infty, \tag{3.10}$$

as $r \longrightarrow \infty$. Let

$$M(r, f) = |f(z_r)|, \tag{3.11}$$

where $z_r = re^{i\theta(r)}, \theta(r) \in [0, 2\pi)$. This together with (3.11) and Lemma 2.5 implies that there exist subsets $E_j \subset (1, \infty)$ with finite logarithmic measure, i.e., $\int_{E_j} \frac{dt}{t} < \infty$, where $1 \leq j \leq k$, such that

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{\nu(r, f)}{z_r}\right)^j (1 + o(1)), \tag{3.12}$$

as $r \notin E_j$ and $r \longrightarrow \infty$. By (3.11) and the Cauchy's inequality we have

$$\frac{P(z_r)}{f(z_r)} \longrightarrow 0, \tag{3.13}$$

as $|z_r| \longrightarrow \infty$. From (1.1) we have

$$\frac{L[f(z)] - P(z)}{f(z) - P(z)} = \frac{\frac{L[f(z)]}{f(z)} - \frac{P(z)}{f(z)}}{1 - \frac{P(z)}{f(z)}}. \tag{3.14}$$

From (1.1), (3.9), (3.12)-(3.14) we get

$$\frac{L[f(z_r)] - P(z_r)}{f(z_r) - P(z_r)} = \left(\frac{\nu(r,f)}{z_r}\right)^k (1 + o(1)), \tag{3.15}$$

as $|z_r| = r \notin \cup_{j=1}^n E_j$ and $r \longrightarrow \infty$. From (3.1) and (3.15) we have

$$\left(\frac{\nu(r,f)}{z_r}\right)^k (1 + o(1)) = e^{\alpha_1(z_r)}, \tag{3.16}$$

as $|z_r| = r \notin \cup_{j=1}^n E_j$ and $r \longrightarrow \infty$. From (3.16) we get

$$\left(\frac{\nu(r,f)}{|z_r|}\right)^k \le 2M(r, e^{\alpha_1}), \tag{3.17}$$

as $|z_r| = r \notin \cup_{j=1}^n E_j$ and $r \longrightarrow \infty$. By rewriting (3.17) we get

$$\{\nu(r,f)\}^k \le 2r^k M(r, e^{\alpha_1}), \tag{3.18}$$

as $|z_r| = r \notin \cup_{j=1}^n E_j$ and $r \longrightarrow \infty$. By (3.18) and the standard reasoning of removing exceptional set (see[17, Lemma 1.1.2]) we know that there exists some sufficiently large positive number r_0 such that

$$\{\nu(r,f)\}^k \le 2r^{k\beta} M(r^\beta, e^{\alpha_1}), \tag{3.19}$$

as $r \ge r_0$, where $\beta > 1$ is an arbitrary positive number. From (3.19) we get

$$\begin{aligned}
\liminf_{r\to\infty} \frac{\log\log \nu(r,f)}{\log r} &= \liminf_{r\to\infty} \frac{\log\log \{\nu(r,f)\}^k}{\log r} \\
&\le \liminf_{r\to\infty} \frac{\log\log \{2r^{k\beta} M(r^\beta, e^{\alpha_1})\}}{\log r} \\
&= \beta \liminf_{r\to\infty} \frac{\log\log M(r^\beta, e^{\alpha_1})}{\log r^\beta}.
\end{aligned} \tag{3.20}$$

From (3.20), Definition 1.1 and Lemma 2.4 we get

$$\mu_2(f) \le \beta\mu(e^{\alpha_1}). \tag{3.21}$$

By letting $\beta \to 1^+$ on two sides of (3.21) we have

$$\mu_2(f) \leq \mu(e^{\alpha_1}). \tag{3.22}$$

Similarly, from (3.19), Definition 1.1 and Lemma 2.6 we can get

$$\rho_2(f) \leq \rho(e^{\alpha_1}). \tag{3.23}$$

Noting that $\mu_2(f) \leq \rho_2(f)$ and $\mu(e^{\alpha_1}) = \rho(e^{\alpha_1})$, we can get from (3.22) and (3.23) that

$$\mu_2(f) \leq \rho_2(f) \leq \mu(e^{\alpha_1}) = \rho(e^{\alpha_1}). \tag{3.24}$$

From (3.5) and (3.24) we get the conclusion of Theorem 1.1.

Theorem 1.1 is thus completely proved.

Proof of Theorem 1.2. By the conditions of Theorem 1.2 we have

$$\frac{L[f(z)] - P(z)}{f(z) - P(z)} = \frac{e^{\alpha_2(z)}}{(z-\omega_1)^k(z-\omega_2)^k \cdots (z-\omega_{n-1})^k(z-\omega_n)^k}, \tag{3.25}$$

where α_2 is an entire function. Let

$$h(z) = \frac{e^{\alpha_2(z)}}{(z-\omega_1)^k(z-\omega_2)^k \cdots (z-\omega_{n-1})^k(z-\omega_n)^k}. \tag{3.26}$$

Then, from (3.26) we know that (3.25) can be rewritten as

$$\frac{L[f] - P}{f - P} = h. \tag{3.27}$$

From (3.26) we have

$$\mu(e^{\alpha_2}) = \rho(e^{\alpha_2}) = \mu(h) = \rho(h). \tag{3.28}$$

Proceeding as in the proof of Theorem 1.1, we can get from (3.27)

$$\mu(h) = \rho(h) \leq \mu_2(f) \leq \rho_2(f). \tag{3.29}$$

Next we let

$$F = P_0 f, \tag{3.30}$$

where P_0 is a nonconstant polynomial such that P_0 and $1/f$ share 0 CM. Then, F is a transcendental entire function. By calculating we get from (3.30) that

$$\frac{f^{(k)}}{f} = \frac{F^{(k)}}{F} + \frac{kR_0'}{R_0} \cdot \frac{F^{(k-1)}}{F} + \cdots + \binom{k}{l} \frac{R_0^{(l)}}{R_0} \cdot \frac{F^{(k-l)}}{F} + \cdots + \frac{kR_0^{(k-1)}}{R_0} \cdot \frac{F'}{F} + \frac{R_0^{(k)}}{R_0}, \tag{3.31}$$

where and in what follows,

$$R_0 = \frac{1}{P_0}. \tag{3.32}$$

By calculating we get from (3.32) and the definition of P_0 that

$$\frac{R_0'(z)}{R_0(z)} = \frac{m_1}{z-\omega_1} + \frac{m_2}{z-\omega_2} + \cdots + \frac{m_n}{z-\omega_n}, \tag{3.33}$$

where $m_1, m_2, \cdots, m_n$ are negative integers. By mathematical induction we get from (3.33) that

$$\frac{R_0^{(j)}(z)}{R_0(z)} = \frac{\{(-1)^{j-1}(j-1)! \sum_{l=1}^{n} m_l\}(1+o(1))}{z^j}, \tag{3.34}$$

as $|z| \longrightarrow \infty$, where j is a positive integer satisfying $1 \leq j \leq k$. Noting that F is a transcendental entire function, we know from Lemma 2.7 and the proposition of the central index in [15, P.33-35] that

$$\nu(r, F) \longrightarrow +\infty. \tag{3.35}$$

Let

$$M(r, F) = |F(z_r)|, \tag{3.36}$$

where $z_r = re^{i\theta(r)}$, and that $\theta(r) \in [0, 2\pi)$ is some nonnegative real number. From (3.36) and Lemma 2.5 we know that there exists some subset $E_j \subset (1, \infty)$ with finite logarithmic measure, i.e., $\int_{E_j} \frac{dt}{t} < \infty$, such that for some point $z_r = re^{i\theta(r)}$, $\theta(r) \in [0, 2\pi)$, as $|z_r| = r \notin E_j$ and $M(r, F) = |F(z_r)|$, we have

$$\frac{F^{(j)}(z_r)}{F(z_r)} = \left(\frac{\nu(r, F)}{z_r}\right)^j \{1 + o(1)\}. \tag{3.37}$$

From (3.31), (3.34)-(3.37) we get

$$\begin{aligned}
&\frac{f^{(k)}(z_r)}{f(z_r)} = \\
&\left.\frac{F^{(k)}}{F} + \frac{kR_0'}{R_0} \cdot \frac{F^{(k-1)}}{F} + \cdots + \binom{k}{l} \frac{R_0^{(l)}}{R_0} \cdot \frac{F^{(k-l)}}{F} + \cdots + \frac{kR_0^{(k-1)}}{R_0} \cdot \frac{F'}{F} + \frac{R_0^{(k)}}{R_0}\right|_{z=z_r} \\
&= \frac{\{\nu(r, F)\}^k\{1 + o(1)\} + N_n \sum_{l=1}^{k-1} \binom{k}{l} \sum_{l=1}^{k-1} (-1)^{l-1}(l-1)!\{\nu(r, F)\}^{k-l}\{1 + o(1)\}}{z_r^k} \\
&= \left(\frac{\nu(r, F)}{z_r}\right)^k \{1 + o(1)\},
\end{aligned} \tag{3.38}$$

as $r \notin \cup_{j=1}^k E_j$ and $r \longrightarrow \infty$, where $N_n = \sum_{l=1}^{n} m_l$. Similarly

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{\nu(r, F)}{z_r}\right)^j \{1 + o(1)\}, \tag{3.39}$$

as $r \notin \cup_{j=1}^k E_j$ and $r \longrightarrow \infty$, where $1 \leq j \leq k - 1$. From (1.1) we have

$$\frac{L[f(z)] - P(z)}{f(z) - P(z)} = \frac{\frac{f^{(k)}(z)}{f(z)} + a_{k-1} \cdot \frac{f^{(k-1)}(z)}{f(z)} + \cdots + a_1 \cdot \frac{f'(z)}{f(z)} + a_0 - \frac{P(z)}{f(z)}}{1 - \frac{P(z)}{f(z)}}. \tag{3.40}$$

From (3.30), (3.36) and the Cauchy's inequality we get

$$\frac{P(z_r)}{f(z_r)} = \frac{P_0(z_r)P(z_r)}{F(z_r)} \longrightarrow 0, \tag{3.41}$$

as $|z_r| = r \longrightarrow \infty$. We discuss the following two cases:

Case 1. Suppose that

$$\liminf_{r\longrightarrow\infty} \frac{\log\nu(r,F)}{\log r} > 1. \tag{3.42}$$

From (3.25)-(3.27) and (3.38)-(3.42) we have

$$\begin{aligned}
\left(\frac{\nu(r,F)}{2|z_r|}\right)^k &\leq \left|\left(\frac{\nu(r,F)}{z_r}\right)^k \{1+o(1)\}\right| \\
&= \left|\frac{\frac{f^{(k)}(z)}{f(z)} + a_{k-1}\cdot\frac{f^{(k-1)}(z)}{f(z)} + \cdots + a_1\cdot\frac{f'(z)}{f(z)} + a_0 - \frac{P(z)}{f(z)}}{1-\frac{P(z)}{f(z)}}\right|_{z=z_r} \\
&= |h(z_r)| \\
&\leq 2\left|e^{\alpha_2(z_r)}\right| \\
&\leq 2M(r,e^{\alpha_2}),
\end{aligned} \tag{3.43}$$

as $|z_r| = r \notin \cup_{j=1}^k E_j$ and $r \longrightarrow \infty$. From (3.43) we have

$$\{\nu(r,F)\}^k \leq 2^{k+1}r^k M(r,e^{\alpha_2}). \tag{3.44}$$

as $|z_r| = r \notin \cup_{j=1}^k E_j$ and $r \longrightarrow \infty$. By (3.44) and the standard reasoning of removing exceptional set (see[17, Lemma 1.1.2]) we know that there exists some positive number r_0 such that

$$\{\nu(r,F)\}^k \leq 2^{k+1}r^{k\beta}M(r^\beta,e^{\alpha_2}), \tag{3.45}$$

as $r \geq r_0$, where $\beta > 1$ is an arbitrary positive number. This together with Definition 1.1, Lemma 2.4 and Lemma 2.6 gives

$$\mu_2(F) \leq \rho_2(F) \leq \mu(e^{\alpha_2}) = \rho(e^{\alpha_2}). \tag{3.46}$$

By (3.30) we have $\mu_2(f) = \mu_2(F)$ and $\rho_2(f) = \rho_2(F)$. This together with (3.46) gives

$$\mu_2(f) \leq \rho_2(f) \leq \rho(e^{\alpha_2}). \tag{3.47}$$

From (3.28), (3.29) and (3.47) we get the conclusion of Theorem 1.2.

Case 2. Suppose that

$$\liminf_{r\longrightarrow\infty} \frac{\log\nu(r,F)}{\log r} \leq 1. \tag{3.48}$$

From (3.48) and Lemma 2.4 we have $\mu(F) \leq 1$, and so $\mu_2(F) = 0$. Combining this with (3.30), we get $\mu_2(F) = \mu_2(f) = 0$. This together with (3.29) implies that e^{α_2} is a constant. Hence

$$\mu_2(f) = \rho(h) = 0. \tag{3.49}$$

We discuss the following two subcases:

Subcase 2.1. Suppose that there exists some subset $I \subset \mathbb{R}^+$ with logarithmic measure $\log mesI = +\infty$, such that

$$\lim_{\substack{r\to\infty \\ r\in I}} \frac{\log \nu(r,F)}{\log r} > 1. \tag{3.50}$$

Then, from (3.50) we can see that there exist some sequence of positive number $r_n \in I \setminus \cup_{j=1}^{k} E_j$, such that

$$\lim_{r_n\to\infty} \frac{\nu(r_n,F)}{r_n} = +\infty. \tag{3.51}$$

Proceeding as in the proof of (3.43), we can get from (3.25)-(3.27) and (3.51) that

$$\left(\frac{\nu(r_n,F)}{2r_n}\right)^k \leq 2\left|e^{\alpha_2(z_{r_n})}\right|, \tag{3.52}$$

From (3.51) and (3.52) we deduce

$$\lim_{r_n\to\infty} \left|e^{\alpha_2(z_{r_n})}\right| = +\infty,$$

which contradicts the fact that e^{α_2} is a constant.

Subcase 2.2. Suppose that there exists some subset $E \subset \mathbb{R}^+$ with logarithmic measure $\log mesE < +\infty$, such that

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\log \nu(r,F)}{\log r} \leq 1. \tag{3.53}$$

Then, from (3.53) we have

$$\nu(r,F) \leq r^2, \tag{3.54}$$

as $r \longrightarrow \infty$ and $r \notin E$. By (3.54) and the standard reasoning of removing exceptional set (see[17, Lemma 1.1.2]) we know that there exists some positive number r_0 such that

$$\nu(r,F) \leq r^{2\beta}, \tag{3.55}$$

as $r \geq r_0$, where $\beta > 1$ is an arbitrary positive number. From (3.55) and Lemma 2.7 we deduce $\rho(f) \leq 2$, which implies that $\rho_2(f) = 0$. This together with (3.28), (3.29) and (3.49) reveals the conclusion of Theorem 1.2.

Theorem 1.2 is thus completely proved.

Proof of Theorem 1.3. First of all, by Lemma 2.8 and the condition that f is a transcendental meromorphic function we have

$$T(r,f) \leq \overline{N}(r,f) + N\left(r, \frac{1}{f(z)-z}\right) + \overline{N}\left(r, \frac{1}{L'[f](z)-z}\right) + O(\log r + \log T(r,f)), \tag{3.56}$$

as $r \notin E$ and $r \to \infty$, where $E \subset \mathbb{R}^+$ is some subset with linear measure $mesE < +\infty$.

Let z_0 be a zero of $f(z) - z$ with multiplicity≥ 2. Then $f(z_0) = z_0$ and $f'(z_0) = 1$. Combining this with the condition that $f(z) - z$ and $L[f(z)] - z$ share 0 IM, we have $L[f(z_0)] = z_0$. Hence $1 + a_0 z_0 = z_0$, and so $f(z) - z$ has at most one zero with multiplicity≥ 2. Therefore

$$N\left(r, \frac{1}{f(z)-z}\right) = \overline{N}\left(r, \frac{1}{f(z)-z}\right) + O(\log r) \tag{3.57}$$

$$= \overline{N}\left(r, \frac{1}{L[f(z)]-z}\right) + O(\log r). \tag{3.58}$$

By the condition that $L[f(z)] - z$ and $L'[f(z)] - z$ share 0 CM, we have

$$\overline{N}\left(r, \frac{1}{L'[f](z)-z}\right) = \overline{N}\left(r, \frac{1}{L[f](z)-z}\right). \tag{3.59}$$

Noting that f is a transcendental meromorphic function that has at most finitely many poles, we can get from (3.56), (3.58) and (3.59) that

$$\begin{aligned} T(r,f) &\leq 2\overline{N}\left(r, \frac{1}{L[f](z)-z}\right) + O(\log r + \log T(r,f)) \\ &\leq 2T(r, L[f]) + O(\log r + \log T(r,f)) \\ &\leq 4T(r,f) + O(\log r + \log T(r,f)), \end{aligned} \tag{3.60}$$

as $r \notin E$ and $r \to \infty$.

From (3.60) and the condition that f is a transcendental meromorphic function we can see that $L[f]$ is a transcendental meromorphic function. Moreover, by (3.60), Definition 1.1 and the standard reasoning of removing exceptional set (see [17, Lemma 1.1.1]) we can get

$$\mu_2(f) = \mu_2(L[f]) \quad \text{and} \quad \rho_2(f) = \rho_2(L[f]). \tag{3.61}$$

We consider the following two cases:

Case 1. Suppose that f has at least one pole in the complex plane, say η_1, η_2, $\cdots$, η_{n-1} and η_n are n distinct poles of f in the complex plane, where $n \geq 1$ is a positive integer. Then, by the condition that $L[f(z)] - z$ and $L'[f(z)] - z$ share 0 CM we have

$$\frac{L'[f(z)]-z}{L[f(z)]-z} = \frac{e^{\alpha_3(z)}}{(z-\eta_1)(z-\eta_2)\cdots(z-\eta_{n-1})(z-\eta_n)}, \tag{3.62}$$

where α_3 is an entire function. By (3.61), (3.62) and Theorem 1.2 we have

$$\mu_2(f) = \rho_2(f) = \rho(e^{\alpha_3}). \tag{3.63}$$

From (3.63) and the condition that $\rho_2(f) < \infty$ is not a positive integer we can see that $e^{\alpha_3} =: c$ is a constant. Therefore, (3.62) can be rewritten as

$$\frac{L'[f(z)]-z}{L[f(z)]-z} = \frac{c}{(z-\eta_1)(z-\eta_2)\cdots(z-\eta_{n-1})(z-\eta_n)}. \tag{3.64}$$

Next we let

$$F_1 = P_1 L[f], \tag{3.65}$$

where P_1 is a nonconstant polynomial such that P_1 and $1/L[f]$ share 0 CM. Then, F_1 is a transcendental entire function. By calculating we get from (3.65) that

$$\frac{L'[f]}{L[f]} = \frac{F_1'}{F_1} + \frac{R_1'}{R_1}, \tag{3.66}$$

where and in what follows,

$$R_1 = \frac{1}{P_1}. \tag{3.67}$$

By calculating we get from (3.67) and the definition of P_1 that

$$\frac{R_1'(z)}{R_1(z)} = \frac{p_1}{z-\eta_1} + \frac{p_2}{z-\eta_2} + \cdots + \frac{p_n}{z-\eta_n} = \frac{\sum\limits_{l=1}^{n} p_l}{z}\{1+o(1)\}, \tag{3.68}$$

as $|z| \longrightarrow \infty$, where $p_1, p_2, \cdots, p_n$ are negative integers. Noting that F_1 is a transcendental entire function, we know from Lemma 2.7 and the proposition of the central index in [15, P.33-35] that

$$\nu(r, F_1) \longrightarrow +\infty, \tag{3.69}$$

as $r \to \infty$. Let

$$M(r, F_1) = |F_1(z_r)|, \tag{3.70}$$

where $z_r = re^{i\theta(r)}$, and that $\theta(r) \in [0, 2\pi)$ is some nonnegative real number. From (3.70) and Lemma 2.5 we know that there exists some subset $E \subset (1, \infty)$ with finite logarithmic measure, i.e., $\int_E \frac{dt}{t} < \infty$, such that

$$\frac{F_1'(z_r)}{F_1(z_r)} = \frac{\nu(r, F_1)}{z_r}\{1+o(1)\}, \tag{3.71}$$

as $|z_r| = r \notin E$ and $r \longrightarrow \infty$. From (3.66), (3.68)-(3.71) we have

$$\frac{L'[f(z_r)]}{L[f(z_r)]} = \frac{F_1'(z_r)}{F_1(z_r)} + \frac{R_1'(z_r)}{R_1(z_r)} = \frac{\nu(r, F_1) + \sum\limits_{l=1}^{n} p_l}{z_r} \cdot \{1+o(1)\} = \frac{\nu(r, F_1)}{z_r}\{1+o(1)\}, \tag{3.72}$$

as $r \notin E$ and $r \longrightarrow \infty$. From (3.65), (3.70) and the Cauchy's inequality we get

$$\frac{z_r}{L[f(z_r)]} = \frac{z_r P_1(z_r)}{F_1(z_r)} \longrightarrow 0, \tag{3.73}$$

as $|z_r| = r \longrightarrow \infty$. Noting that

$$\frac{L'[f(z)] - z}{L[f(z)] - z} = \frac{\frac{L'[f(z)]}{L[f(z)]} - \frac{z}{L[f(z)]}}{1 - \frac{z}{L[f(z)]}},$$

we can get from (3.72) and (3.73) that

$$\frac{L'[f(z_r)] - z_r}{L[f(z_r)] - z_r} = \frac{\nu(r, F_1)}{z_r}\{1 + o(1)\} - \frac{z_r}{L[f(z_r)]}\{1 + o(1)\}, \tag{3.74}$$

as $|z_r| = r \notin E$ and $r \longrightarrow \infty$. From (3.64) and (3.74) we have

$$\begin{aligned} \frac{\nu(r, F_1)}{2} &\leq \left| \frac{cz_r}{(z_r - \eta_1)(z_r - \eta_2)\cdots(z_r - \eta_{n-1})(z_r - \eta_n)} \right| + \left| \frac{2z_r^2}{L[f(z_r)]} \right| \\ &= \left| \frac{cz_r}{(z_r - \eta_1)(z_r - \eta_2)\cdots(z_r - \eta_{n-1})(z_r - \eta_n)} \right| + \left| \frac{2z_r^2 P_1(z_r)}{F_1(z_r)} \right|, \end{aligned}$$

as $|z_r| = r \notin E$ and $r \longrightarrow \infty$. This together with (3.65), (3.70) and the Cauchy's inequality gives

$$\nu(r, F_1) = O(1), \tag{3.75}$$

as $|z_r| = r \notin E$ and $r \longrightarrow \infty$, which contradicts (3.69).

Case 2. Suppose that f is a transcendental entire function. Then, in the same manner as in the proof of (3.64) we have

$$\frac{L'[f(z)] - z}{L[f(z)] - z} \equiv d, \tag{3.76}$$

where $d \neq 0$ is a constant. We discuss the following three subcases.

Subcase 2.1. Suppose that $a_0 = 0$. Then it follows from (1.2) that (3.76) can be rewritten as

$$f''(z) - df'(z) = (1 - d)z. \tag{3.77}$$

From (3.77) we deduce

$$f(z) = d_1 e^{dz} + \frac{d-1}{2d} z^2 + \frac{d-1}{d^2} z + d_2, \tag{3.78}$$

where $d_1 \neq 0$ and d_2 are constants. Thus,

$$f(z) - z = d_1 e^{dz} + \frac{d-1}{2d} z^2 + \frac{d-1-d^2}{d^2} z + d_2, \tag{3.79}$$

$$L[f(z)] - z = d_1 d e^{dz} - \frac{1}{d} z + \frac{d-1}{d^2}. \tag{3.80}$$

Assume that $d \neq 1$. From (3.79), (3.80), Lemma 2.9 and the condition that $f(z) - z$ and $L[f] - z$ share 0 IM, we can get a contradiction. Thus $d = 1$, and so it follows from (3.79) and (3.80) that $f(z) - z = d_1 e^z - z + d_2$ and $L[f(z)] - z = d_1 e^z - z$. Combining this with the condition that $f(z) - z$, $L[f(z)] - z$ share 0 IM we deduce $d_2 = 0$, and so it follows that $f(z) = d_1 e^z$, which reveals the conclusion of Theorem 1.3.

Subcase 2.2. Suppose that $a_0 \neq 0$ and $a_0 = -d$. Then it follows from (1.2) and (3.76) that

$$f''(z) - 2df'(z) + d^2 f(z) = (1 - d)z. \tag{3.81}$$

From (3.81) we deduce

$$f(z) = (d_3 z + d_4)e^{dz} + \frac{1-d}{d^2}z + \frac{2(1-d)}{d^3},$$

where d_3 and d_4 are constants satisfying $d_3 z + d_4 \not\equiv 0$. Thus,

$$f(z) - z = (d_3 z + d_4)e^{dz} + \frac{1-d-d^2}{d^2}z + \frac{2(1-d)}{d^3}, \tag{3.82}$$

$$L[f(z)] - z = d_3 e^{dz} - \frac{1}{d}z + \frac{d-1}{d^2}. \tag{3.83}$$

By Lemma 2.9, (3.82) and (3.83) we can get a contradiction.

Subcase 2.3. Suppose that $a_0 \neq 0$ and $a_0 \neq -d$. Then it follows from (1.2) and (3.76) that

$$f''(z) + (a_0 - d)f'(z) - a_0 d f(z) = (1-d)z. \tag{3.84}$$

From (3.84) we deduce

$$f(z) = d_5 e^{-a_0 z} + d_6 e^{dz} + \frac{d-1}{a_0 d}z + \frac{(a_0-d)(d-1)}{a_0^2 d^2},$$

where d_5 and d_6 are constants satisfying $d_5 e^{-a_0 z} + d_6 e^{dz} \not\equiv 0$. Thus,

$$f(z) - z = d_5 e^{-a_0 z} + d_6 e^{dz} + P_1(z), \tag{3.85}$$

$$L[f(z)] - z = d_6(d + a_0)e^{dz} + P_2(z), \tag{3.86}$$

where

$$P_1(z) = \frac{d-1-a_0 d}{a_0 d}z + \frac{(a_0-d)(d-1)}{a_0^2 d^2},$$

$$P_2(z) = -\frac{1}{d}z + \frac{d-1}{d^2}.$$

If $d_5 = 0$, then $d_6 \neq 0$. By (3.85), (3.86) and Lemma 2.9 we get a contradiction. If $d_6 = 0$, then $d_5 \neq 0$. From (3.85) and (3.86) we obtain a contradiction. Next, we suppose that $d_5 \neq 0$ and $d_6 \neq 0$.

Let z_0 be a zero of $L[f(z)] - z$. From (3.86) we obtain

$$d_6(d + a_0)e^{dz_0} + P_2(z_0) = 0. \tag{3.87}$$

Since $f(z) - z$ and $L[f(z)] - z$ share 0 IM, from (3.85) we deduce

$$d_5 e^{-a_0 z_0} + d_6 e^{dz_0} + P_1(z_0) = 0. \tag{3.88}$$

From (3.87) and (3.88) we have

$$d_5(d + a_0)e^{-a_0 z_0} + (d + a_0)P_1(z_0) - P_2(z_0) = 0. \tag{3.89}$$

Noting that z_0 is a zero of $L[f(z)] - z$, from (3.87) and (3.89) we obtain

$$\overline{N}(r, \frac{1}{d_6(d+a_0)e^{dz} + P_2(z)}) \leq \overline{N}(r, \frac{1}{d_5(d+a_0)e^{-a_0 z} + (d+a_0)P_1(z) - P_2(z)}). \tag{3.90}$$

It is easy to see that

$$T(r, e^{dz}) = \overline{N}(r, \frac{1}{d_6(d+a_0)e^{dz} + P_2(z)}) + O(\log r), \tag{3.91}$$

$$T(r, e^{-a_0 z}) = \overline{N}(r, \frac{1}{d_5(d+a_0)e^{-a_0 z} + (d+a_0)P_1(z) - P_2(z)}) + O(\log r). \tag{3.92}$$

From (3.90)-(3.92) we deduce

$$T(r, e^{dz}) \leq T(r, e^{-a_0 z}) + O(\log r). \tag{3.93}$$

Since

$$T(r, e^{dz}) = \frac{|d|r}{\pi} \quad \text{and} \quad T(r, e^{-a_0 z}) = \frac{|a_0|r}{\pi},$$

from (3.93) we get $|d| \leq |a_0|$. Noting that $d \neq -a_0$, by (3.85), (3.86) and Lemma 2.10 we know that there exists some constant $A > 1$ such that

$$AT(r, L[f]) \leq T(r, f) + O(\log r). \tag{3.94}$$

On the other hand, from (3.86) we have

$$T(r, L[f(z)]) = \overline{N}\left(r, \frac{1}{L[f(z)] - z}\right) + O(\log r). \tag{3.95}$$

By (3.85), (3.86) and the condition that $f(z) - z$ and $L[f(z)] - z$ share 0 IM, we deduce $P_1(z) \not\equiv 0$. Combining this with Lemma 2.11 we get

$$T(r, f(z)) = N\left(r, \frac{1}{f(z) - z}\right) + O(\log r). \tag{3.96}$$

From (3.57) and (3.96) we obtain

$$T(r, f(z)) = \overline{N}\left(r, \frac{1}{f(z) - z}\right) + O(\log r). \tag{3.97}$$

Since $f(z) - z$ and $L[f(z)] - z$ share 0 IM, we have

$$\overline{N}\left(r, \frac{1}{f(z) - z}\right) = \overline{N}\left(r, \frac{1}{L[f(z)] - z}\right). \tag{3.98}$$

From (3.95), (3.97) and (3.98) we obtain

$$T(r, L[f]) = T(r, f) + O(\log r). \tag{3.99}$$

Noting that f is a transcendental entire function, from (3.94) and (3.99) we get a contradiction.

Theorem 1.3 is thus completely proved.

4 Concluding Remarks

We recall the following example:

Example 4.1. Let $f(z) = (e^z - 1)^2$ and $L[f](z) = f^{(3)}(z) - 3f''(z) + \frac{5}{2}f'(z) - f(z)$. Then it is easy to see that $L[f](z) - 1 = (f(z) - 1)e^{-z}$ and $\rho_2(f) + 1 = \rho(f) = \rho(e^{-z}) = 1$. Moreover, by $f'(z) = 2e^z(e^z - 1)$ we can see that $f'(z)$ has infinitely many zeros in the complex plane, and so $f(z) \notin B$.

From Example 4.1, Theorems 1.1 and 1.2, we give the following conjecture:

Conjecture 4.1. If the nonconstant polynomial P in Theorems 1.1 and 1.2 is replaced with a finite value $a \neq 0$, then Theorems 1.1 and 1.2 still hold.

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