On symmetric and periodic solutions of parametric weakly nonlinear ODE with time-reversal symmetries

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Abstract

We show the existence of periodic and symmetric solutions of parametric weakly nonlinear ODE possessing time-reversal symmetries. Local asymptotic behaviours of these solutions are established as well. Concrete examples are presented to illustrate the general theory.

1 Introduction

We consider the systems of differential equations under symmetric assumptions. More concretely, we consider a weakly nonlinear ordinary differential equation of the form

$$\dot{x} = \varepsilon f(x, \mu, t), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R} \tag{1.1}$$

with parameters $\varepsilon \in \mathbb{R}$, $\mu \in \mathbb{R}^k$, where ε is small, and with a C^∞-smooth function $f : \mathbb{R}^{n+k+1} \to \mathbb{R}^n$ symmetric in x, i.e. it holds

$$Af(x, \mu, t) = -f(Ax, \mu, -t - \tau)\,, \tag{1.2}$$

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where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a regular linear map, $\tau \in \mathbb{R}$ is fixed and, moreover, function f is T-periodic on t, i.e. it holds

$$f(x, \mu, t) = f(x, \mu, t + T). \tag{1.3}$$

A survey of dynamical systems with time-reversal symmetries is given in [21]. Note condition (1.2) represents such a kind of symmetry for (1.1).

On the other hand, there are several papers [15, 16, 24, 30, 34, 35] studying ODE with symmetries when (1.2) is replaced with the following assumption

$$Af(Ax, \mu, t) = -f(x, \mu, -t - \tau)\,. \tag{1.4}$$

Moreover, most of these papers suppose additional condition $A^2 = \mathbb{I}$, and then (1.4) is called as property E. Furthermore, clearly property E is our assumption (1.2) with $A^2 = \mathbb{I}$. Consequently, our results are generalizations of some earlier results for weakly nonlinear ordinary differential equations with property E.

Note

$$g(x, \mu, t) := f(x, \mu, t - \tau/2)$$

satisfies (1.2) with $\tau = 0$, so without loss of generality, we suppose

$$Af(x, \mu, t) = -f(Ax, \mu, -t) \tag{1.5}$$

instead of (1.2). We introduce a vector space

$$X := \Big\{ x \in C^1(\mathbb{R}, \mathbb{R}^{n+1}) \mid x(t) = Ax(-t)\; \forall t \in \mathbb{R} \Big\}\,. \tag{1.6}$$

Definition 1. By a *symmetric solution* x of equation (1.1) we mean $x \in X$ satisfying this equation.

The main goal of this paper is to find symmetric and periodic solutions (see Section 4) for equation (1.1) and to study their asymptotic properties (see Sections 5 and 6). We also present examples to illustrate the theory in Section 8.

The results presented in this note are also generalizations of achievements for anti-periodic problems with $A = -\mathbb{I}$ [1, 2], and continuations of [13]. Doubly symmetric solutions of reversible systems are studied in [28]. Symmetric properties of periodic solutions of nonlinear nonautonomous ordinary differential equations are studied also in [9, 10, 11]. We can also apply numerical methods from [31] for computation of symmetric solutions of (1.1). More results on periodic solutions in dynamical systems and ordinary differential equations are presented in [12, 23, 32].

Furthermore, when in addition, f is odd in x, i.e. it holds

$$f(-x, \mu, t) = -f(x, \mu, t)\,, \tag{1.7}$$

then we extend our result to the study of *antisymmetric* and periodic solutions of (1.1), i.e. satisfying (cf Section 7)

$$-x(-t) = Ax(t)\; \forall t \in \mathbb{R} \tag{1.8}$$

instead of $x \in X$.

Finally, results of this paper are closely related to bifurcations of periodic solutions presented in the books [7, 6, 14, 20, 36], but we remind that we also study asymptotic properties of found periodic solutions of (1.1), not just their existence.

2 Classical Results on Existence of Periodic Solutions

Before studying the existence of symmetric and periodic solutions of (1.1), we recall the following classical results [27, 33].

Theorem 1. *If there exist $\bar\eta_0 \in \mathbb{R}^n$ and $\bar\mu_0 \in \mathbb{R}^k$ such that*

$$\int_0^T f(\bar\eta_0, \bar\mu_0, s)\, ds = 0 \quad \text{and} \quad \int_0^T D_{\eta,\mu} f(\bar\eta_0, \bar\mu_0, s)\, ds : \mathbb{R}^{n+k} \to \mathbb{R}^n \quad \text{is onto}. \tag{2.1}$$

Then there are decompositions $\mathbb{R}^k = \bar X_1 \oplus \bar X_2$, $\mathbb{R}^n = \bar Y_1 \oplus \bar Y_2$ with $\dim \bar X_1 + \dim \bar Y_1 = n$ and constants $\bar\varepsilon_0 > 0$, $\bar\delta_1^0 > 0$, $\bar\delta_2^0 > 0$, $\bar\delta_3^0 > 0$, $\bar\delta_4^0 > 0$ along with unique C^∞-smooth functions $\bar\mu_1(\eta_2, \mu_2, \varepsilon) \in \bar X_1$, $\bar\eta_1(\eta_2, \mu_2, \varepsilon) \in \bar Y_1$, $\varepsilon \in (-\bar\varepsilon_0, \bar\varepsilon_0)$, $|\mu_2 - \bar\mu_2^0| < \bar\delta_2^0$, $|\eta_2 - \bar\eta_2^0| < \bar\delta_4^0$ such that $\bar\mu_1(\bar\eta_2^0, \bar\mu_2^0, 0) = \bar\mu_1^0$, $\bar\eta_1(\bar\eta_2^0, \bar\mu_2^0, 0) = \bar\eta_1^0$ for $\bar\mu_0 = (\bar\mu_1^0, \bar\mu_2^0) \in \bar X_1 \times \bar X_2$, $\bar\eta_0 = (\bar\eta_1^0, \bar\eta_2^0) \in \bar Y_1 \times \bar Y_2$ with the following properties: For any $|\mu_1 - \bar\mu_1^0| < \bar\delta_1^0$, $|\mu_2 - \bar\mu_2^0| < \bar\delta_2^0$, $|\eta_1 - \bar\eta_1^0| < \bar\delta_3^0$, $|\eta_2 - \bar\eta_2^0| < \bar\delta_4^0$ and $0 < |\varepsilon| < \bar\varepsilon_0$, equation (1.1) *has a T-periodic solution with $x(0) = (\eta_1, \eta_2)$ if and only if $\mu_1 = \bar\mu_1(\eta_2, \mu_2, \varepsilon)$, $\eta_1 = \bar\eta_1(\eta_2, \mu_2, \varepsilon)$, moreover this solution is unique and located near $\bar\eta_0$.*

Theorem 2. *If there are compact subsets $\Omega \subset \mathbb{R}^n$ and $\Gamma \subset \mathbb{R}^k$ such that*

$$\min_{x\in\Omega, \mu\in\Gamma} \left| \int_0^T f(x, \mu, s)\, ds \right| > 0,$$

then (1.1) *has no T-periodic solutions in Ω for any $\varepsilon \neq 0$ small and $\mu \in \Gamma$.*

3 Existence of symmetric solutions

We suppose for simplicity that f is globally Lipschitz continuous in x. From the property

$$Ax(t) = x(-t) \tag{3.1}$$

we have that

$$Ax(0) = x(0). \tag{3.2}$$

Lemma 1. *A solution x of* (1.1) *is symmetric if and only if it satisfies* (3.2).

Proof. Let us put

$$y(t) := A^{-1}x(-t).$$

Then, taking into account (1.5), we get

$$\dot y(t) = -A^{-1}\dot x(-t) = -A^{-1}\varepsilon f(x(-t), \mu, -t) = \varepsilon f(A^{-1}x(-t), \mu, t) = \varepsilon f(y(t), \mu, t)$$

and

$$y(0) = A^{-1}x(0) = x(0).$$

So $x(t) = y(t)$. Consequently, (3.1) holds. ∎

Remark 1. It follows from the above considerations that any symmetric and T-periodic solution is not asymptotically stable, but it can be stable (cf Example 2). Moreover, if a symmetric and T-periodic solution is hyperbolic then the dimensions of its stable and unstable manifolds are equal and so n is even.

From (3.2) we have

$$x(0) \in \ker(\mathbb{I} - A).$$

Let us consider equation (1.1) with initial value condition

$$x(0) = \eta, \quad \eta \in \ker(\mathbb{I} - A) \tag{3.3}$$

and take its unique C^∞-smooth solution $x(\eta, \varepsilon, \mu, t)$, $x : \ker(\mathbb{I} - A) \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^n$. Summarizing we arrive at the following result.

Theorem 3. *The Cauchy problem* (1.1), (3.3) *has a unique C^∞-smooth solution $x(\eta, \varepsilon, \mu, t)$ which is also symmetric, and any symmetric solution $x(t)$ of* (1.1) *satisfies* (3.3).

4 Existence of symmetric and periodic solutions

If $x(t)$ is T-periodic and satisfying (3.1) then we get

$$x(T/2) = x(-T/2) = Ax(T/2),$$

so

$$x(T/2) \in \ker(\mathbb{I} - A). \tag{4.1}$$

On the other hand, if $x(\eta, \varepsilon, \mu, T/2) \in \ker(\mathbb{I} - A)$ then

$$x(\eta, \varepsilon, \mu, -T/2) = x(\eta, \varepsilon, \mu, T/2),$$

so $x(\eta, \varepsilon, \mu, t)$ is T-periodic. Consequently, in order to find symmetric and periodic solutions of (1.1), we have to study the following equation

$$F(\eta, \mu, \varepsilon) := Sx(\eta, \varepsilon, \mu, T/2) = 0, \tag{4.2}$$

where $\mathbb{I} - S : \mathbb{R}^n \to \ker(\mathbb{I} - A)$ is a A-invariant projection, i.e. $AS = SA$. Let

$$V := \ker(\mathbb{I} - S).$$

Note

$$p := \dim V = n - \dim \ker(\mathbb{I} - A).$$

Since

$$F(\eta, \mu, 0) = S\eta = 0,$$

we solve equation

$$\frac{1}{\varepsilon} F(\eta, \mu, \varepsilon) = 0, \quad \varepsilon \neq 0. \tag{4.3}$$

To state the next results we introduce the following function

$$H_1(\eta, \mu) := D_\varepsilon F(\eta, \mu, 0).$$

Now we suppose that

$$m := \dim \ker(\mathbb{I} - A) + k \geq p. \tag{4.4}$$

Then note $H_1 \in C^\infty(\mathbb{R}^m, \mathbb{R}^p)$.

Remark 2. Let us consider decomposition

$$x(\eta,\varepsilon,\mu,t)=\eta+\varepsilon x_1(\eta,\mu,t)+\varepsilon^2 x_2(\eta,\mu,t)+\varepsilon^3 x_3(\eta,\mu,t)+\dots \tag{4.5}$$

for the Cauchy problem (1.1), (3.3). Then we get

$$\begin{gathered}\dot{x}_1(\eta,\mu,t)+\varepsilon\dot{x}_2(\eta,\mu,t)+\varepsilon^2\dot{x}_3(\eta,\mu,t)+\dots\\ =f\left(\eta+\varepsilon x_1(\eta,\mu,t)+\varepsilon^2 x_2(\eta,\mu,t)+\varepsilon^3 x_3(\eta,\mu,t)+\cdots,\mu,t\right),\\ x_j(\eta,\mu,0)=0\,\forall j\in\mathbb{N}.\end{gathered} \tag{4.6}$$

Putting $\varepsilon=0$ in (4.6), we have

$$x_1(\eta,\mu,t)=\int_0^t f(\eta,\mu,s)ds\,. \tag{4.7}$$

Similarly, differentiating (4.6) by ε once and twice at $\varepsilon=0$, we derive

$$x_2(\eta,\mu,t)=\int_0^t D_x f(\eta,\mu,s)\int_0^s f(\eta,\mu,z)dzds\,, \tag{4.8}$$

and

$$\begin{aligned}x_3(\eta,\mu,t)&=\int_0^t D_x f(\eta,\mu,s)\int_0^s D_x f(\eta,\mu,z)\int_0^z f(\eta,\mu,u)dudzds\\ &+\frac{1}{2}\int_0^t D_{xx} f(\eta,\mu,s)\left(\int_0^s f(\eta,\mu,z)dz,\int_0^s f(\eta,\mu,z)dz\right)ds\,,\end{aligned} \tag{4.9}$$

respectively.

Then, taking into account (4.7), we return to (4.2)

$$H_1(\eta,\mu)=D_\varepsilon F(\eta,\mu,0)=Sx_1(\eta,\mu,T/2)=S\int_0^{T/2} f(\eta,\mu,s)ds.$$

Next, (1.5) implies

$$\begin{aligned}AH_1(\eta,\mu)=AS\int_0^{T/2} f(\eta,\mu,s)ds&=-S\int_0^{T/2} f(A\eta,\mu,-s)ds\\ =-S\int_0^{T/2} f(\eta,\mu,T-s)ds&=-S\int_{T/2}^{T} f(\eta,\mu,s)ds\\ &=-S\int_0^{T} f(\eta,\mu,s)ds+H_1(\eta,\mu).\end{aligned}$$

By using $1\notin\sigma(A/V)$ and $H_1(\eta,\mu)\in V$, we derive

$$H_1(\eta,\mu)=(\mathbb{I}-A)^{-1}S\int_0^{T} f(\eta,\mu,s)ds\,. \tag{4.10}$$

We first study the nondegenerate case:

4.1 The case $\ker(\mathbb{I}-A)=\{0\}$

Then $S=\mathbb{I}$ and by (4.4), $m=k\geq p=n$. Now we can prove the following result.

Theorem 4. *If there exists $\mu_0\in\mathbb{R}^k$ such that*

$$\int_0^{T/2} f(0,\mu_0,s)\,ds=0 \quad \textit{and} \quad \int_0^{T/2} D_\mu f(0,\mu_0,s)\,ds:\mathbb{R}^k\to\mathbb{R}^n \quad \textit{is onto}\,. \tag{4.11}$$

Then there is a decomposition $\mathbb{R}^k=X_1\oplus X_2$ with $\dim X_1=n$ and constants $\varepsilon_0>0$, $\delta_1^0>0$, $\delta_2^0>0$ along with a unique C^∞-smooth function $\mu_1(\mu_2,\varepsilon)\in X_1$, $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, $|\mu_2-\mu_2^0|<\delta_2^0$ such that $\mu_1(\mu_2^0,0)=\mu_1^0$ for $\mu_0=(\mu_1^0,\mu_2^0)\in X_1\times X_2$ with the following properties: For any $|\mu_1-\mu_1^0|<\delta_1^0$, $|\mu_2-\mu_2^0|<\delta_2^0$ and $0<|\varepsilon|<\varepsilon_0$, equation (1.1) has a T-periodic and symmetric solution if and only if $\mu_1=\mu_1(\mu_2,\varepsilon)$, moreover this solution is unique, so that it is given by $x(0,\varepsilon,\mu_1(\mu_2,\varepsilon),\mu_2,t)$ and thus it is located near 0 in $\mathbb{R}^n$.

Proof. By (4.11) there is a decomposition $\mathbb{R}^k=X_1\oplus X_2$ with $\dim X_1=n$ such that

$$\det\left[\int_0^{T/2} D_{\mu_1}f(0,\mu_1^0,\mu_2^0,s)\,ds\right]\neq 0\,,$$

where $\mu=(\mu_1,\mu_2)\in X_1\times X_2$. We set a function

$$G(\mu_1,\mu_2,\varepsilon)=\begin{cases}\frac{1}{\varepsilon}F(0,\mu_1,\mu_2,\varepsilon) & \text{for } \varepsilon\neq 0,\\ H_1(0,\mu_1,\mu_2) & \text{for } \varepsilon=0.\end{cases}$$

Clearly that G is C^∞-smooth. We solve

$$G(\mu_1,\mu_2,\varepsilon)=0. \tag{4.12}$$

From our assumptions we have

$$G(\mu_1^0,\mu_2^0,0)=H_1(0,\mu_1^0,\mu_2^0)=0$$

and

$$\det D_{\mu_1}G(\mu_1^0,\mu_2^0,0)=\det D_{\mu_1}H_1(0,\mu_1^0,\mu_2^0)\neq 0.$$

Now applying Implicit Function Theorem on (4.12) the proof is finished. ∎

Using topological degree methods from [25] we can get the next result.

Theorem 5. *Assume a decomposition $\mathbb{R}^k=X_1\oplus X_2$ with $\dim X_1=n$ and the existence of an open bounded subset $\Omega\subset X_1$ along with $\mu_2^0\in X_2$ such that $0\notin H_1(0,\partial\Omega,\mu_2^0)$ and $\deg(H_1(0,\cdot,\mu_2^0),\Omega,0)\neq 0$, then for any small $\varepsilon\neq 0$ and μ_2 near μ_2^0 there exists $\mu_1(\mu_2,\varepsilon)\in\Omega$ such that (1.1) with $\mu_1=\mu_1(\mu_2,\varepsilon)$ has a T-periodic and symmetric solution.*

Next we have the following result.

Theorem 6. *Assume $\ker(\mathbb{I}-A)=\ker\left(\mathbb{I}-A^2\right)=\{0\}$. Then $x(t)=0$ is the only symmetric solution of (1.1) for any $\varepsilon\neq 0$ small.*

Proof. By (1.5) we obtain $A^2f(0,\mu,t)=f(0,\mu,t)$ and so $f(0,\mu,t)\in\ker\left(\mathbb{I}-A^2\right)$. Hence $f(0,\mu,t)=0$ and the proof is finished. ∎

4.2 The case $\ker(\mathbb{I} - A) \neq \{0\}$

Then $p = \dim V < n$. We recall (4.4). Now we are ready to prove the following result.

Theorem 7. *If there exist $\eta_0 \in \ker(\mathbb{I} - A)$ and $\mu_0 \in \mathbb{R}^k$ such that*

$$S \int_0^{T/2} f(\eta_0, \mu_0, s)\, ds = 0 \quad \text{and} \quad S \int_0^{T/2} D_\mu f(\eta_0, \mu_0, s)\, ds : \mathbb{R}^m \to \mathbb{R}^p \quad \text{is onto}. \tag{4.13}$$

Then there are decompositions $\mathbb{R}^k = X_1 \oplus X_2$, $\ker(\mathbb{I} - A) = Y_1 \oplus Y_2$ with $\dim X_1 + \dim Y_1 = n$ and constants $\varepsilon_0 > 0$, $\delta_1^0 > 0$, $\delta_2^0 > 0$, $\delta_3^0 > 0$, $\delta_4^0 > 0$ along with unique C^∞-smooth functions $\mu_1(\eta_2, \mu_2, \varepsilon) \in X_1$, $\eta_1(\eta_2, \mu_2, \varepsilon) \in Y_1$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $|\mu_2 - \mu_2^0| < \delta_2^0$, $|\eta_2 - \eta_2^0| < \delta_4^0$ such that $\mu_1(\eta_2^0, \mu_2^0, 0) = \mu_1^0$, $\eta_1(\eta_2^0, \mu_2^0, 0) = \eta_1^0$ for $\mu_0 = (\mu_1^0, \mu_2^0) \in X_1 \times X_2$, $\eta_0 = (\eta_1^0, \eta_2^0) \in Y_1 \times Y_2$ with the following properties: For any $|\mu_1 - \mu_1^0| < \delta_1^0$, $|\mu_2 - \mu_2^0| < \delta_2^0$, $|\eta_1 - \eta_1^0| < \delta_3^0$, $|\eta_2 - \eta_2^0| < \delta_4^0$ and $0 < |\varepsilon| < \varepsilon_0$, equation (1.1) with (3.3) has a T-periodic and symmetric solution if and only if $\mu_1 = \mu_1(\eta_2, \mu_2, \varepsilon)$, $\eta_1 = \eta_1(\eta_2, \mu_2, \varepsilon)$, moreover this solution is unique, so that it is given by $x(\varepsilon, \eta_2, \mu_2, t) := x(\eta_1(\eta_2, \mu_2, \varepsilon), \eta_2, \varepsilon, \mu_1(\eta_2, \mu_2, \varepsilon), \mu_2, t)$.

Proof. The proof is very similar to the proof of Theorem 4, so we omit details (cf [5, 7, 20]). ∎

Similarly we can extend Theorem 5, but we leave this to the reader. Next, taking

$$\mathbb{R}^n = \ker(\mathbb{I} - A) \oplus \ker(\mathbb{I} + A) \oplus W \tag{4.14}$$

with $AW = W$ and $\pm 1 \notin \sigma(A_0)$ for $A_0 := A/W$. We note

$$A(\eta, y, z) = (\eta, -y, A_0 z) \tag{4.15}$$

for $\eta \in \ker(\mathbb{I} - A)$, $y \in \ker(\mathbb{I} + A)$ and $z \in W$. Then (1.5) implies

$$\begin{aligned} f_1(\eta, y, z, \mu, t) &= -f_1(\eta, -y, A_0 z, \mu, -t), \\ f_2(\eta, y, z, \mu, t) &= f_2(\eta, -y, A_0 z, \mu, -t), \\ A_0 f_3(\eta, y, z, \mu, t) &= -f_3(\eta, -y, A_0 z, \mu, -t) \end{aligned} \tag{4.16}$$

for

$$\begin{gathered} f(\eta, y, z, \mu, t) = (f_1(\eta, y, z, \mu, t), f_2(\eta, y, z, \mu, t), f_3(\eta, y, z, \mu, t)) \\ \in \ker(\mathbb{I} - A) \times \ker(\mathbb{I} + A) \times W. \end{gathered}$$

Then $S(\eta, y, z) = (0, y, z)$ and $V = \ker(\mathbb{I} + A) \oplus W$. Moreover from

$$A_0^2 f_3(\eta, y, z, \mu, t) = f_3(\eta, y, A_0^2 z, \mu, t),$$

we have

$$A_0^2 f_3(\eta, y, 0, \mu, t) = f_3(\eta, y, 0, \mu, t).$$

So if $\ker(\mathbb{I} - A_0^2) = \{0\}$ then $f_3(\eta, y, 0, \mu, t) = 0$ and symmetric solutions lie in a subspace $\ker(\mathbb{I} - A) \oplus \ker(\mathbb{I} + A)$. Hence a bifurcation function is reduced to

$$\hat{H}_1(\eta, \mu) := \int_0^{T/2} f_2(\eta, 0, 0, \mu, t) dt = \frac{1}{2} \int_0^T f_2(\eta, 0, 0, \mu, t) dt \tag{4.17}$$

instead of $H_1(\eta, \mu)$ (cf (4.10)).

5 Asymptotic properties of symmetric and periodic solutions: The case $A = -\mathbb{I}$

In order to investigate asymptotic properties of symmetric and periodic solutions derived in Section 4, we first consider the case $A = -\mathbb{I}$. Now $S = \mathbb{I}$ and (1.5) has the form

$$-f(x, \mu, t) = -f(-x, \mu, -t),$$

which gives

$$-D_x f(x, \mu, t) = D_x f(-x, \mu, -t). \tag{5.1}$$

Let

$$\phi(\xi, \mu, \varepsilon) := x(\xi, \varepsilon, \mu, T).$$

Note by Theorem 4 that $\xi = 0$ is a fixed point of $\phi(\cdot, \mu, \varepsilon)$ if and only if $\mu = \mu(\mu_2, \varepsilon) := (\mu_1(\mu_2, \varepsilon), \mu_2)$ and it corresponds to a unique symmetric and periodic solution of (1.1) for $\varepsilon \neq 0$ small and μ_2 near μ_2^0. So we set

$$\psi(\xi, \mu_2, \varepsilon) := \phi(\xi, \mu(\mu_2, \varepsilon), \varepsilon),$$

and a linear asymptotic property of $\xi = 0$ for $\psi(\cdot, \mu_2, \varepsilon)$ is determined by the spectrum $\sigma\left(D_\xi \psi(0, \mu_2, \varepsilon)\right)$ of $D_\xi \psi(0, \mu_2, \varepsilon)$. Since (5.1) implies

$$\int_0^T D_x f(0, \mu, t) = 0,$$

the usual first order averaging methods cannot be applied (cf Theorem 13, [27]). For this reason, from (1.1) we derive

$$\begin{aligned} \dot{x}_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), t) &= \varepsilon A_{\mu_2,\varepsilon}(t) x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), t) \\ x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), 0) &= \mathbb{I}, \end{aligned} \tag{5.2}$$

where

$$A_{\mu_2,\varepsilon}(t) := D_x f\left(x(0, \varepsilon, \mu(\mu_2, \varepsilon), t), \mu(\mu_2, \varepsilon), t\right).$$

Next it holds

$$\begin{aligned} -x(0, \varepsilon, \mu(\mu_2, \varepsilon), t) &= x(0, \varepsilon, \mu(\mu_2, \varepsilon), -t), \\ x(0, \varepsilon, \mu(\mu_2, \varepsilon), t + T) &= x(0, \varepsilon, \mu(\mu_2, \varepsilon), t). \end{aligned} \tag{5.3}$$

Then (5.1) and (5.3) imply

$$\begin{aligned} -A_{\mu_2,\varepsilon}(t) &= -D_x f\left(x(0, \varepsilon, \mu(\mu_2, \varepsilon), t), \mu(\mu_2, \varepsilon), t\right) \\ &= D_x f\left(-x(0, \varepsilon, \mu(\mu_2, \varepsilon), t), \mu(\mu_2, \varepsilon), -t\right) \\ &= D_x f\left(x(0, \varepsilon, \mu(\mu_2, \varepsilon), -t), \mu(\mu_2, \varepsilon), -t\right) = A_{\mu_2,\varepsilon}(-t). \end{aligned} \tag{5.4}$$

Since

$$A_{\mu_2,\varepsilon}(t + T) = A_{\mu_2,\varepsilon}(t),$$

from the Floquet theory [19] we have

$$x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), t + T) = x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), t) B_{\mu_2,\varepsilon} \tag{5.5}$$

for a regular matrix $B_{\mu_2,\varepsilon}$. Moreover, by (5.4), clearly $x_\xi(0, \mu(\mu_2, \varepsilon), \varepsilon, -t)$ satisfies (5.2), from the uniqueness of initial value problem, it follows

$$x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), -t) = x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), t)\,.$$

Then (5.5) gives

$$x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), T/2) = x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), -T/2)B_{\mu_2,\varepsilon} = x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), T/2)B_{\mu_2,\varepsilon}$$

and so

$$B_{\mu_2,\varepsilon} = \mathbb{I}\,.$$

Consequently, we arrive at

$$x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), t + T) = x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), t)\,,$$

that is

$$D_\xi\psi(0, \mu_2, \varepsilon) = x_\xi(0, \varepsilon, \mu(\mu_2, \varepsilon), T) = \mathbb{I}\,. \tag{5.6}$$

Summarizing, we cannot apply the linear asymptotic theory in this case for symmetric and periodic solutions. Furthermore, (1.1) implies

$$\psi(\xi, \mu_2, \varepsilon) = \xi + \varepsilon \int_0^T f\,(\xi, \mu_0, t)\,dt + O\left(\varepsilon^2 + |\varepsilon(\mu_2 - \mu_2^0)|\right)\,,$$

which gives

$$D_{\xi\xi}\psi(0, \mu_2, \varepsilon) = \varepsilon \int_0^T D_{xx}f\,(0, \mu_0, t)\,dt + O\left(\varepsilon^2 + |\varepsilon(\mu_2 - \mu_2^0)|\right)\,. \tag{5.7}$$

We immediately arrive at the following result [17, 19, 26].

Theorem 8. *Suppose $n = 1$ in Theorem 4. If in addition*

$$\int_0^T D_{xx}f\,(0, \mu_0, t)\,dt \neq 0\,,$$

then the T-periodic and symmetric solution $x(0, \varepsilon, \mu_1(\mu_2, \varepsilon), \mu_2, t)$ is a saddle-node.

Proof. We have

$$\psi(\xi, \mu_2, \varepsilon) = \xi + \varepsilon\frac{\xi^2}{2} \int_0^T D_{xx}f\,(0, \mu_0, t)\,dt + O\left(\left(\varepsilon^2 + |\varepsilon(\mu_2 - \mu_2^0)|\right)\xi^2 + \varepsilon\xi^3\right)\,,$$

which immediately gives the proof. ∎

The case $n > 1$ is more complicated. We intend to apply some results from papers [3, 4]. First we recall for the reader convenience the following theorem of [4].

Theorem 9. *If a mapping* $F = \left(F^1, F^2\right) \in C^\infty\left(\mathbb{R} \times \mathbb{R}^{n-1}, \mathbb{R} \times \mathbb{R}^{n-1}\right)$ *has a form*

$$\begin{aligned} F^1(x,y) &= x + a_0x^2 + x\langle b_0, y\rangle + \langle A_0y, y\rangle + O\left(|z|^3\right), \\ F_j^2(x,y) &= y + x\langle b_j, y\rangle + \langle A_jy, y\rangle + O\left(|z|^3\right), \quad j = 1, 2, \cdots, n-1, \end{aligned} \tag{5.8}$$

where $\langle\cdot,\cdot\rangle$ *is the usual scalar product on* $\mathbb{R}^{n-1}$, $F^2 = \left(F_1^2, \cdots, F_{n-1}^2\right)$, $x \in \mathbb{R}$, $y = (y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1}$, $z = (x, y) \in \mathbb{R}^n$, $b_0, b_j \in \mathbb{R}^{n-1}$, $A_0, A_j \in L\left(\mathbb{R}^{n-1}\right)$ *are symmetric matrices,* $a_0 < 0$ *and* $\Re\sigma(B) > 0$ *for* $B := (b_1, \cdots, b_{n-1})^* \in L\left(\mathbb{R}^{n-1}\right)$.

Then there is a $t_0 > 0$ *and a local curve* $K \in C^1((-t_0, t_0), \mathbb{R}^n) \cap C^\infty((-t_0, t_0) \setminus \{0\}, \mathbb{R}^n)$ *passing through* $(0,0)$ *which is invariant for F and the dynamics of F restricted on K is equivalent to the local dynamics of a polynomial* $R : \mathbb{R} \to \mathbb{R}$ *with* $R(t) = t + a_0t^2 + O\left(t^3\right)$. *Moreover it holds* $K(t) = (t, 0) + O(t^2)$.

The next theorem gives a condition on F to have the form of (5.8).

Theorem 10. *If a mapping* $F \in C^\infty\left(\mathbb{R}^n, \mathbb{R}^n\right)$ *has a form* $F(0) = 0$, $DF(0) = \mathbb{I}$, $\frac{1}{2}D^2F(0) = \mathcal{B} \neq 0$ *and there are* $\lambda_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, $|x_0| = 1$ *such that* $\mathcal{B}x_0^2 = \lambda_0x_0$. *Then F has a form of* (5.8) *near 0 for the orthogonal decomposition* $\mathbb{R}^n = [x_0] \oplus [x_0]^\perp$.

Proof. Let $P : \mathbb{R}^n \to [x_0]$ be the orthogonal projection. Then $z = xx_0 + y$, $x \in \mathbb{R}$, $y \in [x_0]^\perp$, $F^1 = PF$ and $F^2 = QF$ for $Q = \mathbb{I} - P$. We compute

$$\begin{gathered} F^2(x,y) = QF(xx_0 + y) = Q\left(xx_0 + y + \mathcal{B}(xx_0 + y)^2 + O\left(|z|^3\right)\right) \\ = y + Q\left(x^2\mathcal{B}x_0^2 + 2x\mathcal{B}x_0y + \mathcal{B}y^2\right) + O\left(|z|^3\right) = y + Q\left(x^2\lambda_0x_0 + 2x\mathcal{B}x_0y + \mathcal{B}y^2\right) \\ + O\left(|z|^3\right) = y + 2xQ\mathcal{B}x_0y + Q\mathcal{B}y^2 + O\left(|z|^3\right), \end{gathered}$$

and similarly

$$F^1(x,y) = PF(xx_0 + y) = \left(x + x^2\lambda_0\right)x_0 + 2xP\mathcal{B}x_0y + P\mathcal{B}y^2 + O\left(|z|^3\right).$$

We see that F has a form of (5.8) with $a_0 = \lambda_0$ and $B = 2Q\mathcal{B}x_0 \cdot |[x_0]^\perp$. The proof is finished. ∎

Now we show a perturbation stability condition for (5.8).

Theorem 11. *Let a symmetric quadratic mapping* $\mathcal{B}_0 : \mathbb{R}^n \to \mathbb{R}^n$ *have a form* $\frac{1}{2}D^2F(0)$ *of* (5.8) *with* $a_0 < 0$ *and* $\Re\sigma(B) > 0$. *If a symmetric quadratic mapping* $\mathcal{B}_1 : \mathbb{R}^n \to \mathbb{R}^n$ *is sufficiently near to* $\mathcal{B}_0$ *then* $\mathcal{B}_1$ *has a form of* (5.8) *as well.*

Proof. It is easy to verify that $\mathcal{B}_0$ satisfies assumptions of Theorem 10 for $x_0 = (1, 0, \cdots, 0)$ and $\lambda_0 = a_0$. The vector space of all symmetric and quadratic mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be identified with $\mathbb{R}^M$ for $M := n(n+1)/2$. Then we consider a mapping $\mathcal{H} \in C^\infty\left(\mathbb{R}^{n+1+M}, \mathbb{R}^{n+1}\right)$ defined by

$$\mathcal{H}(z, \lambda, \mathcal{B}) := \left(\mathcal{B}z^2 - \lambda z, |z|^2 - 1\right).$$

Clearly $\mathcal{H}(x_0, a_0, \mathcal{B}_0) = 0$. Next we have

$$\begin{aligned} 0 = D_{(z,\lambda)}\mathcal{H}(x_0, a_0, \mathcal{B}_0)(u, \theta) &= (2\mathcal{B}_0 x_0 u - a_0 u - \theta x_0, 2\langle x_0, u\rangle) \\ &= \left(a_0 v + \langle b_0, w\rangle - \theta, \langle b_j, w\rangle - a_0 w_j, 2v\right) , \\ u = (v, w) \in \mathbb{R} \times \mathbb{R}^{n-1}, &\quad j = 1, 2, \cdots, n-1 , \end{aligned}$$

when $v = 0$, $Bw = a_0 w$ and $\theta = \langle b_0, w\rangle$. Since $\Re\sigma(B) > 0$, we get $w = 0$ and then $\theta = 0$. Consequently, we can apply Implicit Function Theorem for equation

$$\mathcal{H}(z, \lambda, \mathcal{B}_1) = 0$$

to obtain its local C^∞-solution $z = z(\mathcal{B}_1)$ and $\lambda = \lambda(\mathcal{B}_1)$ for any $\mathcal{B}_1$ near B_0. Then $a_0(\mathcal{B}_1) = \lambda(\mathcal{B}_1)$ and $B(\mathcal{B}_1) = 2Q_{\mathcal{B}_1}\mathcal{B}_1 z(\mathcal{B}_1) \cdot |[z(\mathcal{B}_1)]^\perp$ with the orthogonal projection $Q_{\mathcal{B}_1} : \mathbb{R}^n \to [z(\mathcal{B}_1)]^\perp$. Note $a_0(\mathcal{B}_0) = a_0 < 0$ and $B(\mathcal{B}_0) = B$. Hence $\Re\sigma(B(\mathcal{B}_1)) > 0$. The proof of Theorem 11 is finished. ∎

Now we can prove the following result.

Theorem 12. *Suppose $n > 1$. Let the assumptions of Theorem 4 be satisfied. If in addition*

$$\mathcal{B} := \frac{1}{2}\int_0^T D_{xx} f\,(0, \mu_0, t)\, dt$$

has a negative eigenvalue with eigenvector x_0 such that $\Re\sigma(B) > 0$ for $B := 2Q\mathcal{B}x_0 \cdot |[x_0]^\perp$ with the orthogonal projection $Q : \mathbb{R}^n \to [x_0]^\perp$ then the T-periodic and symmetric solution $x(0, \varepsilon, \mu_1(\mu_2, \varepsilon), \mu_2, t)$ has a local saddle-node dynamics. Hence it is unstable.

Proof. The proof follows directly from (5.7) and Theorems 9, 10 and 11. ∎

Concrete examples are presented in Section 8.1.

6 Asymptotic properties of symmetric and periodic solutions: The case $A \neq -\mathbb{I}$

6.1 Hyperbolicity of periodic solutions

To study stability of the T-periodic and symmetric solution of equation (1.1) we recall the approach of [9, 10, 29]. For this aim we consider a C^∞-mapping

$$\Phi_{\varepsilon,\eta_2,\mu_2}(\eta) := x\,(\eta, \varepsilon, \mu_1(\eta_2, \mu_2, \varepsilon), \mu_2, T) \quad \text{for } \eta \in \mathbb{R}^n.$$

Note $\Phi_{\varepsilon,\eta_2,\mu_2}(\eta(\varepsilon, \eta_2, \mu_2)) = \eta(\varepsilon, \eta_2, \mu_2)$ for $\eta(\varepsilon, \eta_2, \mu_2) := (\eta_1(\eta_2, \mu_2, \varepsilon), \eta_2)$. By Remark 2, its linearization at $\eta(\varepsilon, \eta_2, \mu_2)$ has the decomposition (cf (4.5) and (4.7))

$$\begin{aligned} D\Phi_{\varepsilon,\eta_2,\mu_2}(\eta(\varepsilon, \eta_2, \mu_2)) = \mathbb{I} + \varepsilon\int_0^T D_x f(\eta_0, \mu_0, s) ds \\ + O\left(\varepsilon^2 + |\varepsilon(\eta_2 - \eta_2^0)| + |\varepsilon(\mu_2 - \mu_2^0)|\right). \end{aligned} \tag{6.1}$$

By following [9, 27, 29] we obtain the following well-known result.

Theorem 13. *For any $\varepsilon > 0$ sufficiently small, the symmetric and T-periodic solution $x(\varepsilon, \eta_2, \mu_2, t)$ of (1.1) from Theorem 7 has the following asymptotic properties:*

- *If $\Re\left\{\sigma\left(\int_0^T D_x f(\eta_0, \mu_0, s)\, ds\right)\right\} \subset (-\infty, 0)$ then $x(\varepsilon, \eta_2, \mu_2, t)$ is asymptotically stable.*
- *If $\Re\left\{\sigma\left(\int_0^T D_x f(\eta_0, \mu_0, s)\, ds\right)\right\} \cap (0, \infty) \neq \emptyset$ then $x(\varepsilon, \eta_2, \mu_2, t)$ is unstable.*
- *If $\Re\left\{\sigma\left(\int_0^T D_x f(\eta_0, \mu_0, s)\, ds\right)\right\} \subset (0, \infty)$ then $x(\varepsilon, \eta_2, \mu_2, t)$ is a repeller.*
- *If $\Re\left\{\sigma\left(\int_0^T D_x f(\eta_0, \mu_0, s)\, ds\right)\right\} \cap \{0\} = \emptyset$ then $x(\varepsilon, \eta_2, \mu_2, t)$ is hyperbolic with the same hyperbolicity type as $\int_0^T D_x f(\eta_0, \mu_0, s)\, ds$.*

By (4.16) after some computations we derive

$$\begin{gathered} D_\eta f_1(\eta,0,0,\mu,t) = -D_\eta f_1(\eta,0,0,\mu,-t),\ D_y f_1(\eta,0,0,\mu,t) = D_y f_1(\eta,0,0,\mu,-t) \\ D_z f_1(\eta,0,0,\mu,t) = -D_z f_1(\eta,0,0,\mu,-t)A_0, D_\eta f_2(\eta,0,0,\mu,t) = D_\eta f_2(\eta,0,0,\mu,-t) \\ D_y f_2(\eta,0,0,\mu,t) = -D_y f_2(\eta,0,0,\mu,-t), D_z f_2(\eta,0,0,\mu,t) = D_z f_2(\eta,0,0,\mu,-t)A_0 \\ A_0 D_\eta f_3(\eta,0,0,\mu,t) = -D_\eta f_3(\eta,0,0,\mu,-t) \\ A_0 D_y f_3(\eta,0,0,\mu,t) = D_y f_3(\eta,0,0,\mu,-t) \\ A_0 D_z f_3(\eta,0,0,\mu,t) = -D_z f_3(\eta,0,0,\mu,-t)A_0. \end{gathered} \tag{6.2}$$

By (6.2) we derive

$$\begin{gathered} \int_0^T D_\eta f_1(\eta,0,0,\mu,t)dt = 0, \quad \int_0^T D_y f_1(\eta,0,0,\mu,t)dt = 2\int_0^{T/2} D_y f_1(\eta,0,0,\mu,t)dt \\ \int_0^T D_z f_1(\eta,0,0,\mu,t)dt\,(\mathbb{I} + A_0) = 0 \\ \int_0^T D_\eta f_2(\eta,0,0,\mu,t)dt = 2\int_0^{T/2} D_\eta f_2(\eta,0,0,\mu,t)dt \\ \int_0^T D_y f_2(\eta,0,0,\mu,t)dt = 0, \quad \int_0^T D_z f_2(\eta,0,0,\mu,t)dt\,(\mathbb{I} - A_0) = 0 \\ (\mathbb{I} + A_0)\int_0^T D_\eta f_3(\eta,0,0,\mu,t)dt = 0, \quad (\mathbb{I} - A_0)\int_0^T D_y f_3(\eta,0,0,\mu,t)dt = 0 \\ A_0 \int_0^T D_z f_3(\eta,0,0,\mu,t)dt = -\int_0^T D_z f_3(\eta,0,0,\mu,t)dt A_0. \end{gathered} \tag{6.3}$$

Then (6.3) and $\pm 1 \notin \sigma(A_0)$ imply

$$\begin{aligned} &\int_0^T D_z f_1(\eta,0,0,\mu,t)dt = 0, \quad \int_0^T D_z f_2(\eta,0,0,\mu,t)dt = 0\,, \\ &\int_0^T D_\eta f_3(\eta,0,0,\mu,t)dt = 0, \quad \int_0^T D_y f_3(\eta,0,0,\mu,t)dt = 0\,. \end{aligned} \tag{6.4}$$

Summarizing by (6.3) and (6.4) it holds

$$\int_0^T D_x f(\eta_0, \mu_0, s)\, ds =$$

$$\begin{pmatrix} 0 & \int_0^T D_y f_1(\eta_0,\mu_0,t)dt & 0 \\ \int_0^T D_\eta f_2(\eta_0,\mu_0,t)dt & 0 & 0 \\ 0 & 0 & \int_0^T D_z f_3(\eta_0,\mu_0,t)dt \end{pmatrix}$$

for matrices

$$\begin{aligned} &\int_0^T D_y f_1(\eta_0,\mu_0,t)dt : \ker(\mathbb{I} + A) \to \ker(\mathbb{I} - A)\,, \\ &\int_0^T D_\eta f_2(\eta_0,\mu_0,t)dt : \ker(\mathbb{I} - A) \to \ker(\mathbb{I} + A)\,, \\ &\int_0^T D_z f_3(\eta_0,\mu_0,t)dt : W \to W\,. \end{aligned}$$

Clearly, if $\int_0^T D_x f(\eta_0, \mu_0, s)\, ds$ is hyperbolic then $\dim\ker(\mathbb{I} + A) = \dim\ker(\mathbb{I} - A)$. On the other hand, if $\dim\ker(\mathbb{I} + A) = \dim\ker(\mathbb{I} - A) \neq 0$ then $\int_0^T D_x f(\eta_0, \mu_0, s)\, ds$ is hyperbolic if and only if

$$\Re\left\{\lambda \in \mathbb{C} \mid \lambda^2 \in \sigma\left(\int_0^T D_y f_1(\eta_0, \mu_0, t)dt \int_0^T D_\eta f_2(\eta_0, \mu_0, t)dt\right)\right\} \cap \{0\} = \emptyset,$$
$$\Re\left(\sigma\left(\int_0^T D_z f_3(\eta_0, \mu_0, t)dt\right)\right) \cap \{0\} = \emptyset.$$

Of course when $\dim\ker(\mathbb{I} + A) = \dim\ker(\mathbb{I} - A) = 0$ then we suppose

$$\Re\left\{\sigma\left(\int_0^T D_z f_3(0, \mu_0, t)dt\right)\right\} \cap \{0\} = \emptyset.$$

6.2 k-Hyperbolicity

To study more sophisticated hyperbolicity of periodic solutions of equation (1.1) we need the following results from [9, 29].

Definition 2. A continuous matrix function $L_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ of $\varepsilon \geq 0$ and such that $L_0 = \mathbb{I}$, is *k-hyperbolic* if for every matrix function N_ε defined for $\varepsilon \geq 0$ satisfying $N_\varepsilon = o(\varepsilon^k)$, there exists an interval $0 < \varepsilon < \varepsilon_1$ in which $L_\varepsilon + N_\varepsilon$ is hyperbolic of the same type (i.e., with the same number of eigenvalues on each side of the unit circle).

Definition 3. A continuous matrix function $L_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ of $\varepsilon \geq 0$ and such that $L_0 = \mathbb{I}$, is *strongly k-hyperbolic* if there exists a continuous real matrix C_ε defined in an interval $0 \leq \varepsilon < \varepsilon_0$ such that C_ε is regular (even for $\varepsilon = 0$) and such that

$$C^{-1} L_\varepsilon C_\varepsilon = \begin{pmatrix} A_\varepsilon & 0 \\ 0 & B_\varepsilon \end{pmatrix}$$

for $0 < \varepsilon < \varepsilon_0$, where A_ε, B_ε are $r \times r$ and $s \times s$ blocks, respectively, and $\|A_\varepsilon\| < 1 - c\varepsilon^k$, $\|B_\varepsilon^{-1}\| < 1 - c\varepsilon^k$, for some $c > 0$.

Theorem 14 ([29, Theorem 2.2])**.** *If $L_\varepsilon = \mathbb{I} + \varepsilon L_1 + \cdots + \varepsilon^k L_k$, if the eigenvalues of L_1 are distinct numbers on the unit circle, and if the eigenvalues $\lambda_i(\varepsilon)$ of L_ε suitably numbered satisfy $|\lambda_i(\varepsilon)| < 1 - c\varepsilon^k$ for $i = 1, \ldots r$, $|\lambda_i(\varepsilon)| > 1 + c\varepsilon^k$ for $i = r+1, \ldots, n$, for some constant $c > 0$ and $\varepsilon > 0$ small, then L_ε is strongly k-hyperbolic.*

If $m = n$ then $\Phi_{\varepsilon,\eta_2,\mu_2}$ and $\eta(\varepsilon, \eta_2, \mu_2))$ depend only on ε, so we have Φ_ε and $\eta(\varepsilon)$. Now we can improve Theorem 13 as follows.

Theorem 15. *Suppose $m = n$. Let $D\Phi_\varepsilon(\eta(\varepsilon)) = \mathbb{I} + \varepsilon M_1 + \cdots + \varepsilon^k M_k + o(\varepsilon^k)$. Suppose that all eigenvalues of M_1 $\left(= \int_0^T D_x f(\eta_0, \mu_0, s)ds\right)$ are distinct complex numbers on the unit circle. If the eigenvalues $\lambda_i(\varepsilon)$ of $\mathbb{I} + \varepsilon M_1 + \cdots + \varepsilon^k M_k$ suitably numbered satisfy $|\lambda_i(\varepsilon)| < 1 - c\varepsilon^k$ for $i = 1, \ldots r$, $|\lambda_i(\varepsilon)| > 1 + c\varepsilon^k$ for $i = r+1, \ldots, n$, for some constant $c > 0$ and $\varepsilon > 0$ small. Then the symmetric and T-periodic solution $x_\varepsilon(t)$ of* (1.1) *from Theorem 7 is hyperbolic for any $\varepsilon > 0$ small.*

6.3 A particular case

We consider the splitting (4.14) with

$$\dim\ker(\mathbb{I}-A)=\dim\ker(\mathbb{I}-A_0^2)=0 \quad \text{and} \quad \dim\ker(\mathbb{I}+A)\neq 0\,. \tag{6.5}$$

So we do not have variable η in (4.15) and function f_1 in (4.16) as well. Moreover we know that the only symmetric solution of (1.1) has the form

$$x(0,0,\varepsilon,\mu,t)=(y(0,0,\varepsilon,\mu,t),z(0,0,\varepsilon,\mu,t))=(y(0,0,\varepsilon,\mu,t),0) \tag{6.6}$$

with $y(0,\varepsilon,\mu,-t)=-y(0,\varepsilon,\mu,t)$. Next (4.16) implies

$$\begin{aligned} D_yf_2(y,0,\mu,t) &= -D_yf_2(-y,0,\mu,-t)\\ D_zf_2(y,0,\mu,t) &= D_zf_2(-y,0,\mu,-t)A_0\\ A_0D_yf_3(y,0,\mu,t) &= D_yf_3(-y,0,\mu,-t)\\ A_0D_zf_3(y,0,\mu,t) &= -D_zf_3(-y,0,\mu,-t)A_0\,. \end{aligned} \tag{6.7}$$

From (6.7), we derive

$$\begin{aligned} D_zf_2(y,0,\mu,t) &= D_zf_2(-y,0,\mu,-t)A_0 = D_zf_2(y,0,\mu,t)A_0^2\\ &\Rightarrow D_zf_2(y,0,\mu,t)\left(\mathbb{I}-A_0^2\right)=0\,,\\ A_0^2D_yf_3(y,0,\mu,t) &= A_0D_yf_3(-y,0,\mu,-t)=D_yf_3(y,0,\mu,t)\\ &\Rightarrow \left(\mathbb{I}-A_0^2\right)D_yf_3(y,0,\mu,t)=0\,. \end{aligned} \tag{6.8}$$

Since $\mathbb{I}-A_0^2:W\to W$ is an isomorphism, (6.8) gives

$$D_zf_2(y,0,\mu,t)=0,\quad D_yf_3(y,0,\mu,t)=0\,.$$

Consequently, the variational equation of (1.1) along the symmetric solution (6.6) has the form

$$\begin{aligned} D_y\dot y(0,0,\varepsilon,\mu,t) &= \varepsilon D_yf_2\left(y(0,0,\varepsilon,\mu,t),0,\mu,t\right)D_yy(0,0,\varepsilon,\mu,t)\\ D_yy(0,0,\varepsilon,\mu,0) &= \mathbb{I}\,,\\ D_z\dot y(0,0,\varepsilon,\mu,t) &= \varepsilon D_yf_2\left(y(0,0,\varepsilon,\mu,t),0,\mu,t\right)D_zy(0,0,\varepsilon,\mu,t)\\ D_zy(0,0,\varepsilon,\mu,0) &= 0\,,\\ D_y\dot z(0,0,\varepsilon,\mu,t) &= \varepsilon D_zf_3\left(y(0,0,\varepsilon,\mu,t),0,\mu(\varepsilon),t\right)D_yz(0,0,\varepsilon,\mu,t)\\ D_yz(0,0,\varepsilon,\mu,0) &= 0\,,\\ D_z\dot z(0,0,\varepsilon,\mu,t) &= \varepsilon D_zf_3\left(y(0,0,\varepsilon,\mu,t),0,\mu(\varepsilon),t\right)D_zz(0,0,\varepsilon,\mu,t)\\ D_zz(0,0,\varepsilon,\mu,0) &= \mathbb{I}\,, \end{aligned} \tag{6.9}$$

which yields to $D_zy(0,0,\varepsilon,\mu,t)=0$ and $D_yz(0,0,\varepsilon,\mu,t)=0$ for any $t\in\mathbb{R}$. Moreover, since by (6.7)

$$-D_yf_2\left(y(0,0,\varepsilon,\mu,t),0,\mu,t\right)=D_yf_2\left(y(0,0,\varepsilon,\mu,-t),0,\mu,-t\right)$$

the first equation of (6.9) implies $D_y y(0,0,\varepsilon,\mu,t) = D_y y(0,0,\varepsilon,\mu,-t)$ for any $t \in \mathbb{R}$. By assuming a T-periodicity of $y(0,0,\varepsilon,\mu,t)$ in t, then like in Section 5 we arrive at $D_y y(0,0,\varepsilon,\mu,T) = \mathbb{I}$. Consequently, it holds

$$D_{(y,z)}x(0,0,\varepsilon,\mu,T) = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & D_z z(0,0,\varepsilon,\mu,T) \end{pmatrix}.$$

Hence we again cannot apply Theorem 13. On the other hand, we have

$$\int_0^T D_{(y,z)}f(0,0,\mu,t)dt = \begin{pmatrix} 0 & 0 \\ 0 & \int_0^T D_z f_3(0,0,\mu,t)dt \end{pmatrix}.$$

Hence if

$$\begin{aligned} &\int_0^T f(0,0,\mu_0,t)dt = 0 \text{ for some } \mu_0 \in \mathbb{R}^k \\ \text{and} \quad &\Re\sigma\left(\int_0^T D_z f_3(0,0,\mu_0,t)dt\right) \cap \{0\} = \emptyset, \end{aligned} \tag{6.10}$$

then we can apply a local center manifold method to (1.1) near $x \sim 0$ and $\mu \sim \mu_0$ to get the situation of Section 5. Note (4.16) gives

$$\int_0^T f_2(0,0,\mu,t)dt = 2\int_0^{T/2} f_2(0,0,\mu,t)dt, \quad (\mathbb{I} + A_0)\int_0^T f_3(0,0,\mu,t)dt = 0,$$

which implies $\int_0^T f_3(0,0,\mu,t)dt = 0$. Consequently the equation $\int_0^T f(0,0,\mu_0,t)dt = 0$ is equivalent to $\int_0^{T/2} f_2(0,0,\mu_0,t)dt = 0$ (cf (4.17)).

Next we assume that A is unitary, i.e. $\|A\| = 1$. This holds among others when $A^p = \mathbb{I}$ for some $p \in \mathbb{N}$ by taking a new scalar product on $\mathbb{R}^n$ given by [9]

$$(x_1, x_2) := \sum_{j=1}^p \langle A^j x_1, A^j x_2 \rangle.$$

Let $r \in \mathbb{N}$. Now we apply a local center manifold method to (1.1) near $x \sim 0$ and $\mu \sim \mu_0$ to get a local C^r-mapping $\Phi(y,\varepsilon,\mu,t)$ which is T-periodic in t, $y \sim 0$, $\varepsilon \sim 0$, $\mu \sim \mu_0$, $\Phi \in W$ and satisfying

$$A_0\Phi(y,\varepsilon,\mu,t) = \Phi(-y,\varepsilon,\mu,-t) \tag{6.11}$$

along with

$$\varepsilon f_3(y,\Phi(y,\varepsilon,\mu,t),\mu,t) = \varepsilon D_y\Phi(y,\varepsilon,\mu,t) f_2(y,\Phi(y,\varepsilon,\mu,t),\mu,t) + D_t\Phi(y,\varepsilon,\mu,t). \tag{6.12}$$

Expanding

$$\Phi(y,\varepsilon,\mu,t) = \Phi_0(y,\mu) + O(\varepsilon),$$

and using (6.12) we derive

$$\bar{f}_3(y,\Phi_0(y,\mu),\mu) = D_y\Phi_0(y,\mu)\bar{f}_2(y,\Phi_0(y,\mu),\mu), \tag{6.13}$$

where $\bar{f}_i := \int_0^T f_i(y,z,t)dt$, $i = 1,2$. Note

$$\Phi_0(0,\mu_0) = 0 \quad \text{and} \quad D_y\Phi_0(0,\mu_0) = 0. \tag{6.14}$$

Of course, (6.13) means that $\Phi_0(y,\mu)$ is a graph of a local center manifold of the averaged equation of (1.1) given by $\dot{x} = \varepsilon\bar{f}(x,\mu)$ for $x \sim 0$ and $\mu \sim \mu_0$. The reduced ODE on the local center manifold is given by

$$\dot{y} = \varepsilon g(y,\varepsilon,\mu,t) := \varepsilon f_2\left(y,\Phi(y,\varepsilon,\mu,t),\mu,t\right) \,. \tag{6.15}$$

Note (4.16) and (6.11) imply

$$\begin{aligned} g(-y,\varepsilon,\mu,-t) &= f_2\left(-y,\Phi(-y,\varepsilon,\mu,-t),\mu,-t\right) = f_2\left(-y,A_0\Phi(y,\varepsilon,\mu,t),\mu,-t\right) \\ &= f_2\left(y,\Phi(y,\varepsilon,\mu,t),\mu,t\right) = g(y,\varepsilon,\mu,t)\,. \end{aligned}$$

Next we compute

$$D_y\bar{g}(y,0,\mu) = D_y\bar{f}_2\left(y,\Phi_0(y,\mu),\mu\right) + D_z\bar{f}_2\left(y,\Phi_0(y,\mu),\mu\right)D_y\Phi_0(y,\mu)$$

and

$$\begin{aligned} D_{yy}\bar{g}(y,0,\mu) = D_{yy}\bar{f}_2\left(y,\Phi_0(y,\mu),\mu\right) &+ 2D_{yz}\bar{f}_2\left(y,\Phi_0(y,\mu),\mu\right)D_y\Phi_0(y,\mu) \\ &+ D_{zz}\bar{f}_2\left(y,\Phi_0(y,\mu),\mu\right)\left(D_y\Phi_0(y,\mu),D_y\Phi_0(y,\mu)\right) \\ &+ D_y\bar{f}_2\left(y,\Phi_0(y,\mu),\mu\right)D_{yy}\Phi_0(y,\mu)\,. \end{aligned}$$

Using (6.3) and (6.14), we obtain

$$D_y\bar{g}(0,0,\mu_0) = 0 \quad \text{and} \quad D_{yy}\bar{g}(0,0,\mu_0) = D_{yy}\bar{f}_2\left(0,0,\mu_0\right) \,. \tag{6.16}$$

Similarly we derive (cf (6.4))

$$\begin{aligned} &\int_0^{T/2} g(0,0,\mu_0,t)dt = \int_0^{T/2} f_2\left(0,\Phi_0(0,\mu_0),\mu_0,t\right)dt = \frac{1}{2}\bar{f}(0,0,\mu_0) = 0\,, \\ &\int_0^{T/2} D_\mu g(0,0,\mu_0,t)dt = \int_0^{T/2} D_z f_2\left(0,\Phi_0(0,\mu_0),\mu_0,t\right)dt D_\mu\Phi_0(0,\mu_0) \\ &\qquad + \int_0^{T/2} D_\mu f_2\left(0,\Phi_0(0,\mu_0),\mu_0,t\right)dt = \frac{1}{2}D_\mu\bar{f}_2(0,0,\mu_0)\,. \end{aligned} \tag{6.17}$$

Consequently, if $k \geq \dim\ker(\mathbb{I}+A)$ then we can apply results of Sections 4.1 and 5 to this particular case (6.15).

Next by Section 4.1, when $\dim\ker(\mathbb{I}-A_0^2) = 0$ then $f_3\left(y,0,\mu\right) = 0$, so the reduced equation is now

$$\dot{y} = \varepsilon f_2\left(y,0,\mu,t\right) \tag{6.18}$$

and symmetric solutions lie in $\ker(\mathbb{I}+A)$. A 3-dimensional example is given in Section 8.2.4.

Finally, when $\dim\ker(\mathbb{I}+A) = 1$ then we can apply Theorems 4 and 8 to show that for any $\varepsilon \neq 0$ small, there is a surface S_ε of codimension 1 splitting $\mathbb{R}^k$ near μ_0 in two parts $P_{1,\varepsilon}$ and $P_{2,\varepsilon}$ such that: if $\mu \in P_{1,\varepsilon}$ then (1.1) has no small periodic solutions, if $\mu \in S_\varepsilon$ then (1.1) has a unique small periodic solution which is in addition symmetric and unstable, and if $\mu \in P_{2,\varepsilon}$ then (1.1) has exactly two small periodic solutions $x_{1,\varepsilon}(t)$ and $x_{2,\varepsilon}(t)$, which are hyperbolic and nonsymmetric but satisfying $x_{1,\varepsilon}(t) = Ax_{2,\varepsilon}(-t)$ and $x_{2,\varepsilon}(t) = Ax_{1,\varepsilon}(-t)$. So $x_{i,\varepsilon}(t) = A^2x_{i,\varepsilon}(t)$, $i = 1,2$, i.e. $x_{i,\varepsilon}(t) \in \ker(\mathbb{I}-A^2)$ for any $t \in \mathbb{R}$. Note $\ker(\mathbb{I}+A) \subset \ker(\mathbb{I}-A^2)$. This is a saddle-node bifurcation with symmetries.

7 Antisymmetric and periodic solutions

Assuming in addition (1.7), we can directly extend the above results to antiperiodic solutions (cf (1.8)). So we only state some results without proofs.

Theorem 16. *The Cauchy problem* (1.1) *with*

$$x(0) = \theta \in \ker(\mathbb{I} + A) \tag{7.1}$$

has a unique C^∞*-smooth solution* $x(\theta, \varepsilon, \mu, t)$ *which is also antisymmetric, and any antisymmetric solution* $x(t)$ *of* (1.1) *satisfies* (7.1).

We see that in order to study antisymmetric solutions, it is enough to replace $\ker(\mathbb{I} - A)$ with $\ker(\mathbb{I} + A)$ in the arguments dealing for the symmetric case, and so the projection $\mathbb{I} - S$ is replaced with an A-invariant projection $\mathbb{I} - \widetilde{S} : \mathbb{R}^n \to \ker(\mathbb{I} + A)$ in the above sections.

8 Applications

In this section, we present concrete weakly nonlinear ODE to illustrate our theory. We separately consider two cases when either $A = -\mathbb{I}$ or $A \neq -\mathbb{I}$. We start with the first one.

8.1 The case $A = -\mathbb{I}$

8.1.1 Scalar equations

Let us consider scalar equation (1.1) with a form

$$\dot{x} = \varepsilon \left(\cos x + \mu(1 + \sin t) \right), \quad \tau = -\pi, \quad Ax = -x. \tag{8.1}$$

It is easy to see that condition (1.5) is satisfied. Really, we verify

$$\begin{aligned} Af(x, \mu, t) &= -(\cos x + \mu(1 + \sin t)) \\ &= -(\cos(-x) + \mu(1 + \sin(-t + \pi))) = -f(Ax, \mu, -t - \tau). \end{aligned}$$

We have that $\ker(\mathbb{I} - A) = \{0\}$, so now $\eta_0 = 0$. Then

$$H_1(\mu) = H_1(0, \mu) = \int_0^\pi (\cos 0 + \mu(1 + \sin(s + \pi/2)) \, ds = \pi(1 + \mu).$$

Since $H_1(\mu_0) = 0$ if and only if $\mu_0 = -1$ and $H_1'(-1) = \pi \neq 0$, we can apply Theorem 4 to get a unique symmetric and 2π-periodic solution $x_\varepsilon(t) = x(0, \mu(\varepsilon), \varepsilon, t)$ of (8.1) (only for $\mu = \mu(\varepsilon)$) with $\mu(0) = -1$. So it holds

$$-x(0, \varepsilon, \mu(\varepsilon), t) = x(0, \varepsilon, \mu(\varepsilon), -t + \pi), \quad x(0, \varepsilon, \mu(\varepsilon), t + 2\pi) = x(0, \varepsilon, \mu(\varepsilon), t),$$

which imply

$$x(0, \varepsilon, \mu(\varepsilon), 3\pi/2 + t) = -x(0, \varepsilon, \mu(\varepsilon), 3\pi/2 - t). \tag{8.2}$$

Next, since

$$\int_0^{2\pi} D_x f(0,-1,s)ds = 0,$$

we cannot apply the usual first order averaging methods for establishing asymptotic properties of $x_\varepsilon(t)$. So we need to study in more details the mapping (cf Section 5 and Theorem 8)

$$\Phi_\varepsilon(\eta) = x(\eta,\varepsilon,\mu(\varepsilon),5\pi/2) \quad \text{for } \eta \in \mathbb{R}.$$

Note $\Phi_\varepsilon(0) = 0$ and $x(t) = x(\eta,\varepsilon,\mu(\varepsilon),t)$ is the solution of the Cauchy problem

$$\begin{aligned} \dot{x} = \varepsilon\left(\cos x + \mu(\varepsilon)(1+\sin t)\right), \\ x(\pi/2) = \eta. \end{aligned} \tag{8.3}$$

Consequently, we have $\Phi_\varepsilon(\eta + 2\pi) = \Phi_\varepsilon(\eta) + 2\pi$, and so $\Phi_\varepsilon : S^1 \to S^1$ for the unit circle. Moreover, $\Phi_\varepsilon(\eta)$ has the only fixed point $\eta_0 = 0$ in S^1. Now we compute $\Phi_\varepsilon'(0) = D_\eta x(0,\varepsilon,\mu(\varepsilon),5\pi/2)$. By using (8.3) we get

$$\begin{aligned} \dot{D}_\eta x(0,\varepsilon,\mu(\varepsilon),t) = -\varepsilon \sin\left(x(0,\varepsilon,\mu(\varepsilon),t)\right) D_\eta x(0,\varepsilon,\mu(\varepsilon),5\pi/2), \\ D_\eta x(0,\varepsilon,\mu(\varepsilon),\pi/2) = 1. \end{aligned} \tag{8.4}$$

Then using (8.2), we obtain

$$\Phi_\varepsilon'(0) = \mathrm{e}^{-\varepsilon\int_{\pi/2}^{5\pi/2}\sin(x(0,\varepsilon,\mu(\varepsilon),s))ds} = 1.$$

Next, (8.3) also implies

$$\Phi_\varepsilon(\eta) = \eta + 2\pi\varepsilon(\cos\eta - 1) + O(\varepsilon^2),$$

which gives

$$\Phi_\varepsilon''(0) = -4\pi\varepsilon + O(\varepsilon^2) < 0$$

for $\varepsilon > 0$ small. Summarizing we see [17, 19, 26] that 0 is a global saddle-node of $\Phi_\varepsilon : S^1 \to S^1$ for any $\varepsilon > 0$ small: it is attracting from the right and repelling from the left. The orientation of the dynamics of $\Phi_\varepsilon : S^1 \to S^1$ is reverse for $\varepsilon < 0$ small.

8.1.2 Planar equations

Example 1. First we consider the system

$$\begin{aligned} \dot{x} = \varepsilon\left(-(x+y+\sin t)x + \mu_1\right) \\ \dot{y} = \varepsilon\left((x+y+\sin t)y + \mu_2\right) \end{aligned} \tag{8.5}$$

with $\mu = (\mu_1,\mu_2) \in \mathbb{R}^2$. Now $k = n = 2$, $T = 2\pi$ and $\int_0^\pi f(0,\mu,t)\,dt = \pi\mu$. So assumptions of (4.11) are satisfied. On the other hand, (8.5) has a trivial symmetric solution $x = 0$, $y = 0$ for any ε and $\mu = 0$. The uniqueness of $\mu(\varepsilon)$ implies $\mu(\varepsilon) = 0$, and hence (8.5) has no symmetric and periodic solutions for any $\mu \neq 0$ and $\varepsilon \neq 0$ small. Next, we get

$$\mathcal{B}(x,y)^2 = \left(-x^2 - xy, y^2 + xy\right).$$

Since now

$$\mathcal{B}\left((x_1,y_1),(x_2,y_2)\right) = \left(-x_1x_2 - \frac{x_1y_2+x_2y_1}{2}, y_1y_2 + \frac{x_1y_2+x_2y_1}{2}\right),$$
$$x_0 = (1,0), \quad \lambda_0 = -1, \quad [x_0]^\perp = [(0,1)], \quad 2Q\mathcal{B}\left((1,0),(0,y)\right) = y\,.$$

Clearly Theorem 12 can be applied. So the symmetric and periodic solution $(x,y) = 0$ of (8.5) with $\mu = 0$ is unstable for any $\varepsilon \neq 0$ small. In order to find general periodic solutions of (8.5), we apply Theorems 1 and 2. So we solve the averaged equation

$$\begin{aligned} -(x+y)x + \mu_1 &= 0 \\ (x+y)y + \mu_2 &= 0 \end{aligned} \tag{8.6}$$

which implies $(x+y)^2 = \mu_1 - \mu_2$. So we need $\mu_1 \geq \mu_2$. If $\mu_1 = \mu_2 \neq 0$, then (8.6) has no solution as well. If $\mu_1 > \mu_2$, then we derive

$$\begin{aligned} x_1 &= \frac{\mu_1}{\sqrt{\mu_1-\mu_2}}, \quad y_1 = -\frac{\mu_2}{\sqrt{\mu_1-\mu_2}}, \\ x_2 &= -\frac{\mu_1}{\sqrt{\mu_1-\mu_2}}, \quad y_2 = \frac{\mu_2}{\sqrt{\mu_1-\mu_2}}. \end{aligned} \tag{8.7}$$

Next, the characteristic polynomial of the linearization of (8.6) is as follows

$$-2(x+y)^2 + (x-y)\lambda + \lambda^2\,.$$

Since $(x_1+y_1)^2 = (x_2+y_2)^2 = \mu_1 - \mu_2 > 0$, we see that for any $\mu_1 > \mu_2$ both (8.7) give rise to hyperbolic/unstable 2π-periodic solutions $z_1(t)$ and $z_2(t)$ of (8.5) for $\varepsilon \neq 0$ located near (8.7), respectively. Moreover $z_2(-t) = -z_1(t)$. Here $z = (x,y) \in \mathbb{R}^2$. If $\mu_1 \leq \mu_2$ and $\mu \neq 0$ then (8.5) has no 2π-periodic solutions for $\varepsilon \neq 0$ in any bounded domains.

Example 2. Now we modify the system (8.5) as follows

$$\begin{aligned} \dot{x} &= \varepsilon\left(-(x+y+\sin t)y + \mu_1\right) \\ \dot{y} &= \varepsilon\left((x+y+\sin t)x + \mu_2\right) \end{aligned} \tag{8.8}$$

with $\mu = (\mu_1,\mu_2) \in \mathbb{R}^2$. We again derive $\mu(\varepsilon) = 0$ and (8.8) has a symmetric and periodic solution only if $\mu = 0$ and it is a zero one. So we study

$$\begin{aligned} \dot{x} &= -\varepsilon\,(x+y+\sin t)\,y \\ \dot{y} &= \varepsilon\,(x+y+\sin t)\,x\,. \end{aligned} \tag{8.9}$$

Clearly any solution of (8.9) satisfies $x^2(t) + y^2(t) = x^2(0) + y^2(0)$. So the symmetric and periodic solution $(x,y) = 0$ of (8.9) is uniformly stable for any $\varepsilon \neq 0$ small, but not asymptotically (cf Remark 1). In order to find general periodic solutions of (8.8), we apply Theorems 1 and 2. So we solve the averaged equation

$$\begin{aligned} -(x+y)y + \mu_1 &= 0 \\ (x+y)x + \mu_2 &= 0 \end{aligned} \tag{8.10}$$

which implies $(x+y)^2 = \mu_1 - \mu_2$. So we need $\mu_1 - \mu_2 > 0$. If $\mu_1 - \mu_2 \le 0$ and $\mu \neq 0$, then (8.10) has no solution. If $\mu_1 - \mu_2 > 0$, then we derive

$$\begin{aligned} x_1 &= -\frac{\mu_2}{\sqrt{\mu_1 - \mu_2}}, \quad y_1 = \frac{\mu_1}{\sqrt{\mu_1 - \mu_2}}, \\ x_2 &= \frac{\mu_2}{\sqrt{\mu_1 - \mu_2}}, \quad y_2 = -\frac{\mu_1}{\sqrt{\mu_1 - \mu_2}}. \end{aligned} \tag{8.11}$$

Next, the characteristic polynomial of the linearization of (8.10) is as follows

$$2(x+y)^2 + (y-x)\lambda + \lambda^2 .$$

Since $(x_1+y_1)^2 = (x_2+y_2)^2 = \mu_1 - \mu_2 > 0$ and $y_1 - x_1 = -(y_2 - x_2) = \frac{\mu_1+\mu_2}{\sqrt{\mu_1-\mu_2}}$, we see that for any $\mu_1 - \mu_2 > 0$ and $\mu_1 + \mu_2 \neq 0$ both (8.11) give rise to hyperbolic 2π-periodic solutions $z_1(t)$ and $z_2(t)$ of (8.5) for $\varepsilon \neq 0$ located near (8.11), respectively. Moreover $z_2(-t) = -z_1(t)$, and $z_1(t)$ is asymptotically stable (a repeller) and $z_2(t)$ is a repeller (asymptotically stable), when $\mu_1 + \mu_2 > (<)\, 0$, respectively, and $\varepsilon > 0$ small; it is opposite for $\varepsilon < 0$. If $\mu_1 - \mu_2 \le 0$ and $\mu \neq 0$ then (8.5) has no 2π-periodic solutions for $\varepsilon \neq 0$ in any bounded domains.

Now we proceed with the second possibility.

8.2 The case $A \neq -\mathbb{I}$

8.2.1 Planar equations with an involution symmetry

Let us consider a planar differential equation

$$\begin{aligned} \dot{x}_1 &= \varepsilon\left(f_1(x_1,x_2) + \mu h_1(t)\right) \\ \dot{x}_2 &= \varepsilon\left(f_2(x_1,x_2) + \mu h_2(t)\right) \end{aligned} \tag{8.12}$$

with C^∞-smooth functions $f_{1,2}$, $h_{1,2}$, $\dim \mu = k = 1$ and with

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} .$$

Note $A^2 = \mathbb{I}$, so A is an involution. Then symmetry condition (1.2) implies

$$\begin{aligned} f_1(x_1,x_2) = f_2(-x_2,-x_1), \quad f_2(x_1,x_2) = f_1(-x_2,-x_1), \\ h_1(t) = h_2(-t-\tau), \quad h_2(t) = h_1(-t-\tau) . \end{aligned} \tag{8.13}$$

Symmetry conditions (8.13) are satisfied, for instance, to the following polynomials

$$\begin{aligned} f_1(x_1,x_2) &= a_0x_1 + b_0x_2 + \sum_{j,p,j+p>1}^{m} \left(a_{jp}x_1^jx_2^p + b_{pj}x_1^px_2^j\right), \\ f_2(x_1,x_2) &= -b_0x_1 - a_0x_2 + \sum_{j,p,j+p>1}^{m} (-1)^{j+p}\left(b_{pj}x_1^jx_2^p + a_{jp}x_1^px_2^j\right), \\ h_1(t) &= \sin t, \quad h_2(t) = -\sin t , \end{aligned} \tag{8.14}$$

and $\tau = 0$. Since in general polynomials (8.14) are difficult to handle, we consider the following particular case

$$\begin{aligned} \dot{x}_1 &= \varepsilon \left(ax_1 - x_2 + x_1^2 x_2 - b x_1 x_2^2 + \mu \sin t \right) , \\ \dot{x}_2 &= \varepsilon \left(x_1 - a x_2 + b x_1^2 x_2 - x_1 x_2^2 - \mu \sin t \right) , \end{aligned} \tag{8.15}$$

where $a, b \in \mathbb{R}$ are parameters. Now

$$\ker (\mathbb{I} - A) = [(1,-1)], \quad \ker (\mathbb{I} + A) = [(1,1)]$$

and hence

$$S(x_1, x_2) = \frac{x_1 + x_2}{2}(1,1), \quad (\mathbb{I} - S)(x_1, x_2) = \frac{x_1 - x_2}{2}(1,-1) .$$

We derive

$$H_1(\eta, \mu) = \pi \eta \left(a + 1 - (b+1)\eta^2 \right) \tag{8.16}$$

identifying $\ker(\mathbb{I} - A) = [(1,-1)] \sim \mathbb{R}$. Applying Theorem 7 we obtain the following result.

Theorem 17. *If $a \neq -1$, then* (8.15) *has a unique symmetric and 2π-periodic solution $z_1(t)$ located near $(0,0)$ for any $\varepsilon \neq 0$ small and $\mu \neq 0$ fixed. If $(a+1)(b+1) > 0$ then* (8.15) *has unique symmetric and 2π-periodic solutions $z_2(t)$, $z_3(t)$ located near $\left(\sqrt{\frac{a+1}{b+1}}, -\sqrt{\frac{a+1}{b+1}} \right)$ and $\left(-\sqrt{\frac{a+1}{b+1}}, \sqrt{\frac{a+1}{b+1}} \right)$, respectively. Here $z = (x_1, x_2) \in \mathbb{R}^2$.*

We intend to find more 2π-periodic solutions of (8.15). For this reason, we solve by Theorem 1 the averaged equation of (8.15) over $[0, 2\pi]$ given by

$$\begin{aligned} a x_1 - x_2 + x_1^2 x_2 - b x_1 x_2^2 &= 0 , \\ x_1 - a x_2 + b x_1^2 x_2 - x_1 x_2^2 &= 0 , \end{aligned} \tag{8.17}$$

which gives

$$\begin{aligned} (x_1 - x_2)(1 + a + (1+b) x_1 x_2) &= 0 , \\ (x_1 + x_2)(1 - a + (b-1) x_1 x_2) &= 0 . \end{aligned} \tag{8.18}$$

For $x_1 \neq \pm x_2$ from (8.18) we derive $ab = 1$. Hence we suppose $ab \neq 1$. Then:

Either $x_1 = -x_2$ and (8.17) implies

$$x_1 \left(a + 1 - (1+b) x_1^2 \right) = 0 .$$

If $a \neq -1$ and $(a+1)(b+1) > 0$, we obtain the following 3 solutions

$$x_{1,1} = x_{2,1} = 0 ; \quad x_{1,2} = -x_{2,2} = -\sqrt{\frac{a+1}{b+1}} ; \quad x_{1,3} = -x_{2,3} = \sqrt{\frac{a+1}{b+1}} , \tag{8.19}$$

which give symmetric and periodic solutions $z_1(t)$, $z_2(t)$ and $z_3(t)$ from Theorem 17, respectively.

Or $x_1 = x_2 \neq 0$ and (8.17) implies

$$x_1 \left(a - 1 + (1 - b)x_1^2 \right) = 0 .$$

If $a \neq 1$ and $(a-1)(b-1) > 0$, we obtain the following 2 solutions

$$x_{1,4} = x_{2,4} = -\sqrt{\frac{a-1}{b-1}}; \quad x_{1,5} = x_{2,5} = \sqrt{\frac{a-1}{b-1}}, \tag{8.20}$$

which give another periodic solutions $z_4(t)$ and $z_5(t)$, respectively, which are not symmetric. But since A is an involution, from the proof of Lemma 1, we see that $z_5(t) = Az_4(-t)$.

Now we study the hyperbolicity of these periodic solutions by applying Theorem 13. So we find eigenvalues of the matrix

$$\begin{pmatrix} a + 2x_1x_2 - bx_2^2 & x_1^2 - 1 - 2bx_1x_2 \\ 1 + 2bx_1x_2 - x_2^2 & bx_1^2 - a - 2x_1x_2 \end{pmatrix}$$

in points (8.19) and (8.20), which are as follows:

$$\pm\sqrt{a^2-1}, \quad \pm 2\sqrt{\frac{(1-ab)(a+1)}{b+1}}, \quad \pm 2\sqrt{\frac{(a-1)(1-ab)}{b-1}} .$$

Summarizing we obtain the following result.

Theorem 18. *For any $\varepsilon \neq 0$ small and $\mu \neq 0$ fixed, the following holds:*

- *$z_1(t)$ is hyperbolic for $|a| > 1$.*
- *$z_2(t)$ and $z_3(t)$ are hyperbolic for $1 > ab$ and $a > -1$, $b > -1$.*
- *$z_4(t)$ and $z_5(t)$ are hyperbolic for $1 > a$ and $1 > b$.*

We note that all periodic solutions $z_1(t), \cdots, z_5(t)$ cannot be simultaneously hyperbolic.

8.2.2 Odd and planar equations with an involution symmetry

Now we consider modified planar ODEs from Section 8.2.1 which is in addition odd of the form

$$\begin{aligned} \dot{x}_1 &= \varepsilon \left(ax_1 - x_2 + x_1^2x_2 - bx_1x_2^2 + \mu x_1 \sin t \right) \\ \dot{x}_2 &= \varepsilon \left(x_1 - ax_2 + bx_1^2x_2 - x_1x_2^2 + \mu x_2 \sin t \right) . \end{aligned} \tag{8.21}$$

So (8.21) satisfies (1.3) and (1.5) with A from Section 8.2.1 and $T = 2\pi$. First we note that symmetric and periodic solutions are derived in the same way as above, so we get these solutions $\widetilde{z}_1(t)$, $\widetilde{z}_2(t)$ and $\widetilde{z}_3(t)$ located near (8.19), if $a \neq -1$ and $(a+1)(b+1) > 0$. Note $\widetilde{z}_1(t) = 0$. Next to find antisymmetric and periodic solutions, we take $\widetilde{S} = \mathbb{I} - S$ and we derive

$$\widetilde{H}_1(\eta, \mu) = \pi\eta \left(a - 1 - (b-1)\eta^2 \right)$$

identifying $\ker(\mathbb{I} + A) = [(1,1)] \sim \mathbb{R}$. Simple roots of $\widetilde{H}_1(\eta, \mu)$ are $0, \pm\sqrt{\frac{a-1}{b-1}}$ provided $(a-1)(b-1) > 0$, which give antisymmetric and periodic solutions $\widetilde{z}_1(t) = 0$, $\widetilde{z}_4(t)$ and $\widetilde{z}_5(t)$ located near (8.20). Note $A\widetilde{z}_4(t) = -\widetilde{z}_4(-t)$ and $A\widetilde{z}_4(t) = \widetilde{z}_5(-t)$, so $\widetilde{z}_4(t) = -\widetilde{z}_5(t)$. To find the possible remaining periodic solutions (non-symmetric and non-antisymmetric ones), we solve by Theorem 1 the averaged equation of (8.21) over $[0, 2\pi]$ given by (8.17). But we know that there are no more solutions of (8.17). Summarizing, we obtain the following result.

Theorem 19. *If $a \neq -1$, then* (8.21) *has a unique symmetric and 2π-periodic solution $\widetilde{z}_1(t) = 0$ located near $(0,0)$ for any $\varepsilon \neq 0$ small. If $(a+1)(b+1) > 0$ then* (8.21) *has unique symmetric and 2π-periodic solutions $\widetilde{z}_2(t)$, $\widetilde{z}_3(t)$ located near $\left(\sqrt{\frac{a+1}{b+1}}, -\sqrt{\frac{a+1}{b+1}}\right)$ and $\left(-\sqrt{\frac{a+1}{b+1}}, \sqrt{\frac{a+1}{b+1}}\right)$, respectively. If $(a-1)(b-1) > 0$ then* (8.21) *has unique antisymmetric and 2π-periodic solutions $\widetilde{z}_4(t)$, $\widetilde{z}_5(t)$ located near $\left(\sqrt{\frac{a-1}{b-1}}, \sqrt{\frac{a-1}{b-1}}\right)$ and $\left(-\sqrt{\frac{a-1}{b-1}}, -, \sqrt{\frac{a-1}{b-1}}\right)$, respectively. There are no more 2π-periodic solutions. The statement of Theorem 18 remains for this case.*

8.2.3 Planar equations with a rotational symmetry

In this section, we consider planar ODEs of the form

$$\begin{aligned} \dot{x}_1 &= \varepsilon f_1(x_1, x_2, \mu, t) \\ \dot{x}_2 &= \varepsilon f_2(x_1, x_2, \mu, t) , \end{aligned} \tag{8.22}$$

where f_1, f_2 are C^∞-smooth and periodic in T and $\mu \in \mathbb{R}$ is a parameter. We suppose that (8.22) is symmetric (cf. (1.5)) with respect to a rotation matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Then it holds

$$f_1(x_1, x_2, \mu, t) = -f_2(-x_2, x_1, \mu, -t), \quad f_2(x_1, x_2, \mu, t) = f_1(-x_2, x_1, \mu, -t) . \tag{8.23}$$

Then (8.23) gives

$$-f_1(x_1, x_2, \mu, t) = f_1(-x_1, -x_2, \mu, t), \quad -f_2(x_1, x_2, \mu, t) = f_2(-x_1, -x_2, \mu, t) .$$

So (8.22) is also odd (cf (1.7)). Clearly $\ker(\mathbb{I} - A^2) = \ker(\mathbb{I} \pm A) = \{0\}$. So the only symmetric and antisymmetric periodic solution of (8.22) is $z_1(t) = 0$ for $\varepsilon \neq 0$ small (cf Theorem 6). To get more results, we pass to the following concrete ODE

$$\begin{aligned} \dot{x}_1 &= \varepsilon \left(x_1 - x_2 + x_1^2 x_2 - b x_1 x_2^2 + \mu x_1 \sin t \right) \\ \dot{x}_2 &= \varepsilon \left(-x_1 - x_2 + b x_1^2 x_2 + x_1 x_2^2 + \mu x_2 \sin t \right) , \end{aligned} \tag{8.24}$$

where $b, \mu \in \mathbb{R}$ are parameters. For finding further periodic solutions, we again consider the averaged equation

$$\begin{aligned} x_1 - x_2 + x_1^2 x_2 - b x_1 x_2^2 &= 0 \\ -x_1 - x_2 + b x_1^2 x_2 + x_1 x_2^2 &= 0\,. \end{aligned} \tag{8.25}$$

If $x_1 = 0$ then $x_2 = 0$, and $x_2 = 0$ then $x_1 = 0$. So we suppose $x_1 \neq 0$ and $x_2 \neq 0$. Then we take $x_2 = \zeta / x_1$ in (8.25) to derive

$$\begin{aligned} \zeta(1+\zeta) - (1+b\zeta)x_2^2 &= 0 \\ \zeta(b\zeta - 1) - (1-\zeta)x_2^2 &= 0\,, \end{aligned}$$

which implies either $\zeta_+ = \sqrt{\frac{2}{b^2+1}}$ and then

$$\begin{aligned} x_1^{+,+} &= \sqrt{\tfrac{2+2b^2+\sqrt{2}(b-1)\sqrt{b^2+1}}{(1+b)(1+b^2)}}, \quad x_2^{+,-} = \sqrt{\tfrac{2+2b^2-\sqrt{2}(b-1)\sqrt{b^2+1}}{(1+b)(1+b^2)}} \\ x_1^{-,+} &= -\sqrt{\tfrac{2+2b^2+\sqrt{2}(b-1)\sqrt{b^2+1}}{(1+b)(1+b^2)}}, \quad x_2^{-,-} = -\sqrt{\tfrac{2+2b^2-\sqrt{2}(b-1)\sqrt{b^2+1}}{(1+b)(1+b^2)}} \end{aligned} \tag{8.26}$$

or $\zeta_- = -\sqrt{\frac{2}{b^2+1}}$ and then

$$\begin{aligned} x_1^{+,-} &= \sqrt{\tfrac{2+2b^2-\sqrt{2}(b-1)\sqrt{b^2+1}}{(1+b)(1+b^2)}}, \quad x_2^{-,+} = -\sqrt{\tfrac{2+2b^2+\sqrt{2}(b-1)\sqrt{b^2+1}}{(1+b)(1+b^2)}} \\ x_1^{-,-} &= -\sqrt{\tfrac{2+2b^2-\sqrt{2}(b-1)\sqrt{b^2+1}}{(1+b)(1+b^2)}}, \quad x_2^{+,+} = \sqrt{\tfrac{2+2b^2+\sqrt{2}(b-1)\sqrt{b^2+1}}{(1+b)(1+b^2)}} \end{aligned} \tag{8.27}$$

Note $2 + 2b^2 \pm \sqrt{2}(b-1)\sqrt{b^2+1} > 0$ for any $b \in \mathbb{R} \setminus \{-1\}$, so $x_{1,2}^{\pm,\pm}$ are defined only for $b > -1$. Moreover, it holds and

$$x_{1,2}^{\pm,-} \to \pm\infty, \quad x_{1,2}^{\pm,+} \to 0$$

as $b \to -1_+$. Now we study the hyperbolicity of these periodic solutions by applying Theorem 13. So we find the characteristic polynomial of the matrix

$$\begin{pmatrix} 1 + 2x_1x_2 - bx_2^2 & x_1^2 - 1 - 2bx_1x_2 \\ 2bx_1x_2 + x_2^2 - 1 & bx_1^2 - 1 + 2x_1x_2 \end{pmatrix} \tag{8.28}$$

at (8.26), which is

$$\lambda^2 - \lambda \frac{2\sqrt{2}(2+b+b^2)}{(1+b)\sqrt{1+b^2}} + 8\,,$$

and at (8.27), which is

$$\lambda^2 + \lambda \frac{2\sqrt{2}(2+b+b^2)}{(1+b)\sqrt{1+b^2}} + 8\,,$$

Then the eigenvalues $\lambda_\pm^+$ of (8.28) at (8.26) satisfy $\Re\lambda_\pm^+ > 0$ and the eigenvalues $\lambda_\pm^-$ of (8.28) at (8.27) satisfy $\Re\lambda_\pm^- < 0$ for any $b > -1$. Next, the eigenvalues of (8.28) at $x_1^0 = x_2^0 = 0$ are $\pm\sqrt{2}$.

Finally, we easily see that (8.25) has the only solution $x_1 = x_2 = 0$ for $b = -1$. Summarizing, we arrive at the following result.

Theorem 20. *If $b \leq -1$ then* (8.24) *has the only 2π-periodic solution $z_0(t) = 0$ for any $\varepsilon \neq 0$ small which is hyperbolic. If $b > -1$ then* (8.24) *has in addition four 2π-periodic solutions $z_i(t)$, $i = 1, 2, 3, 4$ for any $\varepsilon \neq 0$ small which are neither symmetric nor antisymmetric. Moreover, $z_1(t)$ and $z_2(t)$ are asymptotically stable (repellers) while $z_3(t)$ and $z_4(t)$ are repellers (asymptotically stable) for any small $\varepsilon > 0$ ($\varepsilon < 0$), respectively.*

Note $Az_1(t) = z_4(-t)$, $Az_2(t) = z_3(-t)$, $Az_3(t) = z_1(-t)$, $Az_4(t) = z_2(-t)$, and hence $z_2(t) = -z_1(t)$, $z_4(t) = -z_3(t)$. So the set of solutions $\{z_i(t) \mid t \in \mathbb{R},\ i = 1, 2, 3, 4\}$ is invariant by A.

8.2.4 3-dimensional systems

Now we present an example illustrating the case 6.3 given by the following 3-dimensional ODE

$$\begin{aligned}\dot{y} &= \varepsilon\left(y^2 + \left(z_1^2 + z_2^2\right)\cos t + \mu\left(1 + \cos t\right)\right),\\ \dot{z}_1 &= \varepsilon\left(z_1 - z_2 + \left(z_1^3 + z_2^3\right)y^2 + \mu z_1 y \sin t\right),\\ \dot{z}_2 &= \varepsilon\left(-z_1 - z_2 + \left(z_1^3 - z_2^3\right)y^2 - \mu z_2 y \sin t\right)\end{aligned} \tag{8.29}$$

with $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and $A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then clearly (6.5) holds and $1 = k = \dim\ker(\mathbb{I} + A)$. Next we derive $\bar{f}_2(0, 0, \mu) = 2\pi\mu$ and so we take $\mu_0 = 0$, since $\bar{f}_2(0, 0, 0) = 0$ and $D_\mu \bar{f}_2(0, 0, 0) = 2\pi \neq 0$. Moreover $\sigma\left(D_{(y,z)}\bar{f}(0, 0, 0)\right) = \{0, \pm 2\sqrt{2}\pi\}$ (cf (6.10)) and $\dim\ker(\mathbb{I} - A_0^2) = 0$. Consequently, (6.18) has the form

$$\dot{y} = \varepsilon\left(y^2 + \mu\left(1 + \cos t\right)\right), \tag{8.30}$$

and there is a C^∞-function $\mu(\varepsilon)$ defined for ε small with $\mu(0) = 0$ such that for any $\varepsilon \neq 0$ small, (8.29) possesses a (unique) symmetric and 2π-periodic solution $x_{\varepsilon,\mu}(t)$ only for $\mu = \mu(\varepsilon)$ and $x_{0,\mu(0)}(t) = 0$. On the other hand, (8.29) has a solution $x(t) = 0$ for $\mu = 0$, so the uniqueness implies $x_{\varepsilon,\mu(\varepsilon)}(t) = x_0(t) = 0$ and $\mu(\varepsilon) = \mu_0 = 0$ as well. To study other (nonsymmetric) 2π-periodic solutions of (8.29), we solve the averaged equation

$$\begin{aligned} y^2 + \mu &= 0,\\ z_1 - z_2 + \left(z_1^3 + z_2^3\right)y^2 &= 0,\\ -z_1 - z_2 + \left(z_1^3 - z_2^3\right)y^2 &= 0.\end{aligned} \tag{8.31}$$

We see that there are no solutions for $\mu > 0$, while the only zero one for $\mu = 0$ and this corresponds to the trivial one $x_0(t) = 0$. For $\mu < 0$ we get $y_\pm = \pm\sqrt{-\mu}$ and

$$\begin{aligned} z_1 - z_2 - \left(z_1^3 + z_2^3\right)\mu &= 0,\\ -z_1 - z_2 - \left(z_1^3 - z_2^3\right)\mu &= 0,\end{aligned}$$

which implies $z_2 = -\mu z_1^3$, and then $z_1 \left(1 + \mu^4 z_1^8\right) = 0$, which implies $z_1 = z_2 = 0$. Summarizing, for $\mu < 0$, (8.31) has precisely two solutions $x_\pm^\mu = \pm(\sqrt{-\mu}, 0, 0)$ which gives exactly two 2π-periodic solutions $x_\pm^\mu(t) = (y_\pm^\mu(t), 0, 0)$ of (8.29) which are located near $x_\pm^\mu$ for $\varepsilon \neq 0$ small. Note $x_+^\mu(t) = -x_-^\mu(-t)$. So there are saddle-node and symmetry breaking bifurcations of 2π-periodic solutions of (8.29) as μ is crossing 0. Since $\sigma(D_x \bar{f}(x_\pm^\mu, \mu)) = \{\pm 4\pi\sqrt{-\mu}, -2\sqrt{2}\pi, 2\sqrt{2}\pi\}$, periodic solutions $x_\pm^\mu(t)$ are hyperbolic. These results corresponds with arguments at the end of Section 6.3.

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