An entropy solution for some degenerate or singular obstacle parabolic problems with L^1-data via a sequence of penalized equations

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Abstract

We give an existence result of the obstacle parabolic degenerate or singular problem associated to the equation, $\dfrac{\partial u}{\partial t} + A(u) = f$ in Q_T, where A is a classical Leray-Lions operator acting from the weighted Sobolev space $L^p(0, T, W_0^{1,p}(\Omega, w))$ into its dual $L^{p'}(0, T, W^{-1,p'}(\Omega, w^*))$, while the datum f is assumed to lie in $L^1(Q_T)$. The proof is based on the penalization methods.

1 Introduction

In this paper, we investigate the problem of existence of solutions of the obstacle problems associated to the following initial-boundary value problem:

$$(P_e)\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) = f & \text{in } Q_T = \Omega \times (0,T) \\ u = 0 & \text{on } \Sigma = \partial\Omega \times (0,T) \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, $T > 0$, and we have set Q_T the cylinder $\Omega \times (0,T)$ and Σ its lateral surface.
We assume that $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray-Lions operator defined from the weighted Sobolev space $L^p(0, T, W_0^{1,p}(\Omega, w))$ into its dual $L^{p'}(0, T, W^{-1,p'}(\Omega, w^*))$

Received by the editors September 2009.
Communicated by P. Godin.
2000 *Mathematics Subject Classification* : Primary 47A15; Secondary 46A32, 47D20.
Key words and phrases : Weighted Sobolev spaces, degenerated obstacle, penalized equations.

where $w = \{w_i,\ 0 \le i \le N\}$ is collection of weight functions on Ω, $1 \le p \le \infty$, $w^* = \{w_i^{1-p'},\ 0 \le i \le N\}$, and where $a(x,t,s,\xi) : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathodory function satisfy some suitable hypotheses (see assumption (H_2)). The data $f \in L^1(Q_T)$, $u_0 \in L^1(\Omega)$ and $u_0 \ge 0$.

More precisely, this paper deals with the existence of a solution to the obstacle degenerate or singular parabolic problem associated to (P_e) in the sense of entropy solution:

$$(P_u)\begin{cases} u \ge \psi \ \text{ a.e. in } \ Q_T \\ T_k(u) \in L^p(0,T,W_0^{1,p}(\Omega,w)),\ u \in C([0,T],L^1(\Omega)) \\ \displaystyle\int_\Omega S_k(u-\varphi)(\tau)\,dx + \int_{Q_\tau} \frac{\partial \varphi}{\partial t} T_k(u-\varphi)\,dx\,dt \\ \qquad + \displaystyle\int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-\varphi)\,dx\,dt \\ \le \displaystyle\int_{Q_\tau} f T_k(u-\varphi)\,dx\,dt + \int_\Omega S_k(u_0-\varphi(x,0))\,dx \qquad \forall \tau \in [0,T],\ \forall\, k>0 \\ \varphi \in K_\psi \cap L^\infty(Q_T) \cap C([0,T],L^1(\Omega)) \ \text{ such that } \dfrac{\partial \varphi}{\partial t} \in L^{p'}(W^{-1,p'}(\Omega,w^*)). \end{cases}$$

where $S_k(t) = \displaystyle\int_0^t T_k(s)\,ds$, $K_\psi = \Big\{u \in L^p(0,T,W_0^{1,p}(\Omega,w)),\ u \ge \psi \text{ a.e. in } Q_T\Big\}$. and $\psi \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega,w)$.

The aim of our work is to investigate the relationship between the obstacle problem (P_u) and some penalized sequence of approximate equations. More precisely letting $\{f_\epsilon\}$ and u_0^ϵ be a standard approximation of f and u_0 (that is $f_\epsilon \to f$ in $L^1(Q)$ and $u_0^\epsilon \to u_0$ in $L^1(\Omega)$), and considering the following penalized sequence of approximate equations:

$$(P_e^\epsilon)\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} - \operatorname{div}(a(x,t,u_\epsilon,\nabla u_\epsilon)) - \dfrac{1}{\epsilon} T_\epsilon(u_\epsilon-\psi)^- = f_\epsilon \ \text{ in } \ Q_T \\ u_\epsilon = 0 \qquad\qquad\qquad \text{on } \Sigma \\ u_\epsilon(0) = u_0^\epsilon \qquad\qquad \text{in } \Omega. \end{cases}$$

We study the possibility to find a solution of (P_u) as a limit of a subsequence $\{u_\epsilon\}$ of solutions of (P_e^ϵ).

The penalized term $\frac{1}{\epsilon}T_\epsilon(u_\epsilon-\psi)^-$ introduced in (P_e^ϵ) play a crucial role in the proof of our main result, in particular this term allows to prove that the solution u of (P_u) belongs in the convex set K_ψ (that is $u \ge \psi$).

A priori estimates of $T_k(u_\epsilon)$ are obtained in the general settings $L^p(0,T,W_0^{1,p}(\Omega, w))$. For the passage to the limit, we prove the strong converge of the truncation of u_ϵ and the almost everywhere convergence of ∇u_ϵ. The model example is an equation,

$$\frac{\partial u}{\partial t} - \Delta_{p,w} u = f,$$

where $\Delta_{p,w}$ is the so-called degenerate or singular $p-$Laplacian operator, that is $\Delta_{p,w}(u) = -\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u)$, where $w(x)$ is some weight function defined on Ω (see section 3 for more details).

In this context of degenerate or singular parabolic problems, existence results for

(P_e) have been proved in [2] when the data f belongs to $L^p(0,T,W^{-1,p'}(\Omega))$ and u_0 is in $L^2(\Omega)$, while the strongly nonlinear variational case is investigated in [3]. For the nondegenerated cases, we refer the reader to ([14], [25], [26]).
Let us mention that our paper can be seen as a continuation of the works ([2], [3]) and as a generalization of the works ([14], [25], [26]).

2 Abstract framework

Let Ω be a bounded open subset of $\mathbb{R}^N$, p be a real number such that $1 < p < \infty$ and $w = \{w_i(x),\ 1 \leq i \leq N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all this section that, there exists

$$r_0 > \max(N,p) \text{ such that } w_i^{\frac{r_0}{r_0-p}} \in L^1_{loc}(\Omega) \tag{2.1}$$

and

$$w_i^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega) \tag{2.2}$$

for any $0 \leq i \leq N$.

We denote by $W^{1,p}(\Omega,w)$ the space of all real-valued functions $u \in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,w_i) \text{ for all } i = 1,...,N.$$

Which is a Banach space under the norm,

$$\|u\|_{1,p,w} = \left[\int_\Omega |u(x)|^p w_0\, dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx\right]^{\frac{1}{p}}. \tag{2.3}$$

The condition (2.1) implies that $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, the condition (2.2) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ i = 0,...,N\}$ and where p' is the conjugate of p, i.e., $p' = \frac{p}{p-1}$. For more details about the weighted Sobolev spaces, we refer the reader to [18].
Now we turn out to give some fundamental results which allow to study the parabolic problems in a general settings of weighted Sobolev spaces.

In order to deal with time derivative, we introduce a time mollification of a function u belonging in some weighted Lebesgue space. Thus we define for all $\mu \geq 0$ and all $(x,t) \in Q_T$: $u_\mu = \mu \int_{-\infty}^t \tilde{u}(x,s)\exp(\mu(s-t))\, ds$ where $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}$.

Proposition 2.1. *(cf. [3])*

1) *If* $u \in L^p(Q_T, w_i)$*, then,* u_μ *is measurable in* Q_T*,* $\dfrac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ *and*

$$\left(\int_Q |u_\mu|^p w_i(x)\, dx\, dt\right)^{\frac{1}{p}} \leq \left(\int_Q |u|^p w_i(x)\, dx\, dt\right)^{\frac{1}{p}},$$

i.e., $\|u_\mu\|_{L^p(Q_T,w_i)} \leq \|u\|_{L^p(Q_T,w_i)}$.

2) *If* $u \in W_0^{1,p}(Q_T, w)$*, then* $u_\mu \to u$ *in* $W_0^{1,p}(Q_T, w)$ *as* $\mu \to +\infty$.

3) *If* $u_n \to u$ *in* $W_0^{1,p}(Q_T, w)$*, then* $(u_n)_\mu \to u_\mu$ *in* $W_0^{1,p}(Q_T, w)$.

Now, we give some imbedding and compactness results in weighted Sobolev Spaces which allow in particular to extend in the settings of weighted Sobolev spaces, some trace results and the Aubin's and Simon's results [27].

Let $V = W_0^{1,p}(\Omega, w)$, $H = L^2(\Omega, \sigma)$ (where σ is a weight function on Ω such that $\sigma \in L^1(\Omega)$ and $\sigma^{-1} \in L^1(\Omega)$) and let $V^* = W^{-1,p'}(\Omega, w^*)$, with $(2 \leq p < \infty)$. Let $X = L^p(0, T, V)$. The dual space of X is $X^* = L^{p'}(0, T, V^*)$ where $\frac{1}{p'} + \frac{1}{p} = 1$ and denoting the space $W_p^1(0, T, V, H) = \{v \in X : v' \in X^*\}$ endowed with the norm

$$\|u\|_{w_p^1} = \|u\|_X + \|u'\|_{X^*}, \tag{2.4}$$

which is a Banach space. Here u' stands for the generalized derivative of u, i.e.,

$$\int_0^T u'(t)\varphi(t)\, dt = -\int_0^T u(t)\varphi'(t)\, dt \ \text{ for all } \ \varphi \in C_0^\infty(0, T).$$

Lemma 2.2. *The Banach space* H *is an Hilbert space and its dual* H' *can be identified with him self, i.e.,* $H' \simeq H$.

Proof. Indeed, let

$$\begin{aligned} F : H \times H &\to \mathbb{R} \\ (f, g) &\mapsto \int_\Omega fg\sigma\, dx. \end{aligned}$$

Remark that F is a symmetric bilinear form, which is also continuous and defined positively, since

$$\int_\Omega fg\sigma\, dx = \int_\Omega f\sigma^{\frac{1}{2}} g\sigma^{\frac{1}{2}}\, dx \leq \left(\int_\Omega |f|^2\sigma\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |g|^2\sigma\, dx\right)^{\frac{1}{2}}.$$

Then, the Banach space H is an Hilbert space. Finally by a standard argument, we can identified H with its dual H' i.e., $H' \simeq H$. ∎

Lemma 2.3. *(cf. [3])*

The evolution triple $V \subseteq H \subseteq V^*$ *is verified.*

Lemma 2.4. *(cf. [3])*

Assume that, $\dfrac{\partial u_n}{\partial t} = h_n + k_n$ *in* $D'(\Omega)$, *where* h_n *and* k_n *are bounded respectively in* $L^{p'}(0,T,W^{1,p'}(\Omega,w^*)$ *and in* $L^1(Q_T)$.

If u_n *is bounded in* $L^p(0,T,W_0^{1,p}(\Omega,w))$, *then* $u_n \to u$ *in* $L^p_{loc}(Q_T,\sigma)$.

Lemma 2.5. *(cf. [3])*

Let $g \in L^r(Q_T,\gamma)$ *and let* $g_n \in L^r(Q_T,\gamma)$, *with* $\|g_n\|_{L^r(Q_T,\gamma)} \leq c, 1 < r < \infty$. *If* $g_n(x) \to g(x)$ *a.e in* Q_T, *then* $g_n \rightharpoonup g$ *in* $L^r(Q_T,\gamma)$, *where* $\rightharpoonup$ *denotes weak convergence and* γ *is a weight function on* Q_T.

Lemma 2.6. *(cf. [29]) Let* $V \subseteq H \subseteq V^*$ *be an evolution triple. Then the imbedding*

$$W_p^1(0,T,V,H) \hookrightarrow C([0,T],H)$$

is continuous.

3 Basic assumptions and Main results

We suppose in all our considerations that, there exists

$$r_0 > \max(N,p) \text{ such that } w_i^{\frac{r_0}{r_0-p}} \in L^1_{loc}(\Omega) \text{ and } w_i^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega),$$

for any $0 \leq i \leq N$. Now we state our basic assumptions:
Assumption $(\mathbf{H_1})$**.** For $2 \leq p < \infty$, we suppose that the expression:

$$|||u||| = \left(\sum_{i=1}^{N}\int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\right)^{\frac{1}{p}} \tag{3.1}$$

is a norm on $W_0^{1,p}(\Omega,w)$ which is equivalent to (2.3) and that there exists a weight function σ on Ω such that,

$$\sigma \in L^1(\Omega) \text{ and } \sigma^{-1} \in L^1(\Omega). \tag{3.2}$$

and for which the Hardy inequality,

$$\left(\int_\Omega |u(x)|^p \sigma\,dx\right)^{\frac{1}{p}} \leq c\left(\sum_{i=1}^{N}\int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\right)^{\frac{1}{p}}, \tag{3.3}$$

holds for every $u \in W_0^{1,p}(\Omega,w)$ with a constant $c > 0$ independent of u. Moreover, the imbedding

$$W_0^{1,p}(\Omega,w) \hookrightarrow\hookrightarrow L^p(\Omega,\sigma) \tag{3.4}$$

expressed by the inequality (3.3) is assumed compact.
Now we give some examples in which the abstract hypothesis (H_1) is satisfied:

Remark 3.1. Assume that $w_0(x) \equiv 1$ and there exists $\nu \in \left]\frac{N}{P}, +\infty\right[\cap \left[\frac{1}{P-1}, +\infty\right[$ such that

$$w_i^{\frac{N}{N-1}},\ w_i^{-\nu} \in L^1(\Omega) \ \text{ for all } \ i = 1, ..., N. \tag{3.5}$$

Then, it's easily seen that the assumptions (3.5) imply that,

$$|||u||| = \left(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx \right)^{\frac{1}{p}} \tag{3.6}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and it's equivalent to (2.3) and that, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^p(\Omega) \tag{3.7}$$

is compact [see [18], pp 46].
Thus the hypotheses (H_1) is satisfied for $\sigma \equiv 1$.

Remark 3.2. If we use the special weight functions w and σ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \text{dist}(x, \partial\Omega)$ and set $w(x) = \sigma(x) = d^\lambda(x)$. In this case, the Hardy inequality reads

$$\left(\int_Q |u|^p w(x)\, dx\, dt \right)^{\frac{1}{p}} \leq c \left(\int_Q |\nabla u|^p w(x)\, dx\, dt \right)^{\frac{1}{p}}$$

for

$$\lambda < (p-1)\frac{(N-1)}{N}. \tag{3.8}$$

The condition (3.8) is sufficient for the compact imbedding (3.4) holds (see for example [17],[18]).

Assumption $(\mathbf{H_2})$. Let $a = (a_i)_{1 \leq i \leq N}$ be a family of Carathodory functions defined on $Q \times \mathbb{R} \times \mathbb{R}^N$ such that for a.e. $(x,t) \in Q$ and all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$

$$|a_i(x,t,s,\xi)| \leq \beta w_i^{\frac{1}{p}}(x) \ \ [c_1(x,t) + \sigma^{\frac{1}{p'}} |s|^{p-1} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x) |\xi_j|^{p-1}], \tag{3.9}$$

for $1 \leq i \leq N$

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) > 0 \ \ for \ \ all \ \ \xi \neq \eta \in \mathbb{R}^N \tag{3.10}$$

$$a(x,t,s,\xi).\xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p, \tag{3.11}$$

where $c_1(x,t)$ is a positive function in $L^{p'}(Q)$, and α, β are strictly positive constants.

Assumption (H_3). Let $K_\psi = \{u \in W_0^{1,p}(\Omega, w);\ u \geq \psi \ \ a.e. \text{ in } \ \Omega\}$. where $\psi : \Omega \to \overline{\mathbb{R}}$ is a measurable function on Ω such that

$$K_\psi \cap L^\infty(\Omega) \neq \emptyset. \tag{3.12}$$

We suppose that

$$f \in L^1(\Omega), \tag{3.13}$$

and

$$u_0 \in L^1(\Omega) \text{ and } u_0 \geq 0 \text{ a.e. in } \Omega. \tag{3.14}$$

We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

Now, we recall the following lemmas:

Lemma 3.3. *Assume that (H_1) holds. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^{1,p}(\Omega, w)$. Then $F(u) \in W_0^{1,p}(\Omega, w)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \quad in \quad \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \quad in \quad \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Proof. See [13] (see also [5]). ■

From the previous lemma, we deduce the following.

Lemma 3.4. *Assume that (H_1) holds. Let $u \in W_0^{1,p}(\Omega, w)$, and let $T_k(u)$ be the usual truncation ($k \in \mathbb{R}^+$), then $T_k(u) \in W_0^{1,p}(\Omega, w)$. Moreover, we have*

$$T_k(u) \to u \ \text{ strongly in } \ W_0^{1,p}(\Omega, w).$$

Proof. See [13] (see also [5]). ■

The following lemma generalizes to the weighted case the analogous Lemma 5 in [12]. For that, we use the method of [12] and [23] which gives the strong convergence of u_n.

Lemma 3.5. *Assume that (H_1) and (H_2) are satisfied and let (u_n) be a sequence in $L^p(0, T, W_0^{1,p}(\Omega, w))$ such that $u_n \rightharpoonup u$ weakly in $L^p(0, T, W_0^{1,p}(\Omega, w))$ and*

$$\int_Q [a(x, t, u_n, \nabla u_n) - a(x, t, u_n, \nabla u)][\nabla u_n - \nabla u] \, dxdt \to 0. \tag{3.15}$$

Then, $u_n \to u$ in $L^p(0, T, W_0^{1,p}(\Omega, w))$.

Proof. See [3]. ■

4 Existence result

This section is devoted to establish the existence theorem.

Theorem 4.1. *Let $u_0 \in L^1(\Omega)$ such that $u_0 \geq 0$. Assume that (H_1), (H_2) and (H_3) hold true. Then there exists at last one solution $u \in C([0,T];L^1(\Omega))$ such that $u(x,0) = u_0$ a.e. and for all $\tau \in]0,T]$,*

$$\begin{cases} T_k(u) \in L^p(0,T,W_0^{1,p}(\Omega,w)), u \geq \psi \text{ a.e. in } \Omega, \\ \int_\Omega S_k(u(\tau)-\varphi(\tau))\,dx + \langle \frac{\partial \varphi}{\partial t}, T_k(u-\varphi)\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-\varphi)\,dx\,dt \\ \qquad \leq \int_{Q_\tau} f T_k(u-\varphi)\,dx\,dt + \int_\Omega S_k(u_0-\varphi(x,0))\,dx \\ \forall\, k>0 \text{ and } \forall\, \varphi \in K_\psi \cap L^\infty(Q) \text{ such that } \frac{\partial \varphi}{\partial t} \in L^{p'}(0,T,W^{-1,p'}(\Omega,w^*)) \end{cases}$$

where $Q_\tau = \Omega \times]0,\tau[$.

Proof. The proof is divided into 3 steps.
Step 1: A priori estimates
Consider the approximate problem

$$(P_\epsilon)\begin{cases} \frac{\partial u_\epsilon}{\partial t} - \operatorname{div}(a(x,t,u_\epsilon,\nabla u_\epsilon)) - \frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- = f_\epsilon \\ u_\epsilon \in L^p(0,T,W_0^{1,p}(\Omega,w)),\ u_\epsilon(x,0) = u_0^\epsilon \end{cases}$$

where $f_\epsilon \to f$ strongly $L^1(Q)$, $u_0^\epsilon \to u_0$ strongly $L^1(\Omega)$.
Thanks to [3], there exists at least one solution of the problem (P_ϵ).
By choosing $T_\gamma(u_\epsilon - T_\beta(u_\epsilon))$, $\beta \geq \|\psi\|_\infty$ as test function in (P_ϵ), we get

$$\begin{aligned} &\langle \frac{\partial u_\epsilon}{\partial t}, T_\gamma(u_\epsilon - T_\beta(u_\epsilon))\rangle + \int_{\{\beta\leq|u_\epsilon|\leq\beta+\gamma\}} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon\,dx\,dt \\ &\quad - \int_Q \frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- T_\gamma(u_\epsilon - T_\beta(u_\epsilon))\,dx\,dt = \int_Q f_\epsilon T_\gamma(u_\epsilon - T_\beta(u_\epsilon))\,dx\,dt \end{aligned} \tag{4.1}$$

On the one hand, we have

$$\langle \frac{\partial u_\epsilon}{\partial t}, T_\gamma(u_\epsilon - T_\beta(u_\epsilon))\rangle = \int_\Omega S_\gamma^\beta(u_\epsilon(T))\,dx - \int_\Omega S_\gamma^\beta(u_\epsilon^0)\,dx \tag{4.2}$$

where $S_\gamma^\beta(s) = \int_0^s T_\gamma(t - T_\beta(t))\,dt$. Since a satisfies (3.11) and by using the fact that $\int_\Omega S_\gamma^\beta(u_\epsilon(T))\,dx \geq 0$ and $|\int_\Omega S_\gamma^\beta(u_\epsilon^0)\,dx| \leq \gamma\|u_\epsilon^0\|$, we get $\forall\, \epsilon > 0$:

$$\begin{aligned} &\alpha \int_{\{\beta\leq|u_\epsilon|\leq\beta+\gamma\}} \sum_{i=1}^N \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^p w_i(x)\,dx\,dt \\ &\qquad\qquad - \frac{1}{\epsilon}\int_Q T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- T_\gamma(u_\epsilon - T_\beta(u_\epsilon))\,dx\,dt \leq c\gamma, \end{aligned} \tag{4.3}$$

where c is a constant which varies from line to line and which depends only the data. It follows that

$$-\int_Q \frac{1}{\epsilon} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- \frac{T_\gamma(u_\epsilon - T_\beta(u_\epsilon))}{\gamma}\, dx\, dt \le c$$

since $-\int_Q \frac{1}{\epsilon} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- \frac{T_\gamma(u_\epsilon - T_\beta(u_\epsilon))}{\gamma}\, dx\ dt \ \ge\ 0$, for every $\beta \ge \|\psi\|_\infty$, we deduce by Fatou's lemma as $\gamma \to 0$ that

$$\int_Q \frac{1}{\epsilon} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- \le c. \tag{4.4}$$

Using in (P_ϵ) the test function $T_\epsilon(u_\epsilon)\chi_{(0,\tau)}$, we get for every $\tau \in (0, T)$,

$$\int_\Omega S_k(u_\epsilon(\tau))\, dx + \int_{Q_\tau} a(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \nabla T_k(u_\epsilon)\, dx\, dt$$
$$-\frac{1}{\epsilon} \int_Q T_{\frac{1}{\epsilon}}((u_\epsilon - \psi)^-) T_k(u_\epsilon)\, dx\, dt \le ck,$$

which gives thanks to (4.4):

$$\int_\Omega S_k(u_\epsilon(\tau))\, dx + \int_{Q_\tau} a(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon))\, dx\, dt \le ck. \tag{4.5}$$

Then,

$$\alpha \int_Q \sum_{i=1}^N \left| \frac{\partial T_k(u_\epsilon)}{\partial x_i} \right|^p w_i(x)\, dx\, dt \le ck, \ \ \forall\, k \ge 1. \tag{4.6}$$

Hence, $T_k(u_\epsilon)$ is bounded in $L^p(0, T, W_0^{1,p}(\Omega, w))$.
Let $k > 0$ large enough and B_R be a ball of Ω, we have,

$$k \operatorname{meas}(\{|u_\epsilon| > k\} \cap B_R \times [0, T]) =$$
$$\int_0^T \int_{\{|u_\epsilon| > k\} \cap B_R} |T_k(u_\epsilon)|\, dx\, dt \le \int_0^T \int_{B_R} |T_k(u_\epsilon)|\, dx\, dt$$
$$\le \left(\int_Q |T_k(u_\epsilon)|^p w_0\, dx\, dt \right)^{\frac{1}{p}} \times \left(\int_0^T \int_{B_R} w_0^{1-p'}\, dx\, dt \right)^{\frac{1}{p'}}$$

then, thanks to (H_1), we deduce that,

$$k \operatorname{meas}(\{|u_\epsilon| > k\} \cap B_R \times [0, T]) \le c \left(\int_Q \sum_{i=1}^N \left| \frac{\partial T_k(u_\epsilon)}{\partial x_i} \right|^p w_i(x)\, dx\, dt \right)^{\frac{1}{p}} \le c\, k^{\frac{1}{p}} \tag{4.7}$$

which implies that, $\operatorname{meas}(\{|u_\epsilon| > k\} \cap B_R \times [0, T]) \le \dfrac{c_1}{k^{1-\frac{1}{p}}}$, $\forall\, k \ge 1$. So, we have,

$$\lim_{k \to +\infty} (\operatorname{meas}(\{(x, t) \in Q : \ |u_\epsilon| > k\} \cap B_R \times [0, T]) = 0 \tag{4.8}$$

uniformly with respect to ϵ.
Consider now a function nondecreasing $\xi_k \in C^2(\mathbb{R})$ such that

$$\begin{cases} \xi_k(s) = s \text{ for } |s| \leq \frac{k}{2} \\ \xi_k(s) = k \text{ for } |s| \geq k. \end{cases}$$

Multiplying the approximate equation by $\xi_k'(u_\epsilon)$, we get

$$\frac{\partial}{\partial t}(\xi_k(u_\epsilon)) - \operatorname{div}(a(x,t,u_\epsilon,\nabla u_\epsilon)\xi_k'(u_\epsilon)) + a(x,t,u_\epsilon,\nabla u_\epsilon)\xi_k''(u_\epsilon)$$
$$-\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}((u_\epsilon - \psi)^-)\xi_k'(u_\epsilon) = f_\epsilon \xi_k'(u_\epsilon),$$

in the sense of distribution.
This implies, thanks to (4.6) and the fact that ξ_k' has compact support, that $\xi_k(u_\epsilon)$ is bounded in $L^p(0,T,W_0^{1,p}(\Omega,w))$, while it's time derivative $\frac{\partial}{\partial t}(\xi_k(u_\epsilon))$ is bounded in $L^{p'}(0,T,W^{-1,p'}(\Omega,w^*)) + L^1(Q_T)$, hence lemma 3.3 allows us to conclude that $\xi_k(u_\epsilon)$ is compact in $L^p_{loc}(Q_T,\sigma)$.
Thus, for a subsequence, it also converges in measure and almost every where in Q_T since we have, for every $\lambda > 0$

$$\begin{aligned} &\operatorname{meas}(\{|u_\epsilon - u_\eta| > \lambda\} \cap B_R \times [0,T]) \leq \operatorname{meas}(\{|u_\epsilon| > \tfrac{k}{2}\} \cap B_R \times [0,T]) \\ &+ \operatorname{meas}(\{|u_\eta| > \frac{k}{2}\} \cap B_R \times [0,T]) + \operatorname{meas}(\{|\xi_k(u_\epsilon) - \xi_k(u_\eta)| > \lambda\} \cap B_R \times [0,T]). \end{aligned} \tag{4.9}$$

Let $\sigma > 0$, then, by (4.8) and the fact that $\xi_k(u_\epsilon)$ is compact in $L^p_{loc}(Q_T,\sigma)$, there exists $k(\sigma) > 0$ such that, $\operatorname{meas}(\{|u_\epsilon - u_\eta| > \lambda\} \cap B_R \times [0,T]) \leq \sigma$ for all $\epsilon, \eta \leq \epsilon_0(k(\sigma),\lambda,R)$. This proves that (u_ϵ) is a Cauchy sequence in measure in $B_R \times [0,T]$, thus converges almost everywhere to some measurable function u. Then for a subsequence denoted again u_ϵ, we can deduce from (4.6) that,

$$T_k(u_\epsilon) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T,W_0^{1,p}(\Omega,w)). \tag{4.10}$$

and then, the compact imbedding (3.4) gives,

$$T_k(u_\epsilon) \to T_k(u) \text{ strongly in } L^p(Q_T,\sigma) \text{ and a.e. in } Q_T. \tag{4.11}$$

Step 2: About the gradient of approximate solutions.
In the sequel and throughout the paper, we will denote $\alpha(\epsilon,\mu,s)$ all quantities (possibly different) such that, $\lim_{s\to\infty}\lim_{\mu\to\infty}\lim_{\epsilon\to+0}\alpha(\epsilon,\mu,s) = 0$. Taking now $T_\eta(u_\epsilon - (T_k(u))_\mu)$, $\eta > 0$ as test function in (P_ϵ), we get

$$\langle \frac{\partial u_\epsilon}{\partial t}, T_\eta(u_\epsilon - (T_k(u))_\mu)\rangle + \int_Q a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_\eta(u_\epsilon - (T_k(u))_\mu)$$
$$-\frac{1}{\epsilon}\int_{Q_T} T_{\frac{1}{\epsilon}}((u_\epsilon - \psi)^-)T_\eta(u_\epsilon - (T_k(u))_\mu)\,dx\,dt \leq c\eta,$$

which implies that,

$$\langle \frac{\partial u_\epsilon}{\partial t}, T_\eta(u_\epsilon - (T_k(u))_\mu)\rangle + \int_{Q_T} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_\eta(u_\epsilon - (T_k(u))_\mu)\,dx\,dt$$
$$\leq \frac{\eta}{\epsilon}\int_{Q_T} T_{\frac{1}{\epsilon}}((u_\epsilon - \psi)^-)\,dx\,dt + c\eta$$

and by (4.4)

$$\begin{aligned}\langle\frac{\partial u_\epsilon}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle + \int_{Q_T} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_\eta(u_\epsilon - (T_k(u)_\mu) \\ \leq c\eta.\end{aligned} \tag{4.12}$$

The first term of the left-hand side of the last inequality reads as,

$$\begin{aligned}\langle\frac{\partial u_\epsilon}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle = \langle\frac{\partial u_\epsilon}{\partial t} - \frac{\partial T_k(u)_\mu}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle \\ +\langle\frac{\partial T_k(u)_\mu}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle.\end{aligned} \tag{4.13}$$

The second term of the last equality can be written as,

$$\langle\frac{\partial u_\epsilon}{\partial t} - \frac{\partial T_k(u)_\mu}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle \tag{4.14}$$

$$= \int_\Omega S_\eta(u_\epsilon(T) - T_k(u)_\mu(T))\, dx - \int_\Omega S_\eta(u_0^\epsilon)\, dx \geq -\eta \int_\Omega |u_0^\epsilon|\, dx \geq -\eta c.$$

The third term can be written as,

$$\langle\frac{\partial T_k(u)_\mu}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle = \mu \int_{Q_T} (T_k(u) - T_k(u)_\mu)(T_\eta(u_\epsilon - T_k(u)_\mu)) \tag{4.15}$$

thus by letting $\epsilon \to 0$ and by using Lebesgue theorem,

$$\lim_{\epsilon\to 0} \int_{Q_T} (T_k(u) - T_k(u)_\mu)(T_\eta(u_\epsilon - T_k(u)_\mu)) = \int_{Q_T} (T_k(u) - T_k(u)_\mu)(T_\eta(u - T_k(u)_\mu)).$$

Consequently,

$$\langle\frac{\partial u_\epsilon}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle \geq \alpha(\epsilon,\mu) - \eta c \tag{4.16}$$

on the other hand,

$$\begin{aligned}&\int_{Q_T} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_\eta(u_\epsilon - T_k(u)_\mu)\, dx\, dt \\ &= \int_{\{|u_\epsilon - T_k(u)_\mu)|<\eta\}} a(x,t,u_\epsilon,\nabla u_\epsilon)(\nabla u_\epsilon - \nabla T_k(u)_\mu)\, dx\, dt \\ &= \int_{\{|T_k(u_\epsilon) - T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon) - \nabla T_k(u)_\mu)\, dx\, dt \\ &\quad + \int_{\{|u_\epsilon|>k\}\cap\{|u_\epsilon - T_k(u)_\mu)|<\eta\}} a(x,t,u_\epsilon,\nabla u_\epsilon)(\nabla u_\epsilon - \nabla T_k(u)_\mu)\, dx\, dt\end{aligned}$$

which implies, by using the fact that

$$\int_{\{|u_\epsilon|>k\}\cap\{|u_\epsilon - T_k(u)_\mu)|<\eta\}} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon\, dx\, dt \geq 0,$$

that

$$\begin{aligned}&\int_{\{|T_k(u_\epsilon) - T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon) - \nabla T_k(u)_\mu)\, dx\, dt \\ &\leq c\eta + \int_{\{|u_\epsilon|>k\}\cap\{|u_\epsilon - T_k(u)_\mu)|<\eta\}} a(x,t,u_\epsilon,\nabla u_\epsilon)|\nabla T_k(u)_\mu|\, dx\, dt.\end{aligned} \tag{4.17}$$

Since $a(x,t,T_{k+\eta}(u_\epsilon),\nabla T_{k+\eta}(u_\epsilon))$ is bounded $\prod_{i=1}^{N} L^{p'}(Q_T,w_i^*)$, there exists some $h_{k+\eta}\in\prod_{i=1}^{N} L^{p'}(Q_T,w_i^*)$ such that, $a(x,t,T_{k+\eta}(u_\epsilon),\nabla T_{k+\eta}(u_\epsilon))\rightharpoonup h_{k+\eta}$ weakly in $\prod_{i=1}^{N} L^{p'}(Q_T,w_i^*)$. Consequently,

$$\begin{aligned}&\int_{\{|u_\epsilon|>k\}\cap\{|u_\epsilon-T_k(u)_\mu)|<\eta\}} a(x,t,u_\epsilon,\nabla u_\epsilon)|\nabla T_k(u)_\mu|\,dx\,dt\\ &\quad=\int_{\{|u|>k\}\cap\{|u-T_k(u)_\mu)|<\eta\}} h_{k+\eta}|\nabla T_k(u)_\mu|\,dx\,dt+\alpha(\epsilon)\end{aligned}$$

thanks to proposition 2.1, one easily has,

$$\int_{\{|u|>k\}\cap\{|u-T_k(u)_\mu)|<\eta\}} h_{k+\eta}|\nabla T_k(u)_\mu|\,dx\,dt=\alpha(\mu).$$

Hence,

$$\begin{aligned}\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon)-\nabla T_k(u)_\mu)\,dx\,dt\\ \le c\eta+\alpha(\epsilon,\mu).\end{aligned}\tag{4.18}$$

On the other hand, note that

$$\begin{aligned}&\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon)-\nabla T_k(u)_\mu)\,dx\,dt\\ &=\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon)-\nabla T_k(u))\,dx\,dt\\ &\quad+\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u)-\nabla T_k(u)_\mu)\,dx\,dt\end{aligned}\tag{4.19}$$

the last integral tends to 0 as $\epsilon\to 0$ and $\mu\to\infty$. Indeed, we have that

$$\begin{aligned}&\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u)-\nabla T_k(u)_\mu)\,dx\,dt\\ &\to\int_{\{|T_k(u)-T_k(u)_\mu)|<\eta\}} h_k(\nabla T_k(u)-\nabla T_k(u)_\mu)\,dx\,dt\ \text{ as }\ \epsilon\to 0.\end{aligned}$$

It is obviously that, $\int_{\{|T_k(u)-T_k(u)_\mu)|<\eta\}} h_k(\nabla T_k(u)-\nabla T_k(u)_\mu)\,dx\,dt\to 0$ as $\mu\to\infty$. We deduce then that,

$$\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon)-\nabla T_k(u))\,dx\,dt\le c\eta+\alpha(\epsilon,\mu).\tag{4.20}$$

Let $A_\epsilon=\Big([a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))-a(x,t,T_k(u_\epsilon),\nabla T_k(u))][\nabla T_k(u_\epsilon)-\nabla T_k(u)]\Big)$, then for any $0<\theta<1$, we write

$$I_\epsilon=\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|\le\eta\}} A_\epsilon^\theta\,dx\,dt+\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|>\eta\}} A_\epsilon^\theta\,dx\,dt$$

since, $a(x, T_k(u_\epsilon), \nabla T_k(u_\epsilon))$ is bounded in $\prod_{i=1}^{N} L^{p'}(Q_T, w_i^{1-p'})$, while $\nabla T_k(u_\epsilon)$ is bounded in $\prod_{i=1}^{N} L^p(Q_T, w_i)$, then by applying the Hölder's inequality, we obtain,

$$I_\epsilon \leq c \left(\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} A_\epsilon \, dx \, dt \right)^\theta \qquad (4.21)$$
$$+ c_2 \operatorname{meas}\{(x,t) \in Q_T : \ |T_k(u_\epsilon) - T_k(u)_\mu)| > \eta\}^{1-\theta}$$

on the other hand, we have,

$$\begin{aligned} &\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} A_\epsilon \, dx \, dt \\ &= \int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon), \nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon) - \nabla T_k(u)) \, dx \, dt \\ &- \int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon), \nabla T_k(u))(\nabla T_k(u_\epsilon) - \nabla T_k(u)) \, dx \, dt \\ &= I_\epsilon^1 + I_\epsilon^2 \end{aligned} \qquad (4.22)$$

using (4.20), we have,

$$I_\epsilon^1 \leq c\eta + \alpha(\epsilon, \mu). \qquad (4.23)$$

Concerning I_ϵ^2 the second term of the right hand side of the (4.22), it is easy to see that

$$I_\epsilon^2 = \alpha(\epsilon) \qquad (4.24)$$

because for all $i = 1, ..., N$, we have, $a_i(x, t, T_k(u_\epsilon), \nabla T_k(u)) \to a_i(x, t, T_k(u), \nabla T_k(u))$ strongly in $L^{p'}(Q_T, w_i^{1-p'})$, while $\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i}$ weakly in $L^p(Q_T, w_i)$.
Combining (4.21), (4.22), (4.23) and (4.24) we get,

$$I_\epsilon \leq c \operatorname{meas}\{|T_k(u_\epsilon) - T_k(u)_\mu| < \eta\}^\theta + c(\alpha(\epsilon, \mu, \eta))^{1-\theta}$$

and by passing to the limit sup over ϵ, μ and η

$$\lim_{\epsilon \to 0} \int_{Q_T} \Big([a(x,t,T_k(u_\epsilon), \nabla T_k(u_\epsilon)) - a(x,t,T_k(u_\epsilon), \nabla T_k(u))] \\ [\nabla T_k(u_\epsilon) - \nabla T_k(u)] \Big)^\theta = 0$$

by Theorem 3.3 of [15] (see also [8], [9]), there exist a subsequence also denoted by u_ϵ such that,

$$\nabla u_\epsilon \longrightarrow \nabla u \ \text{ a.e. in } \ Q_T. \qquad (4.25)$$

Step 3: Passage to the limit
Let $\varphi \in K_\psi \cap L^\infty(\bar{Q})$, choosing $T_k(u_\epsilon - \varphi)_{\chi_{(0,\tau)}}$ as test function in (P_ϵ), we get,

$$\begin{aligned} \langle \frac{\partial u_\epsilon}{\partial t}, T_k(u_\epsilon - \varphi) \rangle_{Q_\tau} &+ \int_{Q_\tau} a(x,t,u_\epsilon, \nabla u_\epsilon) \nabla T_k(u_\epsilon - \varphi) \, dx \, dt \\ &- \frac{1}{\epsilon} \int_{Q_\tau} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- T_k(u_\epsilon - \varphi) \, dx \, dt \\ &= \int_{Q_\tau} f_\epsilon T_k(u_\epsilon - \varphi) \, dx \, dt \end{aligned} \qquad (4.26)$$

since, $-\frac{1}{\epsilon}\int_{Q_\tau} T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- T_k(u_\epsilon-\varphi)\,dx\,dt \geq 0$ and $\frac{\partial u_\epsilon}{\partial t} = \frac{\partial}{\partial t}(u_\epsilon-\varphi)+\frac{\partial \varphi}{\partial t}$, we get

$$\begin{aligned}\int_\Omega S_k(u_\epsilon(\tau)-\varphi(\tau))\,dx + \langle\frac{\partial\varphi}{\partial t}, T_k(u_\epsilon-\varphi)\rangle_{Q_\tau}&\\ +\int_{Q_\tau} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon-\varphi)\,dx\,dt&\\ \leq \int_{Q_\tau} f_\epsilon T_\epsilon(u_\epsilon-\varphi)\,dx\,dt + \int_\Omega S_k(u_\epsilon(0)-\varphi(0))\,dx.&\end{aligned} \tag{4.27}$$

Lemma 4.2. *Assume that the assumptions of Theorem 4.1 hold true. Let u^ϵ be a sequence of solutions of (P_ϵ) converges to u a.e. in Q. Then the sequence u_ϵ is a Cauchy sequence in $C([0,T],L^1(\Omega))$, moreover, $u\in C([0,T],L^1(\Omega))$ and u_ϵ converges to u in $C([0,T],L^1(\Omega))$.*

Proof. See Appendix. ∎

Because of $u_\epsilon \to u$ in $C([0,T],L^1(\Omega))$, then $\forall\ \tau\leq T$, $u_\epsilon(t)\to u(t)$ in $L^1(\Omega)$, thus

$$\int_\Omega S_k(u_\epsilon(\tau)-\varphi(\tau))\,dx \to \int_\Omega S_k(u-\varphi)\,dx \tag{4.28}$$

and

$$\int_\Omega S_k(u_\epsilon(0)-\varphi(0))\,dx \to \int_\Omega S_k(u_0-\varphi(0))\,dx.$$

Let $M=k+\|\varphi\|_\infty$, then, we can write,

$$\begin{aligned}&\int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)\,dx\,dt\\ &\qquad = \int_{Q_T} a(x,t,T_M(u_n),\nabla T_M(u_n))\nabla T_k(u_n-\varphi)\,dx\,dt,\end{aligned}$$

by Fatou's lemma and the convergence of $a(x,t,T_M(u_n),\nabla T_M(u_n))$ to $a(x,t,T_M(u),\nabla T_M(u))$ weakly in $\prod_{i=1}^N L^{p'}(Q,w_i^{1-p'})$, it possible to conclude that

$$\int_{Q_T} a(x,t,T_M(u),\nabla T_M(u))\nabla T_k(u-\varphi) \tag{4.29}$$

$$\leq \liminf_{\epsilon\to 0}\int_\Omega a(x,t,T_M(u_\epsilon),\nabla T_M(u_\epsilon))\nabla T_k(u_\epsilon-\varphi)\,dx\,dt.$$

Moreover, since $\frac{\partial\varphi}{\partial t}\in L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))$ and $\nabla T_k(u_\epsilon-\varphi)\rightharpoonup \nabla T_k(u-\varphi)$ weakly in $\prod_{i=1}^N L^p(Q_T,w_i)$, we get,

$$\int_{Q_\tau}\frac{\partial\varphi}{\partial t}T_k(u_\epsilon-\varphi)\,dx\,dt \to \int_{Q_\tau}\frac{\partial\varphi}{\partial t}T_k(u-\varphi)\,dx\,dt \tag{4.30}$$

$$\int_Q f_\epsilon T_k(u_\epsilon-\varphi)\,dx\,dt \to \int_Q fT_k(u-\varphi)\,dx\,dt. \tag{4.31}$$

Finally, by (4.27)-(4.31) we get,

$$\begin{aligned}\int_\Omega S_k(u_\epsilon(\tau)-\varphi(\tau))\,dx + \langle \frac{\partial\varphi}{\partial t}, T_k(u-\varphi)\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-\varphi)\,dx\,dt \\ \le \int_{Q_\tau} fT_k(u-\varphi)\,dx\,dt + \int_\Omega S_k(u(0)-\varphi(0))\,dx.\end{aligned}$$

By Step 1, Step 2 and Step 3, the proof of Theorem 4.1 is complete. ■

5 Appendix

In this appendix we give the proof of Lemma 4.2.

Proof. Note that $T_l(u)\in K_\psi$, for every $l\ge\|\psi\|_\infty$. Let $\eta_i\ge 0$ converges to u_0 in $L^1(\Omega)$ and $v_\mu^{i,l}=(T_l(u))_\mu+e^{-\mu t}T_l(\eta_i)$. Using the admissible test function $T_k(u_\epsilon-v_\mu^{i,l})$ in (P_ϵ) leads to

$$\begin{aligned}\langle \frac{\partial u_\epsilon}{\partial t}, T_k(u_\epsilon-v_\mu^{i,l})\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon-v_\mu^{i,l})\,dx\,dt \\ -\frac{1}{\epsilon}\int_{Q_\tau} T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- T_k(u_\epsilon-v_\mu^{i,l})\,dx\,dt = \int_{Q_\tau} f_\epsilon T_k(u_\epsilon-v_\mu^{i,l})\,dx\,dt\end{aligned} \tag{5.1}$$

since $\frac{\partial u_\epsilon}{\partial t}=\frac{\partial}{\partial t}(u_\epsilon-v_\mu^{i,l})+\frac{\partial}{\partial t}(v_\mu^{i,l})=\frac{\partial}{\partial t}(u_\epsilon-v_\mu^{i,l})+\mu(T_l(u)-v_\mu^{i,l})$ we deduce,

$$\begin{aligned}\langle \frac{\partial u_\epsilon}{\partial t}, T_k(u_\epsilon-v_\mu^{i,l})\rangle_{Q_\tau} &= \langle \frac{\partial}{\partial t}(u_\epsilon-v_\mu^{i,l}), T_k(u_\epsilon-v_\mu^{i,l})\rangle_{Q_\tau} \\ &+\mu\int_{Q\tau}(T_l(u)-v_\mu^{i,l})T_k(u_\epsilon-v_\mu^{i,l})\,dx\,dt.\end{aligned}$$

Remark that, for every $\tau\in[0,T]$ and when ϵ tends to 0

$$\mu\int_{Q\tau}(T_l(u)-v_\mu^{i,l})T_k(u_\epsilon-v_\mu^{i,l})\,dx\,dt \to \mu\int_{Q\tau}(T_l(u)-v_\mu^{i,l})T_k(u-v_\mu^{i,l})\,dx\,dt \ge 0.$$

On the other hand, by using Lebesgue's convergence theorem and the fact that, $-\int_{Q_\tau}\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-T_k(u_\epsilon-v_\mu^{i,l})\,dx\,dt\ge 0$, we deduce by (5.1) that

$$\begin{aligned}\langle \frac{\partial}{\partial t}(u_\epsilon-v_\mu^{i,l}), T_k(u_\epsilon-v_\mu^{i,l})\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u_\epsilon,\nabla v_\mu^{i,l})\nabla T_k(u_\epsilon-v_\mu^{i,l})\,dx\,dt \\ \le \alpha(\epsilon,\mu,i,l)\end{aligned}$$

for every $\tau\in[0,T]$.

Since for all $j = 1, ..., N$: we have $a_j(x,t,T_{2k+l}(u_\epsilon), \nabla v_\mu^{i,l}) \to a_j(x,t,T_{k+2l}(u), \nabla v_\mu^{i,l})$ strongly in $L^{p'}(Q, w_j^{1-p'})$ while $\frac{\partial}{\partial x_j} T_k(u_\epsilon - v_\mu^{i,l}) \rightharpoonup \frac{\partial}{\partial x_j} T_k(u - v_\mu^{i,l})$ weakly in $L^p(Q, w_j)$, we have,

$$\langle \frac{\partial}{\partial t}(u_\epsilon - v_\mu^{i,l}), T_k(u_\epsilon - v_\mu^{i,l}) \rangle_{Q_\tau} \leq \alpha(\epsilon, \mu, i, l). \tag{5.2}$$

In view of the definition $S_k(z) = \int_0^z T_k(s)\, ds$ and using Lebesgue's convergence theorem, we deduce that $\int_\Omega S_k(u_\epsilon(\tau) - v_\mu^{i,l}(\tau))\ dx \leq \alpha(\epsilon, \mu, i, l)$. Which implies, by writing,

$$\int_\Omega S_k(\frac{u_\epsilon - u_\lambda}{2})\ dx \leq \frac{1}{2}\left(\int_\Omega S_k(u_\epsilon(\tau) - v_\mu^{i,l}(\tau))\ dx + \int_\Omega S_k(u_\lambda(\tau) - v_\mu^{i,l}(\tau))\ dx \right),$$

that

$$\int_\Omega S_k(\frac{u_\epsilon - u_\lambda}{2})\ dx \leq \alpha(\epsilon, \lambda). \tag{5.3}$$

Finally, by Hölder's inequality, we have,

$$\begin{aligned}
\int_\Omega |u_\epsilon - u_\lambda|\ dx &= \int_{\{|u_\epsilon - u_\lambda| \leq 1\}} |u_\epsilon - u_\lambda|\ dx + \int_{\{|u_\epsilon - u_\lambda| > 1\}} |u_\epsilon - u_\lambda|\ dx \\
&\leq \left(\int_{\{|u_\epsilon - u_\lambda| \leq 1\}} |u_\epsilon - u_\lambda|^2\ dx \right)^{\frac{1}{2}} \text{meas}(\Omega)^{\frac{1}{2}} + \int_{\{|u_\epsilon - u_\lambda| > 1\}} |u_\epsilon - u_\lambda|\ dx \\
&\leq \text{meas}(\Omega)^{\frac{1}{2}} \left(\int_{\{|u_\epsilon - u_\lambda| \leq 1\}} 2S_1(u_\epsilon - u_\lambda)\ dx \right)^{\frac{1}{2}} + \int_{\{|u_\epsilon - u_\lambda| > 1\}} 2S_1(u_\epsilon - u_\lambda)\ dx
\end{aligned}$$

since $(\frac{|y|^2}{2})\chi_{\{|y| \leq 1\}} = S_1(y)\chi_{\{|y| \leq 1\}}$ and $(\frac{|y|}{2})\chi_{\{|y|>1\}} \leq (\frac{|y|}{2} + \frac{|y|-1}{2})\chi_{\{|y|>1\}} = S_1(y)\chi_{\{|y|>1\}}$ Due to (5.3) we deduce that, $\int_\Omega |u_\epsilon(\tau) - u_\lambda(\tau)|\ dx \leq \alpha(\epsilon, \lambda)$, not depending on τ, and thus (u_ϵ) is a Cauchy sequence in $C([0,T], L^1(\Omega))$, and since $u_\epsilon \to u$, a.e. in Q, we deduce that $u_\epsilon \to u$, a.e. in $C([0,T], L^1(\Omega))$. ∎

Acknowledgment
The authors would like to thank the anonymous referee for his/her valuable suggestions and comments.

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