Orthogonality of the Meixner-Pollaczek polynomials beyond Favard's theorem*

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Abstract

We extend the family of Meixner-Pollaczek polynomials $\{P_n^{(\lambda)}(\cdot;\phi)\}_{n=0}^{\infty}$, classically defined for $\lambda > 0$ and $0 < \phi < \pi$, to arbitrary complex values of the parameter λ, in such a way that both polynomial systems (the classical and the new *generalized* ones) share the same three term recurrence relation. The values $\lambda_N = (1-N)/2$, with N a positive integer, are the only ones for which no orthogonality condition can be deduced from Favard's theorem. In this paper we introduce a non-standard discrete-continuous inner product with respect to which the generalized Meixner-Pollaczek polynomials $\{P_n^{(\lambda_N)}(\cdot;\phi)\}_{n=0}^{\infty}$ become orthogonal.

1 Introduction

First, let us fix some terminologies, notations and conventions that we will use throughout this paper.

The set of complex numbers will be denoted by $\mathbb{C}$ and i will stand for the imaginary unit ($i^2 = -1$); the set of positive integers will be denoted by $\mathbb{N}$, and $\mathbb{N}_0$ will denote the set of nonnegative integers. All polynomials considered will be complex-valued in one complex variable, and $\mathbb{P}$ will stand for the set of all such polynomials. For each $n \in \mathbb{N}_0$, the subset of $\mathbb{P}$ of all polynomials of degree not greater than n will be denoted by $\mathbb{P}_n$. By a system of *monic* polynomials

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we will mean a sequence $\{P_n\}_{n=0}^{\infty}$ of polynomials satisfying $P_n^{(n)} = n!$ for each $n \in \mathbb{N}_0$. For notational convenience, we will use P_{-1} to denote the null polynomial.

For $n \in \mathbb{N}$, a (square) matrix of order n, with complex entries a_{jk}, will be denoted by $A = (a_{jk})_{j,k=0}^{n-1}$ (the entry a_{jk} will be also called the $(j+1,k+1)$th element of the matrix A), and $(a_j)_{j=0}^{n-1} \in \mathbb{C}^n$ will stand for the matrix of order $1 \times n$ (equivalently, for the vector) $(a_0, a_1, \ldots, a_{n-1})$. The conjugate transpose of a matrix $A = (a_{jk})_{j,k=0}^{n-1}$ will be denoted by using the superscript $*$ (with or without parenthesis, as needed), that is,

$$A^* = (A)^* = (a_{jk})_{j,k=0}^{n-1\,*} = \left((a_{jk})_{j,k=0}^{n-1}\right)^* = (\overline{a_{kj}})_{j,k=0}^{n-1},$$

(the overline denotes, of course, complex conjugation). A square matrix A will be called *Hermitian* whenever $A = A^*$; a Hermitian matrix A will be called *positive definite* whenever $xAx^* > 0$ for each $x \in \mathbb{C}^n \setminus \{0\}$ (as usual, we will identify the only element of a matrix of order 1 with the matrix itself, so the matrix $\left((x_j)_{j=0}^{n-1}\right)\left((a_{jk})_{j,k=0}^{n-1}\right)\left((x_j)_{j=0}^{n-1}\right)^*$ will be identified with its unique entry $\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} a_{jk}x_j\overline{x_k}$). For Hermitian matrices, positive definiteness is equivalent to the requirement that all of its principal minors are positive, and also equivalent to the fact that all its eigenvalues are positive. A *sesquilinear form* in a linear complex space V is a map $(\cdot,\cdot) : V \times V \to \mathbb{C}$ that is linear in its first (left) argument and conjugate-linear in the second (right) one; when this sesquilinear form is positive definite (i.e., when $(x,x) > 0$ for each $x \in \mathbb{C}^n \setminus \{0\}$) the map is called an inner product in V.

The Kronecker delta will be denoted by δ_{ij}, and $(\cdot)_n$ will denote the so-called shifted factorial (also, Pochhammer symbol), defined by

$$(x)_0 = 1, \qquad (x)_{n+1} = x(x+1)\cdots(x+n), \qquad n \in \mathbb{N}_0, \qquad x \in \mathbb{C}.$$

As usual, the binomial coefficient for complex numbers n, k is

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}, \qquad -n, -k, -(n-k) \notin \mathbb{N},$$

and the hypergeometric series ${}_mF_n$ is

$${}_mF_n\left(\begin{array}{c} a_1,\ldots,a_m \\ b_1,\ldots,b_n \end{array}\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_m)_k}{(b_1)_k\cdots(b_n)_k}\frac{x^k}{k!}, \qquad m,n \in \mathbb{N}_0,$$

where $(b_1)_k, \ldots, (b_n)_k \neq 0$ for all $k \in \mathbb{N}_0$. When $m = 0$ ($n = 0$) the numerator (denominator) of $(a_1)_k \cdots (a_m)_k/(b_1)_k \cdots (b_n)_k$ becomes 1. Clearly, if one of the numerator parameters satisfies $-a_j \in \mathbb{N}_0$, then the hypergeometric series is a polynomial of degree $\min\{-a_j : -a_j \in \mathbb{N}_0\}$.

In concluding this first part of the introduction, we recall that the nth iteration of an operator $\Psi : \mathbb{P} \to \mathbb{P}$ is recursively defined by means of

$$\begin{cases} \Psi^0 = I, & (I \text{ is the identity operator}), \\ \Psi^{n+1} = \Psi \circ \Psi^n, & n \in \mathbb{N}_0. \end{cases}$$

Now, let us make a brief survey of non-standard orthogonality in the literature.

By a *non-standard orthogonality* result we will mean an orthogonality statement for a system of monic polynomials $\{P_n^{(\lambda_1,\ldots,\lambda_m)}\}_{n=0}^{\infty}$, with parameters $\lambda_1, \ldots, \lambda_m$, and satisfying the three term recurrence relation

$$xP_n^{(\lambda_1,\ldots,\lambda_m)}(x) = P_{n+1}^{(\lambda_1,\ldots,\lambda_m)}(x) + a_n P_n^{(\lambda_1,\ldots,\lambda_m)}(x) + b_n P_{n-1}^{(\lambda_1,\ldots,\lambda_m)}(x), \quad n \in \mathbb{N}_0,$$

(where $a_n = a_n^{(\lambda_1,\ldots,\lambda_m)} \in \mathbb{C}$, $b_n = b_n^{(\lambda_1,\ldots,\lambda_m)} \in \mathbb{C}$), for those values of the parameters for which b_n vanishes for some $n \geq 1$. This topic has attracted great interest in recent years. In [11], Kwon and Littlejohn state that for each $N \in \mathbb{N}$, the Laguerre polynomials $\{L_n^{(-N)}\}_{n=0}^{\infty}$ form an orthogonal sequence with respect to a positive-definite inner product that can be written as a discrete-continuous bilinear form involving derivatives. A unified approach to the orthogonality of the (generalized) Laguerre polynomials $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$, for arbitrary real α, can be found in [20]. For a given positive integer number N, the orthogonality of the generalized Gegenbauer polynomials $\{C_n^{(-N+1/2)}\}_{n=0}^{\infty}$ is solved in [3], using again a Sobolev inner product, that is, an inner product involving derivatives (the case $N = 1$ is considered also in [12]). Other cases for Jacobi polynomials are solved in [2], where the families $\{P_n^{(-N,\beta)}\}_{n=0}^{\infty}$, $N \in \mathbb{N}$, $-(N+\beta) \notin \mathbb{N}$ and $\{P_n^{(\alpha,-N)}\}_{n=0}^{\infty}$, $N \in \mathbb{N}$, $-(\alpha+N) \notin \mathbb{N}$ are considered, and also in [1], in which the orthogonality for $\{P_n^{(-N,-M)}\}_{n=0}^{\infty}$, $N, M \in \mathbb{N}$ is stated. The orthogonality of the sequence $\{M_n^{(\gamma,\mu)}\}_{n=0}^{\infty}$ of generalized Meixner polynomials, with $\gamma \in \mathbb{R}$ and $0 < \mu < 1$, is given in [4], where a special consideration is taken for the values $\gamma = 1 - N$, $N \in \mathbb{N}$. A non-standard inner product with respect to which the symmetric Meixner-Pollaczek polynomials $\{P_n^{(\lambda)}(\cdot/2; \pi/2)\}_{n=0}^{\infty}$ $(\lambda \in \mathbb{R})$ become orthogonal is introduced in [6]. For (not necessarily symmetric) generalized Meixner-Pollaczek polynomials $\{P_n^{(0)}(\cdot; \phi)\}_{n=0}^{\infty}$ $(0 < \phi < \pi)$ we can find an orthogonality result in [8].

In this paper, we consider suitable modifications of our previous result [14, Theorem 3], adapted to the case of Meixner-Pollaczek polynomials $\{P_n^{(\lambda)}(\cdot; \phi)\}_{n=0}^{\infty}$ with arbitrary complex parameter λ, in order to state the orthogonality of the families $\{P_n^{((1-N)/2)}(\cdot; \phi)\}_{n=0}^{\infty}$, where $N \in \mathbb{N}$, the parameters $(1-N)/2$ being the only ones for which no orthogonality condition is ensured by Favard's theorem. For analogous results in the q-world we refer the reader to [15, 16, 17, 18, 19].

The paper is organized as follows. In Section 2 we extend the monic Meixner-Pollaczek polynomial system $\{P_n^{(\lambda)}(\cdot; \phi)\}_{n=0}^{\infty}$, classically defined for $\lambda > 0$ and $0 < \phi < \pi$, giving an explicit definition that works perfectly well for all $\lambda \in \mathbb{C}$; we will also give in this section two preparatory results, one concerning the roots of the new polynomials, and the other one concerning the action of the iterations of the linear operator $\delta/\delta x$ on them. In Section 3 we define a nonstandard discrete-continuous inner product which yields orthogonality for the extended monic Meixner-Pollaczek polynomials with those "exceptional" values of the parameter λ for which Favard's theorem fails to work, i.e. for $\lambda \in \{0, -1/2, -1, -3/2, \ldots\}$.

2 The generalized Meixner-Pollaczek polynomials

J. Meixner [13] introduced in 1934 a class of polynomials that F. Pollaczek [21] considered independently sixteen years later. This remarkable family of orthogonal polynomials is a generalization of some of the classical ones, and it exhibits in many aspects a singular behaviour (for a brief but enlightening discussion, see [22, pp. 393–400]). These so-called Meixner-Pollaczek polynomials appear in the Askey-scheme of hypergeometric orthogonal polynomials [7, 9].

For each $\lambda > 0$ and each $\phi \in (0, \pi)$, the nth degree monic Meixner-Pollaczek polynomial $P_n^{(\lambda)}(\cdot;\phi)$ can be defined in terms of the hypergeometric series ${}_2F_1$ by means of (see (1.7.1) and (1.7.4) in [9])

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{(2\sin\phi)^n} e^{in\phi} {}_2F_1\left(\begin{array}{c} -n, \lambda+ix \\ 2\lambda \end{array} \middle| 1-e^{-2i\phi}\right), \quad n \in \mathbb{N}_0, \tag{2.1}$$

and they satisfy the three term recurrence relation

$$xP_n^{(\lambda)}(x;\phi) = P_{n+1}^{(\lambda)}(x;\phi) + a_n^{(\lambda,\phi)} P_n^{(\lambda)}(x;\phi) + b_n^{(\lambda,\phi)} P_{n-1}^{(\lambda)}(x;\phi), \quad n \in \mathbb{N}_0, \tag{2.2}$$

where

$$a_n^{(\lambda,\phi)} = -\frac{n+\lambda}{\tan\phi}, \qquad b_n^{(\lambda,\phi)} = \frac{n(n+2\lambda-1)}{4\sin^2\phi}, \tag{2.3}$$

with the agreement that $a_n^{(\lambda,\pi/2)} = \lim_{\phi\to\pi/2} a_n^{(\lambda,\phi)} = 0$. The orthogonality condition is (see (1.7.2) and (1.7.4) in [9])

$$\int_{-\infty}^{\infty} P_m^{(\lambda)}(x;\phi) P_n^{(\lambda)}(x;\phi) \frac{e^{(2\phi-\pi)x}}{2\pi} |\Gamma(\lambda+ix)|^2 dx = \frac{n!\Gamma(n+2\lambda)}{(2\sin\phi)^{2(\lambda+n)}} \delta_{mn}, \tag{2.4}$$

where $m, n \in \mathbb{N}_0$, $\lambda > 0$ and $0 < \phi < \pi$.

Our intention is to accomplish the extension of the monic Meixner-Pollaczek polynomials $\{P_n^{(\lambda)}(\cdot;\phi)\}_{n=0}^{\infty}$ for all complex values of the parameter λ. We first observe that (2.1) does not hold when $-2\lambda \in \mathbb{N}_0$, but after straightforward manipulation of the series we obtain an expression for $P_n^{(\lambda)}(\cdot;\phi)$ which is defined for all $\lambda \in \mathbb{C}$, $\phi \in (0, \pi)$ and $n \in \mathbb{N}_0$.

Definition 2.1. *For each $\lambda \in \mathbb{C}$, each $\phi \in (0, \pi)$ and for all $n \in \mathbb{N}_0$ we define the nth degree monic generalized Meixner-Pollaczek polynomials $P_n^{(\lambda)}(\cdot;\phi)$ by means of*

$$P_n^{(\lambda)}(x;\phi) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2\lambda+k)_{n-k} \frac{e^{in\phi}(1-e^{-2i\phi})^k}{(2\sin\phi)^n} (\lambda+ix)_k, \qquad x \in \mathbb{C}. \tag{2.5}$$

Observe that this new extended family satisfies the same three term recurrence relation (2.2), (2.3) as the classical monic Meixner-Pollaczek polynomials with positive parameter λ. Hence, taking into account that for $n \geq 1$ one has $b_n^{(\lambda,\phi)} = 0$ only when $-2\lambda \in \mathbb{N}_0$, the corresponding orthogonality statement would be that the generalized family of monic Meixner-Pollaczek polynomials is orthogonal with respect to a quasi-definite moment functional (which is positive definite if $\lambda > 0$) if and only if $-2\lambda \in \mathbb{C} \setminus \mathbb{N}_0$.

Now we will give some results that will be essential in the main results of this paper.

As shown in [5, Proposition 6], $P_n^{(0)}(x/2;\pi/2) = (x/2)P_{n-1}^{(1)}(x/2;\pi/2)$ for $n \geq 1$, where we have adapted the original relation to our normalization. In [8, Proposition 13], the author improves this relation and gets (again, in the version of monic polynomials) $P_n^{(0)}(x;\phi) = xP_{n-1}^{(1)}(x;\phi)$, $n \geq 1$. The following result generalizes these ones.

Proposition 2.1. *Let $N \in \mathbb{N}$. For each integer $n \geq N$,*

$$\begin{aligned} P_n^{((1-N)/2)}(x;\phi) &= P_N^{((1-N)/2)}(x;\phi)P_{n-N}^{((1+N)/2)}(x;\phi) \\ &= (-i)^N \Big(\frac{1-N}{2} + ix\Big)_N P_{n-N}^{((1+N)/2)}(x;\phi), \quad \phi \in (0,\pi), \quad x \in \mathbb{C}. \end{aligned} \tag{2.6}$$

Proof. For the sake of brevity we introduce the notation

$$c_k^{(n;\phi)} = \frac{e^{in\phi}(1-e^{-2i\phi})^k}{(2\sin\phi)^n}, \quad n \in \mathbb{N}_0, \quad 0 \leq k \leq n, \quad 0 < \phi < \pi.$$

Using (2.5) we have

$$P_n^{((1-N)/2)}(x;\phi) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}(-N+1+k)_{n-k}c_k^{(n;\phi)}\Big(\frac{1-N}{2}+ix\Big)_k.$$

Since $(-N+1+k)_{n-k} = (-N+k+1)(-N+k+2)\cdots(-N+n)$, if $n \geq N$ (which implies that the last factor in the shifted factorial is a non-negative integer), then for each $k \leq N-1$ (which implies the non-positiveness of the first factor) we have $(-N+1+k)_{n-k} = 0$. Consequently, for $n \geq N$

$$\begin{aligned} P_n^{((1-N)/2)}(x;\phi) &= \sum_{k=N}^{n}(-1)^k \binom{n}{k}(-N+1+k)_{n-k}c_k^{(n;\phi)}\Big(\frac{1-N}{2}+ix\Big)_k \\ &= \sum_{k=0}^{n-N}(-1)^{N+k}\binom{n}{N+k}(k+1)_{n-N-k}c_{N+k}^{(n;\phi)}\Big(\frac{1-N}{2}+ix\Big)_{N+k}. \end{aligned}$$

Taking into account that for $0 \leq k \leq n-N$

$$\begin{aligned} \binom{n}{N+k}(k+1)_{n-N-k} &= \binom{n-N}{k}(1+N+k)_{n-N-k}, \\ c_{N+k}^{(n;\phi)} &= c_N^{(N;\phi)}c_k^{(n-N;\phi)} = i^N c_k^{(n-N;\phi)}, \\ \Big(\frac{1-N}{2}+ix\Big)_{N+k} &= \Big(\frac{1-N}{2}+ix\Big)_N\Big(\frac{1+N}{2}+ix\Big)_k, \end{aligned}$$

we get

$$\begin{aligned} P_n^{((1-N)/2)}(x;\phi) = &\bigg((-1)^N c_N^{(N;\phi)}\Big(\frac{1-N}{2}+ix\Big)_N\bigg) \\ &\bigg(\sum_{k=0}^{n-N}(-1)^k\binom{n-N}{k}(1+N+k)_{n-N-k}c_k^{(n-N;\phi)}\Big(\frac{1+N}{2}+ix\Big)_k\bigg). \end{aligned}$$

Noting that

$$P_N^{((1-N)/2)}(x;\phi) = (-1)^N c_N^{(N;\phi)} \Big(\frac{1-N}{2}+ix\Big)_N = (-i)^N \Big(\frac{1-N}{2}+ix\Big)_N, \quad (2.7)$$

we finally establish the factorization (2.6). ■

Corollary 2.1. *For fixed $N \in \mathbb{N}$, let us denote $x_k^{(N)} = (((2k+1)-N)/2)i$ for $0 \le k \le N-1$. We have*

$$P_n^{((1-N)/2)}\left(x_k^{(N)};\phi\right) = 0, \qquad 0 \le k \le N-1, \qquad n \ge N, \qquad 0 < \phi < \pi.$$

Proof. Since for $0 \le k \le N-1$ the points $x_k^{(N)}$ are the N different (pure imaginary) roots of the equation $((1-N)/2+ix)_N = 0$, from (2.6) we deduce that for $n \ge N$, each $x_k^{(N)}$ is a root of the polynomial $P_n^{((1-N)/2)}(\cdot;\phi)$. ■

We define the *forward shift operator* $\delta : \mathbb{P} \to \mathbb{P}$ as usual (see [9, 0.9.1]), that is, for each polynomial p, the polynomial $\delta(p) := \delta p$ is the one defined by means of $\delta p(x) = p(x+i/2) - p(x-i/2)$, $x \in \mathbb{C}$. It is clear that δ is a linear operator that reduces by one the degree of the evaluated polynomial. For the power functions e_n, defined by $e_n(x) = x^n$ for $n \in \mathbb{N}_0$ and $x \in \mathbb{C}$, it is usual to denote $\delta e_n(x) = \delta x^n$. Then $\delta x = i$, and also $(\delta p(x))/(\delta x) = (1/i)\delta p(x)$. Thus, the symbol $\delta/\delta x$ will stand for the linear operator $-i\delta$.

Using [9, 1.7.7] (in its version for monic polynomials) in (2.5), we can easily verify that the same forward shift relation holds for the generalized monic Meixner-Pollaczek polynomials. That is to say

$$\frac{\delta P_n^{(\lambda)}(x;\phi)}{\delta x} = nP_{n-1}^{(\lambda+\frac{1}{2})}(x;\phi), \qquad n \in \mathbb{N}_0, \qquad \lambda, x \in \mathbb{C}.$$

(where we have used $\delta P_n^{(\lambda)}(x;\phi)$ instead of the more formal $\delta P_n^{(\lambda)}(\cdot;\phi)(x)$, and we will use this convention in the sequel). Therefore,

Proposition 2.2. *Given a fixed nonnegative integer k, we have, for each $n \ge k-1$*

$$\left(\frac{\delta}{\delta x}\right)^k P_n^{(\lambda)}(x;\phi) = \frac{\delta^k P_n^{(\lambda)}(x;\phi)}{(\delta x)^k} = (n-k+1)_k P_{n-k}^{(\lambda+\frac{k}{2})}(x;\phi),$$
$$\lambda, x \in \mathbb{C}, \quad \phi \in (0,\pi).$$

3 Orthogonality of $\{P_n^{((1-N)/2)}(\cdot;\phi)\}_{n=0}^\infty$ for positive integers N

By using the Poisson kernel and the associated Poisson measure, T.K. Araaya [5] gives an orthogonality result for the system $\{P_n^{(\lambda)}(\cdot/2;\pi/2)\}_{n=0}^\infty$ of symmetric Meixner-Pollaczek polynomials with parameter $\lambda = 0$. This result was further extended by the same author [6] to arbitrary real values of the parameter λ, by introducing a non-standard inner product with respect to which the symmetric Meixner-Pollaczek polynomials become orthogonal. In both cases, the technique of the proofs depends strongly both on the use of the generating function (to replace the sequence of symmetric Meixner-Pollaczek polynomials) and on some nice calculus machinery. As far as we know, D. Dominici [8, Remark 17] gives the first non-standard orthogonality statement for (not necessarily symmetric) generalized Meixner-Pollaczek polynomials with parameter $\lambda = 0$.

With the aid of a suitable modification of our result [14, Theorem 3], adapted to the case considered here of generalized monic Meixner-Pollaczek polynomials, we can give orthogonality results that are intimately related with those of Araaya and Dominici.

Definition 3.1. *Let $N \in \mathbb{N}$ and let $\phi \in (0, \pi)$. For a Hermitian and positive definite complex matrix A of order N, and for the points $x_k = x_k^{(N)} = (((2k+1) - N)/2)i$, $0 \le k \le N-1$, we define the inner product $(\cdot, \cdot)_{(N;A;\phi)} : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ by means of*

$$\begin{aligned}(p, q)_{(N;A;\phi)} &= (p(x_0), p(x_1), \dots, p(x_{N-1}))A(q(x_0), q(x_1), \dots, q(x_{N-1}))^* \\ &+ \int_{-\infty}^{\infty} \left(\frac{\delta^N p(x)}{(\delta x)^N}\right) \overline{\left(\frac{\delta^N q(x)}{(\delta x)^N}\right)} \frac{e^{(2\phi - \pi)x}}{2\pi} |\Gamma(1/2 + ix)|^2 dx, \quad p, q \in \mathbb{P}. \quad (3.8)\end{aligned}$$

Our aim is to show that there exist matrices A such that the sequence of generalized monic Meixner-Pollaczek polynomials with parameter λ equal to $(1-N)/2$ is orthogonal with respect to the non-standard inner product introduced above.

Theorem 3.1. *For a fixed positive integer N and for $\phi \in (0, \pi)$, the generalized monic Meixner-Pollaczek polynomials $\{P_n^{((1-N)/2)}(\cdot; \phi)\}_{n=0}^{\infty}$ are orthogonal with respect to the inner product $(\cdot, \cdot)_{(N;A;\phi)}$, where $A = C^{-1}D(C^{-1})^*$, C stands for the matrix $(P_j^{((1-N)/2)}(x_k; \phi))_{j,k=0}^{N-1}$ and D is an arbitrary diagonal matrix of order N with positive entries in its diagonal.*

Proof. First we must verify that A is Hermitian and positive definite.

Let $\{l_j\}_{j=0}^{N-1} \subset \mathbb{P}_{N-1}$ be the set of Lagrange interpolating polynomials with respect to the points $\{x_k\}_{k=0}^{N-1}$ (that is, $l_j(x_k) = \delta_{jk}$ for $0 \le j, k \le N-1\}$). Taking into account that both $\{P_j^{((1-N)/2)}(\cdot; \phi)\}_{j=0}^{N-1}$ and $\{l_j\}_{j=0}^{N-1}$ are bases of $\mathbb{P}_{N-1}$, and also that

$$P_j^{((1-N)/2)}(x; \phi) = \sum_{k=0}^{N-1} P_j^{((1-N)/2)}(x_k; \phi)\, l_k(x), \qquad 0 \le j \le N-1,$$

we can justify that the matrix $C = (P_j^{((1-N)/2)}(x_k; \phi))_{j,k=0}^{N-1}$ is nonsingular.

We recall that a square complex matrix M is positive definite if and only if there exists a nonsingular complex matrix Q such that $M = QQ^*$. Thus, $A = C^{-1}D(C^{-1})^* = (C^{-1}\sqrt{D})(C^{-1}\sqrt{D})^*$ is positive definite (here, $\sqrt{D}$ stands for the diagonal matrix whose diagonal entries are the positive square root of the corresponding diagonal entries of D).

Now we will state the orthogonality in three steps:

i) In case that $0 \leq m, n \leq N-1$, since

$$\left(\frac{\delta}{\delta x}\right)^N P_m^{((1-N)/2)}(x;\phi) = \left(\frac{\delta}{\delta x}\right)^N P_n^{((1-N)/2)}(x;\phi) = 0,$$

we get

$$\begin{aligned}(P_m^{((1-N)/2)}(\cdot;\phi), P_n^{((1-N)/2)}(\cdot;\phi))_{(N;A;\phi)} \\ = \big((P_m^{((1-N)/2)}(x_k;\phi))_{k=0}^{N-1}C^{-1}\big)D\big((P_n^{((1-N)/2)}(x_k;\phi))_{k=0}^{N-1}C^{-1}\big)^* \\ = (\delta_{mk})_{k=0}^{N-1}D\big((\delta_{nk})_{k=0}^{N-1}\big)^* = \kappa_n\delta_{mn},\end{aligned}$$

where κ_n is the (positive) $(n+1, n+1)$th entry of the matrix D.

ii) If $0 \leq m \leq N-1$ and $n \geq N$, then

$$\left(\frac{\delta}{\delta x}\right)^N P_m^{((1-N)/2)}(x;\phi) = 0, \qquad \text{and} \qquad P_n^{((1-N)/2)}(x_k;\phi) = 0, \quad 0 \leq k \leq N-1,$$

so, clearly, we have $(P_m^{((1-N)/2)}(\cdot;\phi), P_n^{((1-N)/2)}(\cdot;\phi))_{(N;A;\phi)} = 0$.

iii) Finally, when $m, n \geq N$, using that $P_{n-N}^{(1/2)}(x;\phi)$ is real for $x \in \mathbb{R}$,

$$\begin{aligned}&(P_m^{((1-N)/2)}(\cdot;\phi), P_n^{((1-N)/2)}(\cdot;\phi))_{(N;A;\phi)} \\ &= \int_{-\infty}^{\infty}\left(\frac{\delta^N P_m^{((1-N)/2)}(x;\phi)}{(\delta x)^N}\right)\overline{\left(\frac{\delta^N P_n^{((1-N)/2)}(x;\phi)}{(\delta x)^N}\right)}\frac{e^{(2\phi-\pi)x}}{2\pi}|\Gamma(1/2+ix)|^2dx \\ &= N!^2\binom{m}{N}\binom{n}{N}\int_{-\infty}^{\infty}P_{m-N}^{(1/2)}(x;\phi)P_{n-N}^{(1/2)}(x;\phi)\frac{e^{(2\phi-\pi)x}}{2\pi}|\Gamma(1/2+ix)|^2dx \\ &= N!^2\binom{n}{N}^2\frac{(n-N)!\Gamma(n-N+1)}{(2\sin\phi)^{2(n-N)+1}}\delta_{mn} = \left(\frac{n!}{(2\sin\phi)^{n-N+1/2}}\right)^2\delta_{mn}. \quad \blacksquare\end{aligned}$$

Let us note that if we choose $D = (j!^2(2\sin\phi)^{2(N-j)-1}\delta_{jk})_{j,k=0}^{N-1}$ in the previous theorem, then we have the closed form for the norms

$$\begin{aligned}||P_n^{((1-N)/2)}(\cdot;\phi)||_{(N;A;\phi)} &= \left((P_n^{((1-N)/2)}(\cdot;\phi), P_n^{((1-N)/2)}(\cdot;\phi))_{(N;A;\phi)}\right)^{1/2} \\ &= \frac{n!}{(2\sin\phi)^{n-N+1/2}}, \qquad n \in \mathbb{N}_0.\end{aligned}$$

We now give a new related orthogonality result in which the discrete part of the sesquilinear form (3.8) changes in such a way that symmetry is gained and explicitness is lost.

Theorem 3.2. *Fixed $N \in \mathbb{N}$ and $\phi \in (0,\pi)$, there exists a Hermitian and positive definite matrix A of order N such that the family $\{P_n^{((1-N)/2)}(\cdot;\phi)\}_{n=0}^{\infty}$ is orthogonal with respect to the inner product $(\cdot,\cdot)_{(\delta;N;A;\phi)}$ defined by*

$$\begin{aligned}(p,q)_{(\delta;N;A;\phi)} &= \left(\delta^k p\,(x_k - (k/2)i)\right)_{k=0}^{N-1} A\left(\left(\delta^k q\,(x_k-(k/2)i)\right)_{k=0}^{N-1}\right)^* \\ &+ \int_{-\infty}^{\infty}\left(\frac{\delta^N p(x)}{(\delta x)^N}\right)\overline{\left(\frac{\delta^N q(x)}{(\delta x)^N}\right)}\frac{e^{(2\phi-\pi)x}}{2\pi}|\Gamma(1/2+ix)|^2dx, \quad p,q \in \mathbb{P}. \quad (3.9)\end{aligned}$$

Proof.

A simple induction argument shows that for each $k \in \mathbb{N}_0$,

$$\delta^k p(x) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} p(x + (k/2 - l)i), \qquad p \in \mathbb{P}, \qquad x \in \mathbb{C}.$$

Therefore, by Corollary 2.1 we have, fixed a nonnegative integer N and for $0 \leq k \leq N-1$ and $n \geq N$,

$$\begin{aligned} \delta^k P_n^{((1-N)/2)}(x_k - (k/2)i;\phi) &= \sum_{l=0}^{k} (-1)^l \binom{k}{l} P_n^{((1-N)/2)}(x_k - li;\phi) \\ &= \sum_{l=0}^{k} (-1)^l \binom{k}{l} P_n^{((1-N)/2)}(x_{k-l};\phi) = 0. \end{aligned}$$

Now consider the set of fundamental polynomials $\{h_j\}_{j=0}^{N-1} \subset \mathbb{P}_{N-1}$ defined by

$$\delta^k h_j(x_k - (k/2)i) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} h_j(x_{k-l}) = \delta_{kj}, \qquad 0 \leq k, j \leq N-1.$$

Clearly, these polynomials exist, are unique, and form a basis of $\mathbb{P}_{N-1}$. Taking into account that both $\{P_j^{((1-N)/2)}(\cdot;\phi)\}_{j=0}^{N-1}$ and $\{h_j\}_{j=0}^{N-1}$ are bases of $\mathbb{P}_{N-1}$, there exists a nonsingular matrix $C^t = (c_{jk})_{j,k=0}^{N-1}$ such that for each $j = 0, 1, \ldots, N-1$,

$$P_j^{((1-N)/2)}(x;\phi) = \sum_{k=0}^{N-1} c_{kj}\, h_k(x),$$

which implies that $C = \left(\delta^k P_j^{((1-N)/2)}(x_k - (k/2)i;\phi)\right)_{j,k=0}^{N-1}$.

Then, for any arbitrary diagonal matrix D of order N with positive elements in the diagonal, and defining $A = C^{-1}D(C^{-1})^*$, we get the desired conclusion following the same reasoning as in Theorem 3.2, replacing in steps i) and ii) $P_m^{((1-N)/2)}(x_k;\phi)$ and $P_n^{((1-N)/2)}(x_k;\phi)$ by $\delta^k P_m^{((1-N)/2)}(x_k - (k/2)i;\phi)$ and $\delta^k P_n^{((1-N)/2)}(x_k - (k/2)i;\phi)$, respectively. ∎

As a final remark, let us say that defining $\widetilde{\delta} : \mathbb{P} \to \mathbb{P}$ by $\widetilde{\delta} p(x) = p(x+i) - p(x)$ we get a result similar to the previous one, where $\delta^k P_n^{((1-N)/2)}$ $(x_k - (k/2)i;\phi)$ is replaced by $\widetilde{\delta}^k P_n^{((1-N)/2)}(((1-N)/2)i;\phi)$ and where $C = \left(\widetilde{\delta}^k P_j^{((1-N)/2)}(((1-N)/2)i;\phi)\right)_{j,k=0}^{N-1}$.

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