Value distribution of p-adic meromorphic functions

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Abstract

Let K be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value. Let f be a transcendental meromorphic function in K. We prove that if all zeroes and poles are of order ≥ 2, then f has no Picard exceptional value different from zero. More generally, if all zeroes and poles are of order $\geq k \geq 3$, then $f^{(k-2)}$ has no exceptional value different from zero. Similarly, a result of this kind is obtained for the $k-th$ derivative when the zeroes of f are at least of order m and the poles of order n, such that $mn > m + n + kn$.

If f admits a sequence of zeroes a_n such that the open disk containing a_n, of diameter $|a_n|$ contains no pole, then f and all its derivatives assume each non-zero value infinitely often. Several corollaries apply to the Hayman conjecture in the non-solved cases. Similar results are obtained concerning "unbounded " meromorphic functions inside an "open" disk.

1 Introduction and results

Notation and definitions: Let K be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value $| \, . \, |$. Given $\alpha \in K$ and $R \in \mathbb{R}_+^*$, we denote by $d(\alpha, R)$ the disk $\{x \in K \mid |x - \alpha| \leq R\}$ and by $d(\alpha, R^-)$ the disk $\{x \in K \mid |x - \alpha| < R\}$, by $\mathcal{A}(K)$ the K-algebra of analytic functions in K (i.e. the set of power series with an infinite radius of convergence) and by $\mathcal{M}(K)$ the field of meromorphic functions in K.

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In the same way, given $\alpha \in K$, $r > 0$ we denote by $\mathcal{A}(d(\alpha, r^-))$ the K-algebra of analytic functions in $d(\alpha, r^-)$ (i.e. the set of power series with a radius of convergence $\geq r$) and by $\mathcal{M}(d(\alpha, r^-))$ the field of fractions of $\mathcal{A}(d(\alpha, r^-))$. We then denote by $\mathcal{A}_b(d(\alpha, r^-))$ the K-algebra of bounded analytic functions in $d(\alpha, r^-)$ and by $\mathcal{M}_b(d(\alpha, r^-))$ the field of fractions of $\mathcal{A}_b(d(\alpha, r^-))$. And we set $\mathcal{A}_u(d(\alpha, r^-)) = \mathcal{A}(d(\alpha, r^-)) \setminus \mathcal{A}_b(d(\alpha, r^-))$ and $\mathcal{M}_u(d(\alpha, r^-)) = \mathcal{M}(d(\alpha, r^-)) \setminus \mathcal{M}_b(d(\alpha, r^-))$. As in complex functions, a meromorphic function is said to be *transcendental* if it is not a rational function.

Recall that we call *exceptional value* or *Picard value* for a meromorphic function f (in K or in a disk $d(a, R^-)$) a value $b \in K$ such that $f - b$ has no zero. Similarly, we call *quasi-exceptional value* for a transcendental meromorphic function f in K or a function $f \in \mathcal{M}_u(d(a, R^-))$ a value $b \in K$ such that $f - b$ has finitely many zeros.

Notation: Let $f \in \mathcal{M}(d(0, R^-))$. For every $r \in]0, R[$, we know that $|f(x)|$ admits a limit when $|x|$ approaches r while keeping different from r. This limit is denoted by $|f|(r)$. Particularly, if $f \in \mathcal{A}(d(0, R^-))$, then $f(x)$ is of the form $\sum_{n=0}^{\infty} a_n x^n$ and then $|f|(r) = \sup_{n \in \mathbb{N}} |a_n| r^n$ [4], [5].
Given $f \in \mathcal{M}(K)$, a value $b \in K$ is called *a special value* for f if $\lim_{r \to +\infty} |f - b|(r) = 0$. Similarly, consider $f \in \mathcal{M}(d(a, R^-))$ and let $g(x) = f(a + x)$. A value $b \in K$ is called *a special value* for f if $\lim_{r \to R} |g - b|(r) = 0$.

Many previous studies were made on Picard's exceptional values for complex and p-adic functions and their derivatives and particularly on various questions related to the famous Hayman Conjecture [1], [6], [7], [9], [11].

Here we mean to study whether the derivatives of a meromorphic function may admit a quasi-exceptional value. Certain study was made on the same topic concerning complex functions in [1], [10]. But the tools used in that study, such as properties of normal families, have no analogue on a p-adic field. Here we shall use other methods, particularly the non-Archimedean Nevanlinna Theory.

Let us now recall the Hayman conjecture. Given a transcendental meromorphic function in $\mathbb{C}$ and $b \in \mathbb{C}^*$, as conjectured by Hayman, we know that $f' + bf^m$ has infinitely many zeroes that are not zeroes of f for every $m \geq 3$, while counterexamples exist for $m = 1, 2$. Now, on a field such as K, we know that given $f \in \mathcal{M}(K)$, transcendental, or $f \in \mathcal{M}_u(d(\alpha, r^-))$, $f' + bf^m$ has infinitely many zeroes that are not zeroes of f for $m = 1$ and every $m \geq 5$. And this is also true for $m = 3, 4$ when K has residue characteristic 0 [9]. But if the residue characteristic of K is different from 0, it is not known whether or not certain particular meromorphic functions might violate the Hayman conjecture. In [2], the first author proposes other hypotheses on a transcendental meromorphic function f to assure that $f' + f^3$ or $f' + f^4$ has infinitely many zeroes that are not zeroes of f.

On the other hand, the problem of exceptional values for a transcendental meromorphic function that is the derivative of another one is an old problem. In a joint paper with A. Escassut [3], we proved that if a transcendental meromorphic function f in K has finitely many multiple poles, then f' has infinitely many zeroes. Here, on the contrary, we will consider functions having multiple zeroes and poles.

Theorem 1: *Let $f \in \mathcal{M}(K)$ be transcendental and be such that each zero is at least of order $k \geq 2$ except finitely many m and each pole is at least of order k, except finitely many w. Suppose that f admits at least s zeroes and t poles of order at least $k+1$ and, when $k > 2$, $f^{(k-2)}$ admits at least u multiple zeroes that are not zeroes of f ($s,\ t,\ u \in \mathbb{N}$). Then for each $b \in K^*$, $f^{(k-2)} - b$ has a number of distinct zeroes $q \geq 2 - \frac{2}{k} + u + \frac{t+s(k-1)}{k(k+1)} - \frac{(w+m(k-1))(k-1)}{k}$.*

Corollary 1.1: *Let $f \in \mathcal{M}(K)$ be transcendental and be such that each zero and each pole is at least of order $k \geq 2$. Then $f^{(k-2)}$ has no exceptional value different from 0.*

Corollary 1.2: *Let $f \in \mathcal{M}(K)$ be transcendental and be such that each zero is at least of order $k \geq 2$ and each pole is at least of order $k \geq 2$ except finitely many for both. If f also satisfies one of the following three conditions, then $f^{(k-2)}$ has no quasi-exceptional value different from 0.*

1) f admits infinitely many zeroes of order $\geq k+1$
2) f admits infinitely many poles of order $\geq k+1$,
3) $f^{(k-2)}$ admits infinitely many multiple zeroes that are not zeroes of f.

Corollary 1.3: *Let $f \in \mathcal{M}(K)$ be transcendental. Then $f'f^2$ has no exceptional value different from 0. Further, if f has infinitely many zeroes or poles of order ≥ 2, then $f'f^2$ has no quasi-exceptional value different from 0.*

Proof: We check that f^3 satisfies the hypothesis of Theorem 1.

Corollary 1.4: *Let $f \in \mathcal{M}(K)$ be transcendental and have infinitely many zeroes or poles of order ≥ 2 or be such that f' admits infinitely many zeroes of order ≥ 2. Then for every $b \in K^*$, $f' - bf^4$ has infinitely many zeroes that are not zeroes of f.*

Theorem 2: *Let $f \in \mathcal{M}_u(d(0, R^-))$ be such that each zero is at least of order $k \geq 2$ and each pole is at least of order k except finitely many, satisfying further at least one of the following three conditions:*

1) f admits a sequence of zeroes (a_n) of order $s_n \geq k+1$ such that
$\lim_{n\to\infty} |a_n| = R$, $\displaystyle\prod_{n=0}^{\infty} \left(\frac{|a_n|}{R}\right)^{s_n} = 0$,

2) f admits a sequence of poles (b_n) of order $t_n \geq k+1$ such that
$\lim_{n\to\infty} |b_n| = R$, $\displaystyle\prod_{n=0}^{\infty} \left(\frac{|b_n|}{R}\right)^{t_n} = 0$,

3) $f^{(k-2)}$ admits a sequence of zeroes (c_n) of order $u_n \geq 2$ that are not zeroes of f such that
$\lim_{n\to\infty} |c_n| = R$, $\displaystyle\prod_{n=0}^{\infty} \left(\frac{|c_n|}{R}\right)^{u_n} = 0$,

Then $f^{(k-2)}$ has no quasi-exceptional value different from 0 .

Corollary 2.1: *Let $f \in \mathcal{M}_u(d(0,R^-)))$ have infinitely many zeroes or poles (a_n) of order $q_n \geq 2$ such that $\prod_{n=0}^{\infty} \left(\frac{|a_n|}{R}\right)^{q_n} = 0$. Then $f'f^2$ has no quasi-exceptional value different from 0 and for every $b \in K^*$, $f' - bf^4$ has infinitely many zeroes that are not zeroes of f.*

Corollary 2.2: *Let $f \in \mathcal{M}_u(d(0,R^-)))$ be such that f' has infinitely many zeroes (a_n) of order $q_n \geq 2$ such that $\prod_{n=0}^{\infty} \left(\frac{|a_n|}{R}\right)^{q_n} = 0$. Then $f'f^2$ has no quasi-exceptional value and for every $b \in K^*$, $f' - bf^4$ has infinitely many zeroes that are not zeroes of f.*

Theorem 3: *Let $f \in \mathcal{M}(K)$ be transcendental (resp. $f \in \mathcal{M}_u(d(0,R^-)))$ and be such that each zero is at least of order $m \geq 3$, except finitely many and each pole is at least of order n except finitely many and let $k \in \mathbf{N}^*$ satisfy $mn > m + n + nk$. Then $f^{(k)}$ has no quasi-exceptional value different from 0.*

Application: *Let $f \in \mathcal{M}(K)$ be transcendental (resp. $f \in \mathcal{M}_u(d(0,R^-)))$ be such that each zero is at least of order $m \geq 5$ and each pole is at least of order 2 except finitely many. Then both f, f' have no quasi-exceptional value different from 0. Moreover, if each pole of f is at least of order 3, then f'' has no quasi-exceptional value different from 0 either.*

Theorem 4: *Let $f \in \mathcal{M}(K)$ be transcendental (resp. $f \in \mathcal{M}(d(a,R^-)))$ admitting a special value $c \neq 0$. There exists $S > 0$ (resp. $S \in]0,R[$) such that for each $b \in K^* \setminus d(0,S)$ (resp. $b \in d(0,R^-) \setminus d(a,S)$), the number of zeroes of f is equal to its number of poles in $d(b,|b|^-)$.*

Corollary 4.1: *Let $f \in \mathcal{M}(K)$ be transcendental (resp. $f \in \mathcal{M}(d(a,R^-)))$, having an infinite sequence $(a_m)_{m\in\mathbf{N}}$ such that for all $m \in \mathbf{N}$, $d(a_m,|a_m|^-)$ does not contain any pole of f. Then f has no special value different from 0.*

Corollary 4.2: *Let $f \in \mathcal{M}(K)$ be transcendental (resp. $f \in \mathcal{M}(d(a,R^-)))$, having an infinite sequence $(b_m)_{m\in\mathbf{N}}$ of poles such that for all $m \in \mathbf{N}$, $d(b_m,|b_m|^-)$ does not contain any zero of f. Then f has no special value different from 0.*

Corollary 4.3: *Let $f \in \mathcal{M}(K)$ be transcendental (resp. $f \in \mathcal{M}(d(a,R^-)))$, having an infinite sequence of zeroes $(a_m)_{m\in\mathbf{N}}$ such that for all $m \in \mathbf{N}$, $d(a_m,|a_m|^-)$ does not contain any pole of f. Then for all n, $k \in \mathbf{N}^*$, $k < n$, $(f^n)^{(k)}$ assumes each value $c \in K^*$ infinitely often.*

Corollary 4.4: *Let $f \in \mathcal{M}(K)$ be transcendental (resp. $f \in \mathcal{M}(d(a,R^-)))$, having an infinite sequence of zeroes $(a_m)_{m\in\mathbf{N}}$ such that for all $m \in \mathbf{N}$, $d(a_m,|a_m|^-)$ does not contain any pole of f. Then for all $n \in \mathbf{N}^*$, $f'f^n$ assumes each value $c \in K^*$ infinitely often.*

Remark: Since the Hayman conjecture concerning $f'f^n$ is solved for $n \geq 3$ [9], Corollary 4.4 actually only applies to the cases $n = 1$ and $n = 2$.

2 The Proofs

Lemmas 1 is well known [4], [5], [8]:

Lemma 1: *Let* $f \in \mathcal{M}(d(0,R^-))$. *Then* $|f^{(k)}|(r) \le \dfrac{|f|(r)}{r^k}$ $\forall r < R$, $\forall k \in \mathbf{N}$.

We shall use the following classical lemma 2 (Corollary 1.7.6 [5])

Lemma 2: *Let* $\widehat{K}$ *be an algebraically closed complete extension of* K *and let* $f \in \mathcal{M}(d(a,R^-))$. *Each zero of* f *in the disk* $\{x \in \widehat{K} \mid |x-a| < R\}$ *is a zero of* f *in* $d(a,R^-)$, *with the same order of multiplicity.*

Let us recall the classical notation of the Nevanlinna Theory:

Notation: Let $f \in \mathcal{M}(d(0,R^-))$ be such that 0 is neither a zero nor a pole of f. Let $(a_n)_{n\in\mathbf{N}}$ be the sequence of zeroes of f with $0 < |a_n| \le |a_{n+1}|$ and let k_n denote the order of the zero a_n. Then we define the counting function of zeroes of f, counting multiplicity as $Z(r,f) = \sum_{|a_n|\le r} k_n(\log r - \log|a_n|)$.

Respectively, let the counting function of zeroes, ignoring multiplicities, be defined as $\overline{Z}(r,f) = \sum_{|a_n|\le r}(\log r - \log|a_n|)$.

Similarly, let $(b_n)_{n\in\mathbf{N}}$ be the sequence of poles of f with $0 < |b_n| \le |b_{n+1}|$ and let q_n be the order of the pole b_n. We denote by $N(r,f)$ the counting function of the poles of f, counting multiplicity $N(r,f) = \sum_{|b_n|\le r} q_n(\log r - \log|b_n|)$.

And we denote by $\overline{N}(r,f)$ the counting function of poles ignoring multiplicities be defined as $\overline{N}(r,f) = \sum_{|b_n|\le r}(\log r - \log|b_n|)$.

Finally, we define the characteristic function $T(r,f)$ as $T(r,f) = \max(Z(r,f) + \log|f(0)|, N(r,f))$.

Lemma 3 comes from classical properties of meromorphic functions [5].

Lemma 3: *Let* $f \in \mathcal{M}(K)$, *(resp.* $f \in \mathcal{M}(d(0,R^-))$*). Then, fixing* $r_0 \in]0,+\infty[$, *(resp.* $r_0 \in]0,R[$*), we have* $\log(|f|(r)) = Z(r,f) - N(r,f) + O(1)$, $\forall r \in]r_0,+\infty[$ *(resp.* $\forall r \in]r_0,R[$*).*

As a corollary of Lemma 1, we have Lemma 4:

Lemma 4: *Let* $f \in \mathcal{M}(K)$, *(resp.* $f \in \mathcal{M}(d(0,R^-))$*) be such that* $f^{(k)}$ *has a quasi-exceptional value* $b \in K^*$. *Then* $Z(r,f) \ge N(r,f) + k\log r + O(1)$.

We will also need Lemma 5 that is classical.

Lemma 5: *Let* $f \in \mathcal{M}(d(0,r^-))$. *If* f *has no zero and no pole in a disk* $d(b,|b|^-)$, *then* $|f(b)| = |f|(|b|)$.

The following Lemma 6 [5] is useful to derive Corollary 4.1 from Theorem 4.

Lemma 6: *Let* $f \in \mathcal{M}(K)$ *be transcendental (resp.* $f \in \mathcal{M}_u(d(a,R^-))$ *). Then* f *admits at most one special value* b. *If* b *is a quasi-exceptional value of* f, *it is a special value. If* f *admits a special value* $b \in K$, *then* f' *admits* 0 *as a special value.*

Proof of Theorem 1: Suppose that f has at least s zeroes and t poles of order $\geq k+1$ and that, if $k>2$, $f^{(k-2)}$ has u zeroes of order ≥ 2 that are not zeroes of f.

Suppose that $f^{(k-2)}$ has a quasi-exceptional value $b \neq 0$. Then $f^{(k-2)}$ is of the form $b + \dfrac{P(x)}{g(x)}$ with $g \in \mathcal{A}(K) \setminus K[x]$ and therefore, there exists R such that $|f^{(k-2)}|(r) = |b|\ \forall r > R$. Particularly, by Lemma 3, $f^{(k-2)}$ admits as many zeroes as many poles in any disk $d(0,r)$ whenever $r \geq R$.

Consequently, by Lemma 4 we have

(1) $Z(r,f) \geq N(r,f) + (k-2)\log r + O(1)$ when $r < R$.

Now, applying the p-adic Nevanlinna Second Main Theorem we have:

$$(2)\quad T(r,f^{(k-2)}) \leq \overline{Z}(r,f^{(k-2)}) + \overline{Z}(r,f^{(k-2)}-b) + \overline{N}(r,f^{(k-2)}) - \log r + O(1).$$

But since b is a quasi-exceptional value of $f^{(k-2)}$, the number q of distinct zeroes of $f^{(k-2)}-b$ is $\leq \deg(P)$ and we have $\overline{Z}(r,f^{(k-2)}-b) \leq q\log r + O(1)$. Consequently, (2) yields:

$Z(r,f^{(k-2)}) \leq \overline{Z}(r,f^{(k-2)}) + \overline{N}(r,f) + (q-1)\log r + O(1)$ i.e.

$$(3)\qquad (Z(r,f^{(k-2)}) - \overline{Z}(r,f^{(k-2)})) \leq \overline{N}(r,f) + (q-1)\log r + O(1)$$

Notice that due to the hypothesis, we have

$$(4)\qquad \overline{Z}(r,f) \leq \frac{Z(r,f)}{k} + \Big(\frac{(k-1)m}{k} - \frac{s}{k(k+1)}\Big)\log r$$

and

$$(5)\qquad \overline{N}(r,f) \leq \frac{N(r,f)}{k} + \Big(\frac{(k-1)w}{k} - \frac{t}{k(k+1)}\Big)\log r.$$

Now, by hypothesis, each zero of f is a zero of order at least k, hence is a zero of $f^{(k-2)}$ of order ≥ 2, except m of them, hence we have

$$Z(r,f^{(k-2)}) - \overline{Z}(r,f^{(k-2)}) \geq Z(r,f) - (k-1)\overline{Z}(r,f) + u\log r + O(1)$$

and then (3) yields

$$Z(r,f) - (k-1)\overline{Z}(r,f) \leq \overline{N}(r,f) + (q-u-1)\log r + O(1)$$

hence by (4) and (5)

$$Z(r,f) \leq (\frac{k-1}{k})Z(r,f) + (k-1)[\frac{(k-1)m}{k} - \frac{s}{k(k+1)}]\log r$$

$$+\frac{N(r,f)}{k}+[\frac{(k-1)w}{k}-\frac{t}{k(k+1)}]\log r+(q-u-1)\log r+O(1)$$

Now by (1), the last inequality yields

$$Z(r,f)\leq\frac{(k-1)}{k}Z(r,f)+(k-1)[\frac{(k-1)m}{k}-\frac{s}{k(k+1)}]\log r$$

$$+\frac{Z(r,f)}{k}-\frac{k-2}{k}\log r+[\frac{(k-1)w}{k}-\frac{t}{k(k+1)}+(q-u-1)]\log r+O(1)$$

hence

$$0\leq[\frac{(k-1)\big((k-1)m+w\big)}{k}-\frac{s(k-1)}{k(k+1)}-\frac{t}{k(k+1)}-\frac{k-2}{k}+(q-u-1)]\log r+O(1)$$

Consequently,

$$q\geq 2-\frac{2}{k}+u+\frac{t+s(k-1)}{k(k+1)}-\frac{(w+m(k-1))(k-1)}{k}$$

which ends the proof.

The following Lemma 7 will be useful in the proof of Theorem 2 and is an immediate consequence of Corollary 1.7.17 [5].

Lemma 7: *Let $f\in\mathcal{A}(d(0,R^-))$ and let $(a_n)_{n\in\mathbb{N}}$ be the sequence of zeroes of f, with respective multiplicity q_n. Then f belongs to $\mathcal{A}_u(d(0,R^-))$ if and only if* $\prod_{n=0}^{\infty}\Big(\frac{|a_n|}{R}\Big)^{q_n}=0.$

Proof of Theorem 2: The proof is similar to this of Theorem 1, with some changes. Thanks to Lemma 2, without loss of generality, we can assume that K is spherically complete. Then we can write $f=\frac{h}{l}$ with $h,\ l\in\mathcal{A}(d(0,R^-))$, having no common zeroes. We shall first show that $f^{(k-2)}$ belongs to $\mathcal{M}_u(d(0,R^-))$.

Suppose that Hypothesis 2) of Theorem 2 is satisfied. Then by Lemma 7, l belongs to $\mathcal{A}_u(d(0,R^-))$ and hence the denominator of $f^{(k-2)}$, in a reduced form, also belongs to $\mathcal{A}_u(d(0,R^-))$ because l divides it in $\mathcal{A}(d(0,R^-))$. Hence $f^{(k-2)}$ belongs to $\mathcal{M}_u(d(0,R^-))$.

If Hypothesis 3) of Theorem 2 is satisfied, by Lemma 7, it is obvious that the numerator of $f^{(k-2)}$, in a reduced form, is unbounded, hence $f^{(k-2)}$ belongs to $\mathcal{M}_u(d(0,R^-))$.

Now, suppose that Hypothesis 1) of Theorem 2 is satisfied. Since $s_n\geq k+1$, we can check that $s_n-k+2\geq\frac{s_n}{k}$ and therefore by Hypothesis 1), we have

$$\prod_{n=0}^{\infty}\Big(\frac{|a_n|}{R}\Big)^{s_n-(k-2)}\leq\prod_{n=0}^{\infty}\Big(\frac{|a_n|}{R}\Big)^{\frac{s_n}{k}}=0.$$

Consequently, by Lemma 7 $f^{(k-2)}$ belongs to $\mathcal{M}_u(d(0,R^-))$.

Suppose $f^{(k-2)}$ has a quasi-exceptional value $b \neq 0$, hence $f^{(k-2)}$ is of the form $b + \dfrac{P(x)}{g(x)}$ and therefore, there exists $S \in]0,R[$ such that $|f^{(k-2)}|(r) = |b|\ \forall r \in]S,R[$. Particularly, $f^{(k-2)}$ admits as many zeroes as many poles in any disk $d(0,r)$ whenever $r \in]S,R[$.

Consequently, $Z(f^{(k-2)},r) = N(f^{(k-2)},r) + O(1)$ when $r \in]S,R[$. On the other hand, by Lemma 1, we have $|f|(r) \geq |f^{(k-2)}|(r)r^{k-2}$, hence finally, by Lemma 3, we have

$$Z(r,f)) \geq N(r,f) + O(1). \tag{1}$$

$T(f^{(k-2)},r) \geq N(f^{(k-2)},r) = N(r,f) + (k-2)\overline{N}(r,f) + O(1)$. Now, applying the p-adic Nevanlinna Main Theorem we have:

$$T(f^{(k-2)},r) \leq \overline{Z}(r,f^{(k-2)}) + \overline{Z}(r,f^{(k-2)} - b) + \overline{N}(r,f^{(k-2)}) + O(1) \tag{2}$$

Now, since b is a quasi-exceptional value and since $\overline{N}(r,f) = \overline{N}(r,f^{(k-2)})$, (2) yields:

$Z(r,f^{(k-2)}) \leq \overline{Z}(r,f^{(k-2)}) + \overline{N}(r,f) + O(1)$ hence

$$Z(r,f^{(k-2)}) - \overline{Z}(r,f^{(k-2)}) \leq \overline{N}(r,f) + O(1) \tag{3}$$

Now, by hypothesis, each zero of f is a zero of order at least k, except finitely many. So, each zero of f of order $\geq k$ is a zero of $f^{(k-2)}$ of order ≥ 2, hence $Z(r,f^{(k-2)}) - \overline{Z}(r,f^{(k-2)}) \geq Z(r,f) - (k-1)\overline{Z}(r,f) + O(1)$, therefore (3) yields

$$Z(r,f) \leq (k-1)\overline{Z}(r,f) + \overline{N}(r,f) + O(1) \tag{4}$$

hence

$$Z(r,f) \leq (k-1)\overline{Z}(r,f) + \frac{N(r,f)}{k} + O(1) \tag{5}$$

Suppose now that f has infinitely many zeroes (a_n) of order $\geq k+1$ satisfying Hypothesis 1). Then we can check that

$$\lim_{r \to R} Z(r,f) - k\overline{Z}(r,f) = +\infty \tag{6}$$

Now, by (1) and (5) we have

$$Z(r,f) \leq (k-1)\overline{Z}(r,f) + \frac{Z(r,f)}{k} + O(1),\ r \in]S,R[$$

hence

$$Z(r,f) - k\overline{Z}(r,f) \leq \frac{Z(r,f)}{k} - \overline{Z}(r,f) + O(1),\ r \in]S,R[.$$

Therefore

$$\frac{(k-1)}{k}\big(Z(r,f) - k\overline{Z}(r,f)\big) \leq O(1),\ r \in]S,R[$$

a contradiction to (6).

Suppose now that f has infinitely many poles of order $\geq k+1$ satisfying Hypothesis 2). Then the function $\theta(r) = \dfrac{N(r,f) - k\overline{N}(r,f)}{k}$ satisfies

(7)
$$\lim_{r\to R} \theta(r) = +\infty.$$

Now, by (5) we have

$$Z(r,f) \leq \frac{(k-1)}{k} Z(r,f) + \frac{N(r,f)}{k} - \theta(r) + O(1)$$

and hence by (1) we obtain

$$Z(r,f) \leq \frac{(k-1)}{k} Z(r,f) + \frac{Z(r,f)}{k} - \theta(r) + O(1),$$

but by (7) the contradiction follows.

Finally, suppose that $f^{(k-2)}$ has infinitely many zeroes that are not zeroes of f, satisfying hypothesis 3). We set again
$\psi(r) = \sum_{n=0}^{\infty}(u_n - 1)(\log r - \log(|c_n|))$. Since $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{|c_n|}{R}\Big)^{u_k - 1} = 0$ the function ψ satisfies

(8)
$$\lim_{r\to R} \psi(r) = +\infty$$

and by construction, we can check that

$$Z(r,f^{(k-2)}) - \overline{Z}(r,f^{(k-2)}) \geq Z(r,f) - (k-1)\overline{Z}(r,f) + \psi(r) + O(1)$$

Then by (3) we obtain $Z(r,f) - (k-1)\overline{Z}(r,f) + \psi(r) \leq \overline{N}(r,f) + O(1)$, hence $\dfrac{Z(r,f)}{k} + \psi(r) \leq \overline{N}(r,f) + O(1)$ and hence by (1), we have $\dfrac{Z(r,f)}{k} + \psi(r) \leq \dfrac{Z(r,f)}{k} + O(1)$, a contradiction by (8). This finishes the proof of Theorem 2.

Proof of Corollary 2.2 : By hypothesis, $f'f^2$ does not admit 0 as a quasi-exceptional value. We set $g = \frac{1}{f}$.

The function f' admits a subsequence $(b_m)_{m\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ satisfying $\displaystyle\prod_{m=0}^{\infty}\Big(\frac{|b_m|}{R}\Big)^{\tau_m} = 0$, with $\tau_m \geq 2$ and such that either all b_m are zeroes f or none of the b_m are zeroes of f.

Suppose the first case holds. That means that each b_m is a zero of f of order $\tau_m + 1 \geq 3$. Then f^3 satisfies 1) in Theorem 2 with $k = 3$. Hence $f'f^2$ has no quasi-exceptional value different from 0. Furthermore each b_m is a pole of $g = \frac{1}{f}$ of order ≥ 3. Then by Corollary 2.1, for all $b \in K^*$, $g'g^2 + b$ admits infinitely many zeroes. Let β be a zero of $g'g^2 + b$ i.e. a zero of $-\dfrac{f'}{f^4} + b$. Then $f(\beta) \neq 0, \infty$, hence β is a zero of $f' - bf^4$, that is not a zero of f, which completes the proof.

Suppose now the second case holds. Then f^3 satisfies 3) in Theorem 2 with $k = 3$, hence $f'f^2$ has no quasi-exceptional value different from 0. Now, $(b_m)_{m\in\mathbb{N}}$ is a sequence of zeroes of g' of order ≥ 2 that are not zeroes of g. We can apply Theorem 2, hypothesis 3) to g^3. This proves that $g'g^2$ has no quasi-exceptional value different from zero. Consequently, given $b \neq 0$, $g'g^2 + b$ has infinitely many zeroes. Consequently $f' - bf^4 = -f^4(g'g^2 + b)$ admits infinitely many zeros that are not zeroes of f.

Proof of Theorem 3: Suppose first $f \in \mathcal{M}(K)$ to be transcendental and suppose $b \neq 0$ is a quasi-exceptional value of $f^{(k)}$. Applying the p-adic Nevanlinna Main Theorem, we have $T(r, f^{(k)}) \leq \overline{Z}(r, f^{(k)}) + \overline{Z}(r, f^{(k)} - b) + \overline{N}(r, f^{(k)}) - \log r + O(1)$.

Now, $Z(r, f^{(k)}) \leq T(r, f^{(k)})$, $\overline{N}(r, f^{(k)}) = \overline{N}(r, f)$ and, since b is a quasi-exceptional value of $f^{(k)}$, $\overline{Z}(r, f^{(k)} - b) \leq O(\log r)$. Consequently,

$$Z(r, f^{(k)}) - \overline{Z}(r, f^{(k)}) \leq \overline{N}(r, f) + O(\log r). \tag{1}$$

And now, since each zero of f has order at least $m \geq k$ we have $Z(r, f^{(k)}) \geq Z(r, f) - k\overline{Z}(r, f) + O(\log r)$ hence by (1) we obtain

$$Z(r, f) - (k+1)\overline{Z}(r, f) \leq \overline{N}(r, f) + O(\log r). \tag{2}$$

Now, since each zero of f is of order at least m and since pole is of order at least n exceptly finitely many, we have
$Z(r, f) - (k+1)\overline{Z}(r, f) \geq (\dfrac{m-k-1}{m})Z(r, f) + O(\log r)$ and $\overline{N}(r, f) \leq \dfrac{N(r, f)}{n} + O(\log r)$ hence (2) yields

$$Z(r, f)(\frac{m-k-1}{m}) \leq \frac{N(r, f)}{n} + O(\log r). \tag{3}$$

Now by Lemma 4, we have $Z(r, f) \geq N(r, f) + O(\log r)$, and hence (3) yields

$$Z(r, f)(\frac{m-k-1}{m}) \leq \frac{Z(r, f)}{n} + O(\log r)$$

hence finally $m - k - 1 \leq mn$, a contradiction to the hypothesis.

Suppose now that f belongs to $\mathcal{M}_u(d(0, R^-))$. The proof is the same by replacing each time $O(\log r)$ by $O(1)$.

By properties of analytic elements, we know the following lemmas 9, 10 given in [4] and [5]:

Lemma 9: *Let $f \in \mathcal{M}(d(0, R^-))$, let $a \in d(0, R^-)$ and let $r = |a|$. Then*

$$\lim_{\substack{|x| \to r, \\ |x| \neq r}} |f(x)| = \lim_{\substack{|x-a| \to r, \\ |x-a| \neq r}} |f(x)| = |f|(r).$$

Lemma 10: *Let $f \in \mathcal{M}(d(0, R^-))$ have q zeroes and t poles in $d(a, s)$ and have no zero and no pole in $\Gamma(0, s, r)$ (with $s < r < R$). Then $|f(x)| = |f|(r)\Big(\frac{|x-a|}{r}\Big)^{q-t}$.*

Proof of Theorem 4: Suppose f has a special value $c \neq 0$. Without loss of generality, we may assume $c = 1$ and $a = 0$. By hypothesis, there exists $S > 0$ (resp. $S \in]0, R[$) such that $|f - 1|(r) < 1$ $\forall r \geq S$ (resp. $|f - 1|(r) < 1$ $\forall r \in]S, R[$). Let $b \in K^*$ be such that $|b| > S$ (resp. $b \in d(0, R^-)$ be such that $S < |b| < R$) and set $r = |b|$. By Lemma 9 we have

$$\lim_{\substack{|x| \to r, \\ |x| \neq r}} |f(x) - 1| = \lim_{\substack{|x-b| \to r, \\ |x-b| \neq r}} |f(x) - 1| = |f - 1|(r)$$

hence $\lim_{\substack{|x-b| \to r, \\ |x-b| \neq r}} |f(x) - 1| < 1$. Thus, there exists $s \in]0, r[$ such that $|f(x) - 1| < 1$ $\forall x \in \Gamma(b, s, r)$ and particularly
(1) $|f(x)| = 1$, $\forall x \in \Gamma(b, s, r)$.
Without loss of generality, we can take $s < r$ but big enough to assure that $d(b, s)$ contains all the zeroes and the poles of f inside $d(b, r^-)$. Let q be the number of zeroes of f in $d(b, s)$ and let t be the number of poles of f in $d(b, s)$ taking multiplicity into account. Then by Lemma 10, we have $|f(x)| = |f|(r)(\frac{|x-b|}{r})^{q-t}$. Consequently, by (1) we have $q = t$.

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