Existence of nontrivial weak solutions for a quasilinear elliptic systems with concave-convex nonlinearities*

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Abstract

In this paper, our main purpose is to establish the existence of nontrivial weak solutions to the following systems:

$$\begin{cases} -\triangle_p u = \lambda V(x)|u|^{r-2}u + F_u(x,u,v), & x \in \Omega, \\ -\triangle_p v = \theta V(x)|v|^{r-2}v + F_v(x,u,v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $\lambda, \theta > 0$, $\triangle_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$ is the s-Laplacian of u. We obtain the existence results in two cases: (i)$1 < r < p < N$; (ii)$1 < p < r < p^*$. The existence results of solutions are obtained by variational methods.

1 Introduction

In this paper, we are interested in finding multiple nontrivial weak solutions to the following quasilinear elliptic systems

$$\begin{cases} -\triangle_p u = \lambda V(x)|u|^{r-2}u + F_u(x,u,v), & x \in \Omega, \\ -\triangle_p v = \theta V(x)|v|^{r-2}v + F_v(x,u,v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbf{R}^N$, $\lambda, \theta > 0$, and $1 < r < p^*$, $r \neq p$, $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = \infty$ if $p \geq N$ is the critical Sobolev exponent, $\triangle_s u = \mathrm{div}(|\nabla u|^{s-2}\nabla u)$ is the s-Laplacian of u.

Recently, more and more attention have been paid to the existence and multiplicity of nontrivial weak solutions for the elliptic problems involving concave-convex nonlinearities and critical Soblev exponent. For $p = 2$, see [2,8,15-16,23], and the references therein. For the quasilinear problems, the corresponding results can be found in [4,17,19,25-26]. By the results of the above papers we know that the number of nontrivial solutions for problem (1.1) is affected by the concave-convex nonlinearities.

If $p = 2$, $u = v$ and $F_u = |u|^{2^*-2}u$, (1.1) can be reduced to

$$\begin{cases} -\triangle u = \lambda V(x)|u|^{r-2}u + |u|^{2^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.2}$$

which is a normal Schrodinger equation and has been widely studied, see[10,12,21].

The solutions of problem (1.2) corresponding to the critical points of the energy functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{r}\int_\Omega V(x)|u|^r dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx$$

defined on $W_0^{1,2}(\Omega)$. When $r = 2$, the pioneer result of Brezis-Nirenberg [8] studied problem (1.2) and showed that under some suitable conditions, probnlem (1.2) possesses a positive solution in $W_0^{1,2}(\Omega)$. For more results see [9,18] and reference therein.

The typically difficulty in dealing with problem (1.2) is that the corresponding functional $I(u)$ doesn't satisfy (PS) condition due to the lack of compactness of the embedding: $H_0^1 \hookrightarrow L^{2^*}(\Omega)$. Hence we couldn't use the standard variational methods. However, if $1 < r < 2$, the situation is quite different, see [6,24]. The main essence is that when $1 < r < 2$, the functional $I(u)$ is sublinear, when λ is small enough, $I(u)$ satisfies $(PS)_c$ condition for $c < 0$, so we can look for critical points of negative critical values of $I(u)$.

Many authors studied the following general p-Laplacian problem

$$\begin{cases} -\triangle_p u = \lambda V(x)|u|^{r-2}u + |u|^{p^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

many results valid for problem (1.2) has been extended to problem (1.3). For example, see [4,19,26]. The main difficulty in extending the results for problem (1.2) to the corresponding results for problem (1.3) is that $W_0^{1,p}(\Omega)$ is not a Hilbert space in general, then more analysis is needed.

We recall some results about problem (1.1) now. When $V(x) \equiv 1$ and $F(x,u,v) = \frac{2}{\alpha+\beta}|u|^\alpha|v|^\beta$, $\alpha + \beta = p^*$, (1.1) becomes the following case

$$\begin{cases} -\triangle_p u = \lambda|u|^{r-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, & x \in \Omega, \\ -\triangle_p v = \theta|v|^{r-2}v + \frac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

when $p = r = 2$, Alves et al [2] considered (1.4) and proved the existence of least energy solutions for any $\lambda, \theta \in (0, \lambda_1)$ and generalized the corresponding results of [8] to the case of systems (1.4), here λ_1 denote the first eigenvalue of operator $-\triangle$. Subsequently, Han [15] considered the existence of multiple positive solutions for(1.4) and in [17] T.S.Hsu studied systems (1.4) when $1 < r < p < N, \alpha + \beta = p^*$, with the help of the Nehari manifold, he proved that problem (1.4) has at least two positive solutions if the pair of the parameters (λ, μ) belongs to a certain subset of $\mathbf{R}^2$. More results for problem (1.1) see [16,23,25] etc..

In this paper, we will consider the existence and infinitely many weak solutions of problem (1.1). Let us denote the Banach space $H = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ throughout this paper, and for the functions $V(x), F(x, u, v)$, we add the following assumptions:

(d_1) Suppose $V(x) \in L^{\frac{p^*}{p^*-r}}(\Omega)$ and $V(x) > \sigma > 0$ in Ω;
(d_2) $F : \overline{\Omega} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}^+$ is a C^1 function and $F(x, tu, tv) = t^{p^*} F(x, u, v)$ $(t > 0), \forall x \in \overline{\Omega},\ (u, v) \in \mathbf{R}^2$;
(d_3) $F(x, 0, v) = F(x, u, 0) = F_v(x, 0, v) = F_u(x, u, 0) = 0$, where $u, v \in \mathbf{R}$;
(d_4) $F(x, u, v)$ is even respect to u, v;
(d_5) there exist $x_0 \in \Omega$ and $k > \frac{N-p}{p-1}$ such that

$$F(x_0, 1, 1) = \max_{x \in \Omega} F(x, 1, 1) \geq 2^{\frac{N}{N-p}} M$$

and

$$F(x, 1, 1) = F(x_0, 1, 1) + o(|x - x_0|^k) \ \text{ as } \ x \to x_0,$$

where $M = \max_{\{(x,s,t) \in \overline{\Omega} \times R^+ \times R^+ : s^p + t^p = 1\}} F(x, s, t) > 0$.

Then we have the following results:

Theorem 1.1 Assume $1 < r < p < N$, and $(d_1) - (d_4)$ hold. Then there is a positive constant Λ^* such that for any $0 < \lambda + \theta \leq \Lambda^*$, problem (1.1) possesses infinitely many weak solutions in H.

Theorem 1.2 Assume $1 < r < p < N$, $(d_1) - (d_3)$ and (d_5) hold. Then there is a positive constant Λ^{**} such that for any $0 < \lambda + \theta \leq \Lambda^{**}$, problem (1.1) possesses a nontrivial weak solutions in H.

Theorem 1.3. Assume $1 < p < r < p^*$ and $(d_1) - (d_3)$ hold. Then there is a a positive constant Λ_*, such that for any $(\lambda + \theta) > \Lambda_*$, problem (1.1) possesses a nontrivial weak solutions in H.

Remark 1.4. In [4], J.G.Azvrero and I.P.Aloson obtained that there exists a nontrivial solution for (1.3) with $V(x) \equiv 1$ by the Mountain Pass Lemma. In fact, Theorem 1.3 is an extension of Theorem 3.2 in [4] to systems (1.1).

Remark 1.5. Assume $1 < p < N$, Ω be a bounded domain in R^N,

$$F(x, u, v) = f_1(x)(|u|^\alpha |v|^\beta + |u|^\beta |v|^\alpha) + f_2(x)|u|^{\frac{p^*}{2}} |v|^{\frac{p^*}{2}},$$

where $\alpha+\beta=p^*$, $f_i \in C^1(\Omega)$ and satisfy $f_i(x_0)=\max_{x\in\Omega} f_i(x)$,

$$f_i(x)=f_i(x_0)+o(|x-x_0|^k) \text{ as } x\to x_0$$

for $i=1,2$. Then it's easy to see that $F(x,u,v)$ satisfy $(d_2)-(d_5)$, and it is not contained in the previous works.

The present paper is organized as follows: in section 2, we give some preliminary results; in section 3- 5, we will give the proofs of Theorem 1.1-1.3 respectively.

2 Preliminaries results

Let H' be dual of H, $\langle,\rangle$ the duality paring between H' and H, the norm on H is given by

$$\|z\|=\|(u,v)\|=(\|u\|_p^p+\|v\|_p^p)^{\frac{1}{p}}$$

and the norm on $L^p(\Omega)\times L^p(\Omega)$ is given by

$$|z|=|(u,v)|=(|u|_p^p+|v|_p^p)^{\frac{1}{p}}$$

where $z=(u,v)\in H$ and $\|\cdot\|_p, |\cdot|_p$ are the norm on $W_0^{1,p}(\Omega)$ and $L^p(\Omega)$ respectively, that is,

$$\|u\|_p=(\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}},\ \ |u|_p=(\int_\Omega |u|^p dx)^{\frac{1}{p}}.$$

Throughout this paper, we denote weak convergence by $\rightharpoonup$, and denote strong convergence by $\to$, also we denote positive constants(possibly different) by C_i.

From (d_2), we have the so-called Euler identity

$$z\cdot\nabla F(x,z)=p^*F(x,z) \tag{2.1}$$

and

$$F(x,z)\le M|z|^{p^*} \text{ for all } z\in R^2, \tag{2.2}$$

where M is given in section 1.

As usually, we also denote by

$$S=\inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}}\frac{\|u\|^p}{(\int_\Omega |u|^{p^*}dx)^{\frac{p}{p^*}}}$$

the best Sobolev constant for the embedding $W_0^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$. It is known that S is independent of Ω and is never achieved except when $\Omega=\mathbf{R}^N$. Define

$$S_F:=\inf_{(u,v)\in H}\{\frac{\|u\|_p^p+\|v\|_p^p}{(\int_\Omega F(x,u,v)dx)^{\frac{p}{p^*}}}:\int_\Omega F(x,u,v)dx>0\}.$$

According to (2.2) and the Minkowski inequality, we have

$$\begin{aligned}(\int_\Omega F(x,u,v)dx)^{\frac{p}{p^*}} &\leq M^{\frac{p}{p^*}}[(\int_\Omega |u|^{p^*}dx)^{\frac{p}{p^*}} + (\int_\Omega |v|^{p^*}dx)^{\frac{p}{p^*}}] \\ &\leq M^{\frac{p}{p^*}}\frac{1}{S}\int_\Omega |\nabla u|^p + |\nabla v|^p dx.\end{aligned}$$

Then we obtain that

$$S_F \geq SM^{-\frac{p}{p^*}} > 0. \tag{2.3}$$

By (2.1), the corresponding energy functional of problem (1.1) is defined by

$$E(z) = E(u,v) = \frac{1}{p}(\|u\|_p^p + \|v\|_p^p) + \frac{1}{r}\int_\Omega V(x)(\lambda|u|^r + \theta|v|^r)dx + \int_\Omega F(x,u,v)dx$$

for $z = (u,v) \in H$. Under the hypotheses of our theorems, it is obvious that E is a C^1 functional. It is well known that any critical point of E in H is a weak solution of problem (1.1). Hence, in order to obtain the nontrivial solutions of problem (1.1), we only need to look for the nontrivial critical points of E in H.

Now, we define the Palais-Smale(PS)-sequence, (PS)-value, and (PS)-conditions in H for E as follows.

Definition 2.1. (I) For $c \in R$, a sequence $\{z_n\} \in H$ is a (PS)$_c$-sequence for E if $E(z_n) = c + o(1)$ and $E'(z_n) = o(1)$ strongly in H' as $n \to \infty$.

(II) $c \in R$ is a (PS)-value for E if there exists a (PS)$_c$-sequence in H for E.

(III) E satisfies the (PS)$_c$-condition in H if every (PS)$_c$-sequence in H for E contains a convergent sub-sequence.

Now we give some results for the proof of main results.

Lemma 2.2. Assume $1 < r < p^*, r \neq p$ and $(d_1) - (d_2)$ hold. If $\{z_n\} \subset H$ is a (PS)$_c$ secquence for E, then $\{z_n\}$ is bounded in H.

Proof. Let $z_n = (u_n, v_n)$ be a (PS)$_c$ secquence for E. We argue by contradiction. Assume that $\|z_n\| \to \infty$. Let

$$\overline{z}_n = (\overline{u}_n, \overline{v}_n) = \frac{z_n}{\|z_n\|} = (\frac{u_n}{\|z_n\|}, \frac{v_n}{\|z_n\|}).$$

Then $\|\overline{z}_n\| = 1$, we may assume that $\overline{z}_n \rightharpoonup \overline{z} = (\overline{u}, \overline{v})$ in H. Thus we have that

$$\overline{u}_n \to \overline{u},\ \overline{v}_n \to \overline{v} \ \text{ in } \ L^s(\Omega),\ 1 \leq s < p^*$$

and

$$\int_\Omega \lambda V(x)|\overline{u}_n|^r + \theta V(x)|\overline{v}_n|^r dx = \int_\Omega \lambda V(x)|\overline{u}|^r + \theta V(x)|\overline{v}|^r dx + o(1). \tag{2.4}$$

Since $\{z_n\} \subset H$ is a (PS)$_c$ secquence for E and $\|z_n\| \to \infty$, we have

$$\frac{1}{p}\int_\Omega |\nabla\overline{u}_n|^p + |\nabla\overline{v}_n|^p dx - \frac{\|z_n\|^{r-p}}{r}\int_\Omega \lambda V(x)|\overline{u}_n|^r + \theta V(x)|\overline{v}_n|^r dx - \|z_n\|^{p^*-p}\int_\Omega F(x,\overline{u}_n,\overline{v}_n)dx = o(1) \tag{2.5}$$

and

$$\int_\Omega |\nabla \overline{u}_n|^p + |\nabla \overline{v}_n|^p dx - \|z_n\|^{r-p} \int_\Omega \lambda V(x)|\overline{u}_n|^r + \theta V(x)|\overline{v}_n|^r dx - \\ p^* \|z_n\|^{p^*-p} \int_\Omega F(x, \overline{u}_n, \overline{v}_n) dx = o(1). \quad (2.6)$$

From (2.4)-(2.6), we can deduce that

$$(\frac{p^*}{p} - 1) \int_\Omega |\nabla \overline{u}_n|^p + |\nabla \overline{v}_n|^p dx = \\ (\frac{p^*}{r} - 1)\|z_n\|^{r-p} \int_\Omega \lambda V(x)|\overline{u}|^r + \theta V(x)|\overline{v}|^r dx + o(1).$$

Since $1 < r < p^*, r \neq p$ and $\|z_n\| \to \infty$, we deduce that when $r < p$

$$\int_\Omega |\nabla \overline{u}_n|^p + |\nabla \overline{v}_n|^p dx \to 0, \ \text{ as } n \to \infty,$$

when $r > p$

$$\int_\Omega |\nabla \overline{u}_n|^p + |\nabla \overline{v}_n|^p dx \to \infty, \ \text{ as } n \to \infty,$$

which is contrary to the fact $\|\overline{z}_n\| = 1$.

Lemma 2.3. Assume $1 < r < p$ and $(d_1) - (d_3)$ hold. If $\{z_n\} \subset H$ is a $(\mathrm{PS})_c$ secquence for E, then there exist $z \in H$ and $B > 0$ such that

$$E(z) \geq -B(\lambda + \theta)^{\frac{p}{p-r}},$$

where B will be given later.

Proof. By Lemma 2.2, we know that z_n is bounded in H, there is a $z = (u, v) \in H$ and a subsequence of $\{z_n\}$, sitll denoted by $\{z_n\}$ such that

$$\begin{cases} z_n = (u_n, v_n) \rightharpoonup (u, v) = z, & \text{in } H; \\ z_n = (u_n, v_n) \to (u, v) = z, & \text{in } L^s(\Omega) \times L^s(\Omega), \ 1 \leq s < p^*; \\ z_n = (u_n, v_n) \to (u, v) = z, & \text{a.e. in } \Omega; \\ \nabla z_n = (\nabla u_n, \nabla v_n) \to (\nabla u, \nabla v) = \nabla z, & \text{a.e. in } \Omega. \end{cases}$$

Consequently, passing to the limit in $\langle E'(z_n), (\phi, \varphi)\rangle$ as $n \to \infty$, together with $(d_1) - (d_3)$, we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi dx - \lambda \int_\Omega V(x)|u|^{r-2} u \phi dx - \int_\Omega F_u(x, u, v)\phi dx = 0$$

and

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi dx - \theta \int_\Omega V(x)|v|^{r-2} v \varphi dx - \int_\Omega F_v(x, u, v)\varphi dx = 0$$

for all $(\phi, \varphi) \in H$.

It shows that z is a critical point of E, then we have $\langle E'(z), z\rangle = 0$ and

$$\|z\|^p - \int_\Omega \lambda V(x)|u|^r + \theta V(x)|u|^r dx = p^* \int_\Omega F(x,u,v)dx.$$

Thus,

$$E(z) = (\frac{1}{p} - \frac{1}{p^*})\|z\|^p - (\frac{1}{r} - \frac{1}{p^*}) \int_\Omega \lambda V(x)|u|^r + \theta V(x)|v|^r dx.$$

By the Hölder inequality, the Young inequality, and the Sobolev embedding theorem, we have

$$\begin{aligned} E(z) &\geq \frac{1}{N}\|z\|^p - (\frac{1}{r} - \frac{1}{p^*})|V(x)|_{\frac{p^*}{p^*-r}} (\lambda|u|^r_{p^*} + \theta|v|^r_{p^*}) \\ &\geq \frac{1}{N}\|z\|^p - (\frac{1}{r} - \frac{1}{p^*})S^{-\frac{r}{p}}|V(x)|_{\frac{p^*}{p^*-r}} (\lambda + \theta)\|z\|^r_p. \end{aligned}$$

Consider the following function,

$$g(x) = C_1 x^p - C_2(\lambda+\theta)x^r, \ \ x > 0$$

where $C_1 = \frac{1}{N}, C_2 = (\frac{1}{r} - \frac{1}{p^*})S_q^{-\frac{r}{q}}|V(x)|_{\frac{p^*}{p^*-r}}$ are positive constants. It is easy to see the function obtains its absolute minimum(for $x > 0$) at point $x_0 = (\frac{C_2 r(\lambda+\theta)}{C_1 p})^{\frac{1}{p-r}}$, then we have

$$g(x) \geq g(x_0) = -B(\lambda+\theta)^{\frac{p}{p-r}},$$

where $B = C_1^{\frac{-r}{p-r}} C_2^{\frac{p}{p-r}} (\frac{r}{p})^{\frac{r}{p-r}} (1 - \frac{r}{p}) > 0$ is independent of λ, θ. Then we obtain

$$E(z) \geq -B(\lambda+\theta)^{\frac{p}{p-r}}. \square$$

In addition, we need the following version of the Brezis-Lieb lemma[7].

Lemma 2.4. Assume that $G \in C^1(\overline{\Omega}, R^2)$ with $G(x,0,0) = 0$ and $|\frac{\partial G}{\partial u}(z)|$, $|\frac{\partial G}{\partial v}(z)| \leq C|z|^{s-1}$ for some $1 \leq s < \infty$. Let z_n be a bounded sequence in $L^s(\Omega) \times L^s(\Omega)$, and such that $z_n \rightharpoonup z$ in H. Then, as $n \to \infty$,

$$\int_\Omega G(x, z_n)dx = \int_\Omega G(x, z_n - z)dx + \int_\Omega G(x,z)dx + o(1).$$

Lemma 2.5. Assume $1 < r < p$ and $(d_1) - (d_3)$ hold. Then E satisfies the (PS)$_c$ condition with c satisfying

$$c < \frac{1}{N}(S_F p^{*^{-\frac{p}{p^*}}})^{\frac{N}{p}} - B(\lambda+\theta)^{\frac{p}{p-r}}. \tag{2.7}$$

Proof. Suppose $\{z_n = (u_n, v_n)\} \subset H$ is a (PS)$_c$ sequence of E with $c < \frac{1}{N}(S_F p^{*^{-\frac{p}{p^*}}})^{\frac{N}{p}} - B(\lambda+\theta)^{\frac{p}{p-r}}$, i.e.,

$$E(z_n) = c + o(1), E'(z_n) = o(1),$$

by Lemma 2.2, we may assume there exist a subsequence of $\{z_n\}$ and $z=(u,v)\in H$ such that $z_n \rightharpoonup z$ in H. By the argument in Lemma 2.3, we have

$$\int_\Omega \lambda V(x)|u_n|^r+\theta V(x)|v_n|^r dx=\int_\Omega \lambda V(x)|u|^r+\theta V(x)|v|^r dx+o(1)$$

and

$$\langle E'(z),z\rangle=0.$$

Let $\widetilde{v}_n=u_n-u, \widetilde{v}_n=v_n-v$ and $\widetilde{z}_n=(\widetilde{u}_n,\widetilde{v}_n)$. Then by Lemma 2.4, we deduce that

$$\|\widetilde{z}_n\|^p=\|z_n\|^p-\|z\|^p+o(1). \tag{2.8}$$

Since (d_2) and (d_3) hold, it follows from Lemma 2.4 that

$$\int_\Omega F(x,z_n)dx=\int_\Omega F(x,\widetilde{z}_n)dx+\int_\Omega F(x,z)dx+o(1). \tag{2.9}$$

From $E(z_n)=c+o(1)$, $E'(z_n)=o(1)$ and (2.8),(2.9), we obtain

$$\frac{1}{p}\|\widetilde{z}_n\|^p-\int_\Omega F(x,\widetilde{z}_n)dx=c-E(z)+o(1) \tag{2.10}$$

and

$$\|\widetilde{z}_n\|^p-p^*\int_\Omega F(x,\widetilde{z}_n)dx=o(1). \tag{2.11}$$

From (2.11), we may suppose that

$$\|\widetilde{z}_n\|^p\to l,\ \int_\Omega F(x,\widetilde{z}_n)dx\to\frac{l}{p^*},$$

if $l=0$, then we have $z_n\to z$ in H, we complete the proof. On the contrary, we ssume $l>0$, by the definition of S_F, we have

$$\|\widetilde{z}_n\|^p\ge S_F(\int_\Omega F(x,\widetilde{z}_n)dx)^{\frac{p}{p^*}},$$

then as $n\to\infty$ we obtain that

$$l\ge(S_Fp^{*-\frac{p}{p^*}})^{\frac{N}{p}}.$$

On the other hand, from (2.10) and Lemma 2.3, we have that

$$c=\frac{l}{p}-\frac{l}{p^*}+E(z)\ge\frac{1}{N}(S_Fp^{*-\frac{p}{p^*}})^{\frac{N}{p}}-B(\lambda+\theta)^{\frac{p}{p-r}},$$

which contradicts $c<\frac{1}{N}(S_Fp^{*-\frac{p}{p^*}})^{\frac{N}{p}}-B(\lambda+\theta)^{\frac{p}{p-r}}$. ■

The following is the classical Deformation Lemma:

Lemma 2.6.(see[1]) Let $f\in C^1(X,\mathbf{R})$ and satisfy (PS) condition. If $c\in\mathbf{R}$ and N is any neighborhood of $K_c\doteq\{u\in X|f(u)=c,f'(u)=0\}$, there exist $\eta(t,x)\equiv\eta_t(x)\in C([0,1]\times X,X)$ and constants $\overline{\epsilon}>\epsilon>0$ such that

(1) $\eta_0(x) = x$ for all $x \in X$,
(2) $\eta_t(x) = x$ for all $x \in f^{-1}[c - \overline{\epsilon}, c + \overline{\epsilon}]$,
(3) $\eta_t(x)$ is a homeomorphism of X onto X for all $t \in [0,1]$,
(4) $f(\eta_t(x)) \leq f(x)$ for all $x \in X, t \in [0,1]$,
(5) $\eta_1(A_{c+\epsilon} - N) \subset A_{c+\epsilon}$, where $A_c = \{x \in X | f(x) \leq c\}$ for any $c \in R$,
(6) if $K_c = \emptyset, \eta_1(A_{c+\epsilon}) \subset A_{c-\epsilon}$,
(7) if f is even, η_t is odd in x.

Remark 2.7. Lemma 2.6 is also true if f satisfies $(PS)_c$ condition for $c < c_0$ for some $c_0 \in \mathbf{R}$.

At the end of this section, we recall some concepts in minimax theory.

Let X be a Banach space, and

$$\Sigma = \{A \subset X \setminus \{0\} | A \text{ is closed, } -A = A\},$$

and

$$\Sigma_k = \{A \in \Sigma | \gamma(A) \geq k\},$$

where $\gamma(A)$ is the Z_2 genus of A, that is

$$\gamma(A) = \begin{cases} \inf\{n : \text{there exist odd, continuous } h : A \to \mathbf{R}^n \setminus \{0\}\}, \\ +\infty, \text{ if it doesn't exist odd, continuous } h : A \to \mathbf{R}^n \setminus \{0\}, \forall n \in Z_+, \\ 0, \text{ if } A = \emptyset. \end{cases}$$

The main properties of genus are contained in the following lemma.

Lemma 2.8.(see[20]) Let $A, B \in \Sigma$. Then
(1) If there exists $f \in C(A, B)$, odd, then $\gamma(A) \leq \gamma(B)$.
(2) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.
(3) If there exists an odd homeomorphism between A and B, then $\gamma(A) = \gamma(B)$.
(4) If S^{N-1} is the sphere in $\mathbf{R}^N$, then $\gamma(S^{N-1}) = N$.
(5) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
(6) If $\gamma(A) < \infty$, then $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.
(7) If A is compact, then $\gamma(A) < \infty$, and there exists $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$, where $N_\delta(A) = \{x \in X | d(x, A) \leq \delta\}$.
(8) If X_0 is a subspace of X with codimension k, and $\gamma(A) > k$, then $A \cap X_0 \neq \emptyset$.

3 Proof of Theorem 1.1

We will prove the existence of infinitely many solutions for systems(1.1) in this section. We try to use Lusternik-Schnirelman's theory for Z_2-invariant functional (see [20]). But since the functional $E(z)$ defined in section 2 is not bounded from below, so we following [4] to consider a truncated functional $E_\infty(z)$ which will be constructed later.

At first, let's consider the functional $E(z)$, using the Sobolev's inequality with the hypothesis $1 < r < p < N$, we obtain

$$\begin{aligned} E(z) &= \frac{1}{p}\|z\|^p - \frac{1}{r}\int_\Omega \lambda V(x)|u|^r + \theta V(x)|u|^r dx - \int_\Omega F(x,z)dx \\ &\geq \frac{1}{p}\|z\|^p - \frac{1}{r}S^{-\frac{r}{p}}|V(x)|_{\frac{p^*}{p^*-r}}(\lambda+\theta)\|z\|_p^r - S_F^{-\frac{p^*}{p}}\|z\|^{p^*} \\ &= C_3\|z\|^p - C_4(\lambda+\theta)\|z\|^r - C_5\|z\|^{p^*} \end{aligned}$$

where $C_3 = \frac{1}{p}, C_4 = \frac{1}{r}S^{-\frac{r}{p}}|V(x)|_{\frac{p^*}{p^*-r}}, C_5 = S_F^{-\frac{p^*}{p}}$ are all positive constants.

We now consider function

$$h(x) = C_3x^p - C_4(\lambda+\theta)x^r - C_5x^{p^*},\ x > 0$$

by the hypothesis $1 < r < p < p^*$, we easily know that there exists a $\Lambda^* > 0$ such that for any $0 < (\lambda+\theta) \leq \Lambda^*$, we have the following results hold:

(a) $h(x)$ reaches its positive maximum;

(b) $\frac{1}{N}(S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}} - B(\lambda+\theta)^{\frac{p}{p-r}} \geq 0$, where B is given in Lemma 2.3.

From the structure of $h(x)$, we see that there are two positive solutions $R_1 < R_2$ of $h(x) = 0$. Then we can easily know that

$$h(x)\begin{cases} < 0,\ x \in (0,R_1)\cup(R_2,\infty), \\ > 0,\ x \in (R_1,R_2). \end{cases} \tag{3.1}$$

We let $\tau : R^+ \to [0,1]$ be C^∞ and nonincreasing function such that

$$\tau(x) = 1,\ \text{ if } x \in (0,R_1),$$

$$\tau(x) = 0,\ \text{ if } x \in (R_2,\infty).$$

Let $\varphi(z) = \tau(\|z\|)$, we consider the truncated functional

$$E_\infty(z) = \frac{1}{p}\|z\|^p - \frac{1}{r}\int_\Omega \lambda V(x)|u|^r + \theta V(x)|v|^r dx - \int_\Omega F(x,z)\varphi(z)dx,$$

similar as above, we consider the function

$$\overline{h}(x) = C_3x^p - C_4(\lambda+\theta)x^r - C_5x^{p^*}\tau(x),$$

and have that

$$E_\infty(z) \geq \overline{h}(\|z\|). \tag{3.2}$$

By further analysis, we can see that $\overline{h}(x) \geq h(x)$, for all $x \in (0,\infty)$; and $\overline{h}(x) = h(x)$, for $x \in (0,R_1]$; and $\overline{h}(x) \geq 0$, for $x \in [R_2,\infty)$. So we have that $E(z) = E_\infty(z)$ when $\|z\| \in (0,R_1]$, and since $\tau \in C^\infty$, we get $E_\infty(z) \in C^1(H,R)$. Also we obtain the following results.

Lemma 3.1. (1) If $E_\infty(z) < 0$, then $\|z\| \in (0, R_1)$, and $E(w) = E_\infty(w)$ for all w in a small enough neighborhood of z.

(2) There exists a $\Lambda^* > 0$, such that when $0 < (\lambda + \theta) \leq \Lambda^*$, $E_\infty(z)$ satisfies the $(PS)_c$ condition for $c < 0$.

Proof. We prove (1) by contradiction, assume $E_\infty(z) < 0$ and $\|z\| \in [R_1, \infty)$. Then if $\|z\| \in [R_1, R_2]$, by (3.1),(3.2), we see that

$$E_\infty(z) \geq \overline{h}(\|z\|) \geq h(\|z\|) \geq 0.$$

If $\|z\| \in (R_2, \infty)$, by (3.2) and above analysis, we also have that

$$E_\infty(z) \geq \overline{h}(\|z\|) \geq 0.$$

Thus $\|z\| \in (0, R_1)$, (1) holds.

Now, we prove (2), let Λ^* as above. If $c < 0$ and $\{z_n\} \subset H$ is a $(PS)_c$ sequence of E_∞, then we may assume that $E_\infty(z_n) < 0$ and $E'_\infty(z_n) = o(1)$, by (1), $\|z_n\| \in (0, R_1)$, hence $E(z_n) = E_\infty(z_n)$ and $E'(z_n) = E'_\infty(z_n)$. Since (b) hold when $0 < (\lambda + \theta) \leq \Lambda^*$, By Lemma 2.5, $E(z)$ satisfies the $(PS)_c$ condition for $c < 0$. Thus $E_\infty(z)$ satisfies the $(PS)_c$ condition for $c < 0$, (2) holds. ∎

Now we prove our main result via genus.

Proof of Theorem 1.1. Let $\Sigma_k = \{A \subset H - \{(0,0)\}, A \text{ is closed}, A = -A, \gamma(A) \geq k\}$, $c_k = \inf_{A \in \Sigma_k} \sup_{z \in A} E_\infty(z)$, $K_c = \{z \in H |\ E_\infty(z) = c, E'_\infty(z) = 0\}$, and suppose that $0 < (\lambda + \theta) \leq \Lambda^*$, Λ^* is as above.

We claim that if $k, l \in N$ are such that $c = c_k = c_{k+1} = \cdots = c_{k+l}$, then $\gamma(K_c) \geq l + 1$.

In fact, we assume

$$E_\infty^{-\varepsilon} = \{z \in H |\ E_\infty(z) \leq -\varepsilon\},$$

we will show for any $k \in N$, there exist an $\varepsilon = \varepsilon(k) > 0$, such that

$$\gamma(E_\infty^{-\varepsilon}(z)) \geq k.$$

Fix $k \in N$, denote H_k be an k-dimensional subspace of H, choose $z = (u, v) \in H_k$, with $\|z\| = 1$, for $0 < \rho < R_1$, we have

$$E(\rho z) = E_\infty(\rho z) = \frac{1}{p}\rho^p - \frac{\rho^r}{r}\int_\Omega \lambda V(x)|u|^r + \theta V(x)|v|^r dx - \rho^{p^*}\int_\Omega F(x, z)dx. \tag{3.3}$$

For H_k is a finite dimension space, all the norms in H_k are equivalent. So we can define

$$\alpha_k = \sup\{|z|_{p^*}^{p^*} |\ z \in H_k, \|z\| = 1\} < \infty, \tag{3.4}$$

$$\beta_k = \inf\{|z|_r^r |\ z \in H_k, \|z\| = 1\} > 0, \tag{3.5}$$

from (3.3)-(3.5), we have

$$E_\infty(\rho z) \leq \frac{1}{p}\rho^p - \sigma\beta_k \frac{\min\{\lambda, \theta\}\rho^r}{r} + \rho^{p^*} M\alpha_k.$$

For any $\varepsilon > 0$ and an $0 < \rho < R_1$ such that $E_\infty(\rho z) \le -\varepsilon$ for $z \in H_k$, $\|z\| = 1$, let $S_\rho = \{z \in H | \, \|z\| = \rho\}$, then $S_\rho \cap H_k \subset E_\infty^{-\varepsilon}$. By Lemma 2.8, we obtain that

$$\gamma(E_\infty^{-\varepsilon}(z)) \ge \gamma(S_\rho \cap H_k) = k. \tag{3.6}$$

Since E_∞ is continuous and even, with (3.6), we have $E_\infty^{-\varepsilon} \in \Sigma_k$ and $c = c_k \le -\varepsilon < 0$. As E_∞ is bounded from below, we see that $c = c_k > -\infty$(This is the main reason that we consider E_∞ instead of E). Then by Lemma 3.1 E_∞ satisfies $(PS)_c$ condition and it is easy to see that K_c is a compact set.

Now we prove our claim by contradiction, suppose on the contrary $\gamma(K_c) \le l$. By Lemma 2.8, there is a closed and symmetric set U with $K_c \subset U$ and $\gamma(U) \le l$. Since $c < 0$, we also can assume that the closed set $U \subset E_\infty^0$. By Lemma 2.6, there exists an odd homeomorphism

$$\eta : \; H \to H$$

such that $\eta(E_\infty^{c+\delta} - U) \subset E_\infty^{c-\delta}$ for some $0 < \delta < -c$.

From the definition of $c = c_{k+l}$, we know that there is an $A \in \Sigma_{k+l}$ such that

$$\sup_{z \in A} E_\infty(z) < c + \delta,$$

i.e., $A \subset E_\infty^{c+\delta}$, and

$$\eta(A - U) \subset \eta(E_\infty^{c+\delta} - U) \subset E_\infty^{c-\delta},$$

that's meaning

$$\sup_{z \in \eta(A-U)} E_\infty(z) \le c - \delta. \tag{3.7}$$

Again by Lemma 2.8, we have

$$\gamma(\eta(\overline{A-U})) \ge \gamma(\overline{A-U}) \ge \gamma(A) - \gamma(U) \ge k.$$

Thus we have $\eta(\overline{A-U}) \in \Sigma_k$ and $\sup_{z \in \eta(\overline{A-U})} E_\infty(z) \ge c_k = c$, which contradicts to (3.7). So we have proved our claim.

Now let's complete the proof of Theorem 1.1. If for all $k \in N$, we have $\Sigma_{k+1} \subset \Sigma_k$, $c_k \le c_{k+1} < 0$. If all c_k are distinct, then $\gamma(K_{c_k}) \ge 1$, and we see that $\{c_k\}$ is a sequence of distinct negative critical values of E_∞; if for some k_0, there is a $l \ge 1$ such that $c = c_{k_0} = c_{k_0+1} = \cdots = c_{k_0+l}$, then by the claim, we have

$$\gamma(K_c) \ge l + 1,$$

which shows that K_c contains infinitely many distinct elements.

By Lemma 3.1, we know $E(z) = E_\infty(z)$ when $E_\infty(z) < 0$, so we show that there are infinitely many critical points of $E(z)$. Theorem 1.1 is proved. ∎

4 Proof of Theorem 1.2.

In this section, we will prove Theorem 1.2 by the following general version of the Mountain Pass Lemma(see[3]).

Lemma 4.1. Let I be a functional on a Banach space H, $I \in C^1(H, R)$. Let us assume that there exists $\rho, R > 0$ such that

(i) $I(z) > \rho$, $\forall z \in H$ with $\|z\| = R$.

(ii) $I(0) = 0$, and $I(w_0) < \rho$ for some $w_0 \in H$, with $\|w_0\| > R$.

Let us define $\Gamma = \{\gamma \in C([0,1], H) |\ \gamma(0) = 0, \gamma(1) = w_0\}$, and

$$\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)). \tag{4.1}$$

Then there exists a sequence $\{z_n\} \subset H$, such that $I(z_n) \to \mu$, and $I'(z_n) \to 0$ in H' (dual of H) as $n \to \infty$.

Define, for $\eta > 0$,

$$u_\eta(x) = \frac{\eta^{\frac{N-p}{p(p-1)}} \psi(x)}{(\eta^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}})^{\frac{N-p}{p}}},$$

where $\psi(x) \in C_0^\infty(B(x_0, 2\delta_0))$ is such that $0 \le \psi(x) \le 1$, $\psi(x) \equiv 1$ on $B(x_0, \delta_0)$ and $|\nabla \psi| \le C$ for some positive constant C.

After a detailed calculation, we have the following estimate

$$\frac{\int_\Omega |\nabla u_\eta|^p dx}{(\int_\Omega |u_\eta|^{p^*} dx)^{\frac{p}{p^*}}} = S + O(\eta^{\frac{N-p}{p-1}}),\ \eta \to 0. \tag{4.2}$$

Now we show that (4.2) is valid. Indeed, we have

$$\nabla u_\eta(x) = \eta^{\frac{N-p}{p(p-1)}} \Big(\frac{\nabla \psi}{(\eta^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}})^{\frac{N-p}{p}}} - \frac{N-p}{p-1} \frac{\psi |x - x_0|^{\frac{2-p}{p-1}} x}{(\eta^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}})^{\frac{N}{p}}} \Big).$$

Let $x = x_0 + \eta y$, by the definition of ψ, we obtain

$$\begin{aligned}
\int_\Omega |\nabla u_\eta|^p dx &= \eta^{\frac{N-p}{p-1}} \int_\Omega \frac{|x - x_0|^{\frac{p}{p-1}}}{(\eta^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}})^N} dx + O(\eta^{\frac{N-p}{p-1}}) \\
&= \eta^{\frac{N-p}{p-1}} \int_{R^N} \frac{|x - x_0|^{\frac{p}{p-1}}}{(\eta^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}})^N} dx + O(\eta^{\frac{N-p}{p-1}}) \\
&= \int_{R^N} \frac{|y|^{\frac{p}{p-1}}}{(1 + |y|^{\frac{p}{p-1}})^N} dy + O(\eta^{\frac{N-p}{p-1}}) \\
&= |\nabla U|^p_{L^p(R^N)} + O(\eta^{\frac{N-p}{p-1}}),
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |u_\eta|^{p^*}dx &= \eta^{\frac{N}{p-1}}\int_\Omega \frac{\psi^{p^*}}{(\eta^{\frac{p}{p-1}}+|x-x_0|^{\frac{p}{p-1}})^N}dx \\
&= \eta^{\frac{N}{p-1}}\int_{B(x_0,\delta_0)} \frac{1}{(\eta^{\frac{p}{p-1}}+|x-x_0|^{\frac{p}{p-1}})^N}dx + O(\eta^{\frac{N}{p-1}}) \\
&= \eta^{\frac{N}{p-1}}\int_{R^N} \frac{1}{(\eta^{\frac{p}{p-1}}+|x-x_0|^{\frac{p}{p-1}})^N}dx + O(\eta^{\frac{N}{p-1}}) \\
&= \int_{R^N} \frac{1}{(1+|y|^{\frac{p}{p-1}})^N}dy + O(\eta^{\frac{N}{p-1}}) \\
&= |U|^{p^*}_{L^{p^*}(R^N)} + O(\eta^{\frac{N}{p-1}}),
\end{aligned}$$

where $U(x) = (1+|x|^{\frac{p}{p-1}})^{-\frac{N-p}{p}} \in W^{1,p}(R^N)$ satisfies

$$\frac{|\nabla U|^p_{L^p(R^N)}}{|U|^p_{L^{p^*}(R^N)}} = S = \inf_{u\in W^{1,p}(R^N)\setminus\{0\}} \frac{|\nabla u|^p_{L^p(R^N)}}{|u|^p_{L^{p^*}(R^N)}}.$$

A direct calculation, we deduce that (4.2) holds.

Proof of Theorem 1.2. By the analysis in section 3, see (a), we know that when $\lambda+\theta<\Lambda^*$ there exist $R,\rho>0$ such that

$$E(z)>\rho, \quad \text{for all } \|z\|=R.$$

On the other hand, since F is positive homogenous of degree p^* and $1<r<p<p^*$, for any $z_0\in H$, it's easy to see that

$$\lim_{t\to\infty} E(tz_0) = -\infty.$$

Choose $t_0>0$ large enough such that $E(t_0z_0)<\rho$ and $\|t_0z_0\|>R$, set $w_0=t_0z_0$, then we know that the functional E has the mountain pass geometry.

From (2.7) and (4.1), we only need to show

$$\mu < \frac{1}{N}(S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}} - B(\lambda+\theta)^{\frac{p}{p-r}}, \tag{4.3}$$

then Lemma 4.1 and Lemma 2.5 give the existence of the critical point of E.

Let we take $z_\eta=(u_\eta,u_\eta)$, and

$$g(t) = J(tz_\eta) = \frac{2t^p}{p}\int_\Omega |\nabla u_\eta|^p dx - t^{p^*}\int_\Omega F(x,1,1)|u_\eta|^{p^*}dx.$$

We can easily see that $g(t)$ attains its maximum at $t_\eta = (\frac{2\int_\Omega |\nabla u_\eta|^p dx}{p^*\int_\Omega F(x,1,1)u_\eta^{p^*}dx})^{\frac{1}{p^*-p}}$. Using the definition of u_η and F, we obtain $t_\eta<\infty$. We also have

$$\sup_{t\geq 0} J(tz_\eta) = J(t_\eta z_\eta) = \Phi(\eta)+\Psi(\eta),$$

where

$$\Phi(\eta) = \frac{2t_\eta^p}{p}|\nabla u_\eta|^p - F(x_0,1,1)t_\eta^{p^*}\int_\Omega |u_\eta|^{p^*}dx,$$

$$\Psi(\eta) = t_\eta^{p^*}\int_\Omega (F(x_0,1,1) - F(x,1,1))|u_\eta|^{p^*}dx.$$

We deduce from (2.3) and (4.2) that

$$\begin{aligned}\Phi(\eta) &\leq \frac{1}{N}p^{*-\frac{N}{p^*}}(F(x_0,1,1))^{-\frac{N-p}{p}}[\frac{2\int_\Omega |\nabla u_\eta|^p dx}{(\int_\Omega |u_\eta|^{p^*}dx)^{\frac{p}{p^*}}}]^{\frac{N}{p}} \\ &= \frac{1}{N}p^{*-\frac{N}{p^*}}(F(x_0,1,1))^{-\frac{N-p}{p}}(2S)^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}) \\ &\leq \frac{1}{N}(S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}),\end{aligned}$$

here we use the assumption that $2^{\frac{N}{N-p}}M \leq F(x_0,1,1)$.

It follows from (d_5) that there exists $\rho_0 \in (0,\delta_0)$ such that

$$0 \leq F(x_0,1,1) - F(x,1,1) \leq |x_0 - x|^k \text{ for all } x \in B(x_0,\rho_0).$$

From $k > \frac{N-p}{p-1}$, noticing that $t_\eta < \infty$, we have

$$\begin{aligned}\Psi(\eta) &= t_\eta^{p^*}\eta^{\frac{N}{p-1}}\int_\Omega \frac{(F(x_0,1,1) - F(x,1,1))\psi^{p^*}}{(\eta^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}})^N}dx \\ &\leq t_\eta^{p^*}\eta^{\frac{N}{p-1}}\int_{R^N\setminus B(x_0,\rho_0)} \frac{2F(x_0,1,1)}{(\eta^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}})^N}dx \\ &\quad + t_\eta^{p^*}\eta^{\frac{N}{p-1}}\int_{B(x_0,\rho_0)} \frac{|x-x_0|^k}{(\eta^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}})^N}dx \\ &\leq 2t_\eta^{p^*}\eta^{\frac{N}{p-1}}F(x_0,1,1)\int_{R^N\setminus B(x_0,\rho_0)} |x-x_0|^{-\frac{pN}{p-1}}dx + \\ &\quad \frac{t_\eta^{p^*}\eta^{\frac{N-p}{p-1}}}{N}\int_{B(x_0,\rho_0)} |x-x_0|^{k-\frac{p(N-1)}{p-1}}dx \\ &= N\omega_N 2t_\eta^{p^*}\eta^{\frac{N}{p-1}}F(x_0,1,1)\int_{\rho_0}^{+\infty} r^{-\frac{N}{p-1}-1}dr + \omega_N t_\eta^{p^*}\eta^{\frac{N-p}{p-1}}\int_0^{\rho_0} r^{k-1-\frac{N-p}{p-1}}dr \\ &= (p-1)\omega_N 2t_\eta^{p^*}\eta^{\frac{N}{p-1}}F(x_0,1,1)\rho_0^{-\frac{N}{p-1}} + \frac{(p-1)\omega_N t_\eta^{p^*}}{k(p-1)-N+p}\eta^{\frac{N-p}{p-1}}\rho_0^{k-\frac{N-p}{p-1}} \\ &= O(\eta^{\frac{N-p}{p-1}}),\end{aligned}$$

where $\omega_N = \frac{2\pi^{\frac{N}{2}}}{N\Gamma(\frac{N}{2})}$ denotes the volume of the unit ball $B(0,1) \subset R^N$.

Then we have

$$\sup_{t\geq 0} J(tz_\eta) \leq \frac{1}{N}(S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}).$$

By the definition of $E(z)$ and u_η, and $(d_1), (d_2)$, we have

$$E(tz_\eta) \le \frac{2}{p} t^p \int_\Omega |\nabla u_\eta|^p dx = \frac{2}{p} t^p [|\nabla U|^p_{L^p(R^N)} + O(\eta^{\frac{N-p}{p-1}})].$$

Then there exist a $T \in (0,1)$ and $\delta_1 > 0$ such that for $\lambda + \theta < \delta_1$

$$\sup_{0 \le t \le T} E(tz_\eta) \le \frac{1}{N} (S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}} - B(\lambda+\theta)^{\frac{p}{p-r}}.$$

For $t \ge T$, we have

$$\begin{aligned} \sup_{t \ge T} E(tz_\eta) &= \sup_{t \ge T} [J(tz_\eta) - \frac{t^r}{r} \int_\Omega \lambda V(x)|u_\eta|^r + \theta V(x)|u_\eta|^r dx] \\ &\le \frac{1}{N} (S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}) - \sigma(\lambda+\theta) \frac{T^r}{r} \int_{B(x_0,\delta_0)} |u_\eta|^r dx. \end{aligned}$$

Let $\eta \in (0, \delta_0]$, then we have

$$\begin{aligned} \int_{B(x_0,\delta_0)} |u_\eta|^r dx &= \eta^{\frac{r(N-p)}{p(p-1)}} \int_{B(x_0,\delta_0)} \frac{1}{(\eta^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}})^{\frac{r(N-p)}{p}}} dx \\ &\ge \eta^{\frac{r(N-p)}{p(p-1)}} \int_{B(x_0,\delta_0)} \frac{1}{(2\delta_0^{\frac{p}{p-1}})^{\frac{r(N-p)}{p}}} dx \\ &= C_6 \eta^{\frac{r(N-p)}{p(p-1)}}, \end{aligned}$$

where $C_6 = \int_{B(x_0,\delta_0)} \frac{1}{(2\delta_0^{\frac{p}{p-1}})^{\frac{r(N-p)}{p}}} dx$.

Then for any $0 < \eta \le \delta_0$, we have

$$\sup_{t \ge T} E(tz_\eta) \le \frac{1}{N} (S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}) - C_6 \sigma \frac{T^r}{r} (\lambda+\theta) \eta^{\frac{r(N-p)}{p(p-1)}}.$$

For any positive constants C_7, C_8 and any $\lambda, \theta > 0$, choose $\eta < \min\{\delta_0, (\frac{C_8(\lambda+\theta)}{C_7})^{\frac{p(p-1)}{(p-r)(N-p)}}\}$, we have

$$\begin{aligned} C_7 \eta^{\frac{N-p}{p-1}} - 2C_8(\lambda+\theta)\eta^{\frac{r(N-p)}{p(p-1)}} &< \delta_0^{\frac{r(N-p)}{p(p-1)}} (C_7 \eta^{\frac{(p-r)(N-p)}{p(p-1)}} - 2C_8(\lambda+\theta)) \\ &< -C_8 \delta_0^{\frac{r(N-p)}{p(p-1)}} (\lambda+\theta), \end{aligned}$$

which implies that there exist $\eta_0 > 0$ and $C_9 > 0$ such that for all $\lambda, \theta > 0$ and $0 < \eta < \eta_0$,

$$O(\eta^{\frac{N-p}{p-1}}) - C_6 \sigma \frac{T^r}{r} (\lambda+\theta) \eta^{\frac{r(N-p)}{p(p-1)}} < -C_9(\lambda+\theta).$$

Then there exists $\delta_2 > 0$ such that when $\lambda + \theta < \delta_2$, we have

$$-C_9(\lambda+\theta) < -B(\lambda+\theta)^{\frac{p}{p-r}}.$$

Then for any $\eta \in (0, \eta_2)$, $\lambda + \theta \in (0, \delta_2)$, we have

$$\sup_{t \geq T} E(tz_\eta) \leq \frac{1}{N}(S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}} - B(\lambda+\theta)^{\frac{p}{p-r}}.$$

Set $\Lambda^{**} = \min\{\delta_1, \delta_2, \Lambda^*\}$, then for all $\lambda + \theta \in (0, \Lambda^{**})$ and $\eta \in (0, \eta_0)$, we have

$$\sup_{t \geq 0} E(tz_\eta) \leq \frac{1}{N}(S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}} - B(\lambda+\theta)^{\frac{p}{p-r}}.$$

Then we obtain (4.3). This completes the proof of theorem 1.2. ■

5 Proof of Theorem 1.3.

We will study the case $1 < p < r < p^*$ and prove theorem 1.3 in this section.

Similar to Lemma 2.5 in section 2, we have the following result.

Lemma 5.1. Assume $1 < p < r < p^*$ and $(d_1) - (d_3)$ hold. Then E satisfies the (PS)$_c$ condition with c satisfying

$$c < \frac{1}{N}(S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}}. \tag{5.1}$$

Proof. By Lemma 2.3, we know that z_n is bounded in H, there is a $z = (u, v) \in H$ and a subsequence of $\{z_n\}$, sitll denoted by $\{z_n\}$ such that $z_n \rightharpoonup z$. A standard argument shows that z is a critical point of E, this implies that $\langle E'(z), z\rangle = 0$ and

$$\|z\|^p - p^* \int_\Omega F(x,z)dx = \int_\Omega \lambda V(x)|u|^r + \theta V(x)|v|^r dx.$$

Thus,

$$E(z) = (\frac{1}{p} - \frac{1}{r})\|z\|^p + (\frac{p^*}{r} - 1)\int_\Omega F(x,z)dx.$$

For $p < r < p^*$, we deduce that $E(z) > 0$ for any $\lambda, \theta > 0$, the following is similar to lemma 2.5. we omit it here. ■

Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3. From (4.1) and (5.1), we only need to show

$$\mu < \frac{1}{N}(S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}}, \tag{5.2}$$

then Lemma 4.1 and Lemma 5.1 give the existence of the critical point of E.

To obtain (5.2), Let us choose $z_0 = (u_0, u_0) \in H$, with

$$\int_\Omega F(x,1,1)u_0^{p^*}dx > 0, \ \lim_{t\to\infty} E(tz_0) = -\infty,$$

then there exists a $t_{\theta\lambda} > 0$ such that $\sup_{t\geq 0} E(tz_0) = E(t_{\theta\lambda}z_0)$ holds, and then $t_{\theta\lambda}$ satisfies

$$0 = t_{\theta\lambda}^{p-1}\|z_0\|^p - (\lambda+\theta)t_{\theta\lambda}^{r-1}\int_\Omega V(x)|u_0|^r dx - p^* t_{\theta\lambda}^{p^*-1}\int_\Omega F(x,1,1)u_0^{p^*}dx,$$

then we get

$$(\lambda+\theta)\int_\Omega V(x)|u_0|^r dx = t_{\theta\lambda}^{p-r}\|z_0\|^p - p^* t_{\theta\lambda}^{p^*-r}\int_\Omega F(x,1,1)u_0^{p^*}dx,$$

from $p < r < p^*$, we get $t_{\theta\lambda} \to 0$ as $(\lambda+\theta) \to \infty$. Then there exists $\Lambda_* > 0$ such that for any $(\lambda+\theta) > \Lambda_*$, we have

$$\sup_{t\geq 0} E(tz_0) < \frac{1}{N}(S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}}.$$

Now we take $w_0 = t_0 z_0$ with t_0 large enough to verify $E(w_0) < 0$, we get

$$\alpha \leq \max_{t\in[0,1]} E(\gamma_0(t)),$$

where $\gamma_0(t) = tw_0$. Therefore,

$$\mu \leq \sup_{t\geq 0} E(tw_0) < \frac{1}{N}(S_F p^{*-\frac{p}{p^*}})^{\frac{N}{p}}.$$

then we have proved (5.2). The proof of Theorem 1.3 is completed. ∎

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