An Ecological Model with Grazing and Constant Yield Harvesting

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Abstract

We study positive solutions to the steady state reaction diffusion equation with Dirichlet boundary condition of the form:

$$\begin{cases} -\Delta u = au - bu^2 - c\dfrac{u^p}{1+u^p} - K, & x \in \Omega \\ u = 0, \quad x \in \partial\Omega. \end{cases} \tag{1}$$

Here $\Delta u = div(\nabla u)$ is the Laplacian of u, a, b, c, p, K are positive constants with $p \geq 2$ and Ω is a smooth bounded region with $\partial\Omega$ in C^2. This model describes the steady states of a logistic growth model with grazing and constant yield harvesting. It also describes the dynamics of the fish population with natural predation and constant yield harvesting. We study the existence of positive solutions to this model. We prove our results by the method of sub-super solutions.

1 Introduction

Consider the nonlinear boundary value problem

$$\begin{cases} -\Delta u = au - bu^2 - c\dfrac{u^p}{1+u^p}, & x \in \Omega \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{2}$$

where $\Delta u = div(\nabla u)$ is is the Laplacian of u,a, b, c, p are positive constants with $p \geq 2$. Ω is a smooth bounded region with $\partial\Omega$ in C^2. Here u is the population

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density and $au - bu^2$ represents logistics growth. This model describes grazing of a fixed number of grazers on a logistically growing species (see [6]-[5]). The herbivore density is assumed to be a constant which is a valid assumption for managed grazing systems and the rate of grazing is given by $\frac{cu^p}{1+u^p}$. At high levels of vegetation density this term saturates to c as the grazing population is a constant. This model has also been applied to describe the dynamics of fish populations (see [5] and [8]). In the case of the fish population the term $\frac{cu^p}{1+u^p}$ corresponds to natural predation. In this paper we introduce a constant yield harvesting term to the model and study the existence of a positive solution $u \in C^2(\Omega) \bigcap C(\bar{\Omega})$ to the problem

$$\begin{cases} -\Delta u = au - bu^2 - c\dfrac{u^p}{1+u^p} - K, & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

It can be easily shown that (3) does not have a positive solution if $a \leq \lambda_1$, where λ_1 is the principle eigenvalue of the operator $-\Delta$ with Dirichlet boundary condition. To see this let $\phi_1 > 0$ be an eigenfunction corresponding to λ_1, i.e.,

$$\begin{cases} -\Delta\phi_1 = \lambda_1\phi_1, & x \in \Omega \\ \phi_1 = 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

Multiplying (3) by ϕ_1 and integrating over Ω, we obtain

$$(a - \lambda_1)\int_\Omega u\phi_1 dx = b\int_\Omega u^2 dx + c\int_\Omega \frac{u^p}{1+u^p}dx + K\int_\Omega dx. \tag{5}$$

The right hand side of the above equation is always positive and hence if $a \leq \lambda_1$ then we have a contradiction.

We will establish our existence results by the method of sub-super solutions. Consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

By a subsolution of (6) we mean a function $\psi \in C^2(\Omega) \bigcap C(\bar{\Omega})$ that satisfies:

$$\begin{cases} -\Delta\psi \leq \lambda f(\psi), & x \in \Omega \\ \psi \leq 0, & x \in \partial\Omega \end{cases} \tag{7}$$

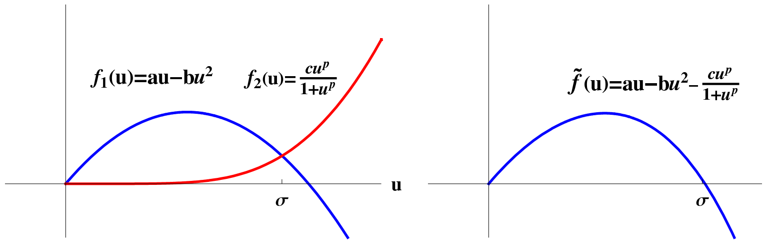
and by a supersolution of (6) we mean a function $\phi \in C^2(\Omega) \bigcap C(\bar{\Omega})$ that satisfies:

$$\begin{cases} -\Delta\phi \geq \lambda f(\phi), & x \in \Omega \\ \phi \geq 0, & x \in \partial\Omega. \end{cases} \tag{8}$$

Then the following lemma holds (see [1]).

Lemma 1.1. *Let ψ be a subsolution of (6) and ϕ be a supersolution of (6) such that $\psi \leq \phi$. Then (6) has a solution u such that $\psi \leq u \leq \phi$.*

In the absence of the constant yield harvesting term K, the boundary value problem always has a positive steady state for $a > \lambda_1$.



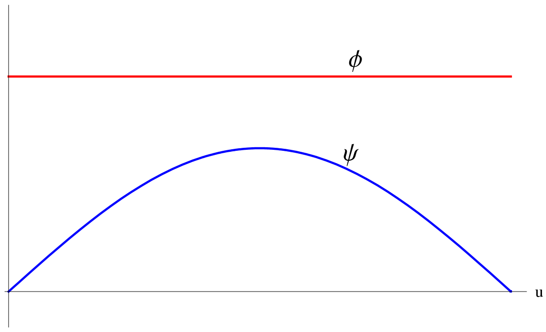
That is, grazing alone does not eliminate the steady states for all positive 'a' values. To see this let $\phi_1 > 0$ be an eigenfunction corresponding to λ_1 and define $\psi = \epsilon\phi_1$; $\epsilon > 0$. Then we get

$$-\Delta\psi := -\epsilon\Delta\phi_1 = \epsilon\lambda_1\phi_1; \ x \in \Omega. \tag{9}$$

Let $H(s) = \tilde{f}(s) - \lambda_1 s$ where $\tilde{f}(s) = as - bs^2 - c\dfrac{s^p}{1+s^p}$. Then $H'(s) = \tilde{f}'(s) - \lambda_1$, $H(0) = 0$ and $H'(0) = a - \lambda_1 > 0$. So for $\epsilon \approx 0$ we have $H(\epsilon\phi_1) = \tilde{f}(\epsilon\phi_1) - \lambda_1(\epsilon\phi_1) \geq 0$. Hence from (9) we have

$$-\Delta\psi = \epsilon\lambda_1\phi_1 \leq \tilde{f}(\epsilon\phi_1) = \tilde{f}(\psi); \ x \in \Omega. \tag{10}$$

Thus ψ is a subsolution. Clearly $\phi = \dfrac{a}{b}$ is a supersolution and we can choose



$\epsilon \approx 0$ such that $0 \leq \psi = \epsilon\phi_1 \leq \dfrac{a}{b} = \phi$. Hence by Lemma 1.1 the boundary value problem (2) has a positive solution.

The diffusive logistic equation with constant yield harvesting, in the absence of grazing was studied in [7]. Here if the harvesting rate is too high then there will be no positive steady states. The authors discuss the existence, uniqueness and stability of the maximal steady state solutions to

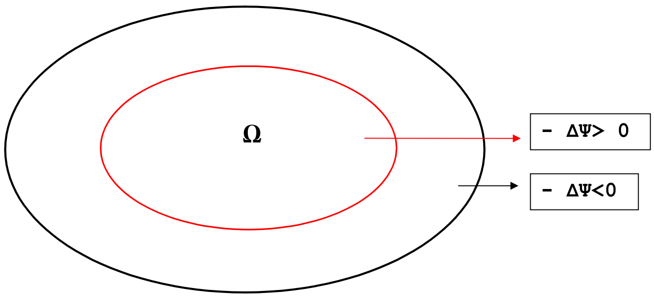
$$\begin{cases} -\Delta u = au - bu^2 - K, & x \in \Omega \\ \quad u = 0, \quad x \in \partial\Omega, \end{cases} \tag{11}$$

where $K \geq 0$ represents the harvesting effort. In particular, in [7] it is shown that if $a > \lambda_1$ and $b > 0$ then there exists a $K_1 = K_1(a, b) > 0$ such that for $0 < K < K_1$, (11) has a positive solution.

Such problems, i.e., problems of the form

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{12}$$

where $f(0) < 0$, are referred as semipositone problems in the literature. Construction of a subsolution is more challenging in semipositone problems (see [2] and [4]). Here our test functions for a positive subsolution must come from positive functions ψ such that $-\Delta\psi < 0$ near $\partial\Omega$ and $-\Delta\psi > 0$ in the interior of Ω.



In this paper we consider (3), that is the model with grazing and constant yield harvesting and prove the following result:

Theorem 1.2. *Let $a > \lambda_1, b > 0$ and $c > 0$ be fixed. Then there exists a $K_0(a,b,c,p) > 0$ such that for $K < K_0(a,b,c,p)$, (3) has a positive solution.*

We provide a proof for this theorem in Section 2.

2 Proof of Theorem 1.2

We start with the construction of a positive subsolution for (3). To get a positive subsolution, we can apply an anti-maximum principle by Clement and Peletier [3], from which we know that there exist a $\delta = \delta(\Omega) > 0$ and a solution z_λ (with $z_\lambda > 0$ in Ω and $\frac{\partial z_\lambda}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the outer unit normal to Ω) of

$$\begin{cases} -\Delta z - \lambda z = -1, & x \in \Omega \\ z = 0, & x \in \partial\Omega, \end{cases} \tag{13}$$

for $\lambda \in (\lambda_1, \lambda_1 + \delta)$. Fix $\lambda^* \in (\lambda_1, min\{a, \lambda_1 + \delta\})$. Let z^*_λ be the solution of (13) when $\lambda = \lambda^*$ and $\alpha = ||z^*_\lambda||_\infty$.

Define $\psi = \mu K z^*_\lambda$ where $\mu \geq 1$ is to be determined later. We will choose μ and $K > 0$ properly so that ψ is a subsolution. We know $-\Delta\psi = -\Delta(\mu K z^*_\lambda) = \lambda^*\psi - \mu K$. Hence ψ is a subsolution if $\lambda^*\psi - \mu K \leq a\psi - b\psi^2 - c\frac{\psi^p}{1+\psi^p} - K$. That is if

$$(a - \lambda^*)\psi - b\psi^2 - c\frac{\psi^p}{1+\psi^p} + (\mu - 1)K \geq 0. \tag{14}$$

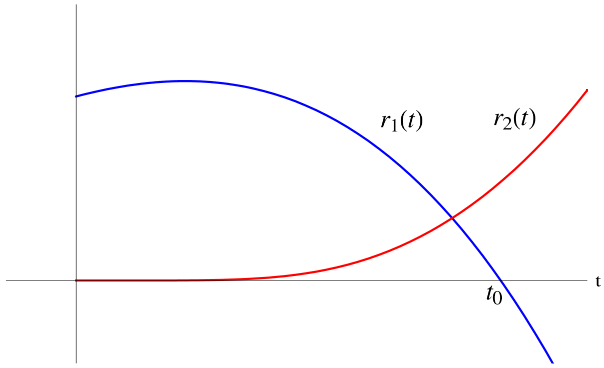
Consider

$$r(t) = (a - \lambda^*)t - bt^2 - c\frac{t^p}{1 + t^p} + (\mu - 1)K. \tag{15}$$

It can be written as $r(t) = r_1(t) + r_2(t)$ where

$$\begin{aligned} r_1(t) &= (a - \lambda^*)t - bt^2 - ct^p + (\mu - 1)K \text{ and} \\ r_2(t) &= \frac{t^{2p}}{1 + t^p}. \end{aligned} \tag{16}$$

Clearly $r_2(t) \geq 0$ for all $t \geq 0$. So if we can find K and μ such that $r_1(t) \geq 0$



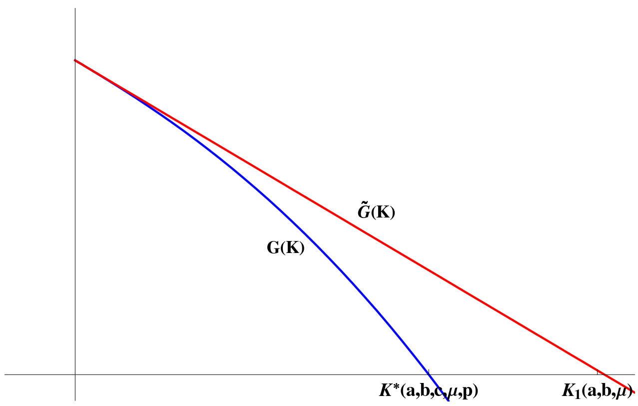
for $0 \leq t \leq \mu K\alpha$ then ψ will be a subsolution. Now $r_1(0) = (\mu - 1)K$, $r_1''(t) = -2b - cp(p-1)t^{p-2} < 0$ and there exists a unique t_0 such that $r_1(t_0) = 0$. This means that ψ is a subsolution if $r_1(\mu K\alpha) \geq 0$, i.e. if

$$(a - \lambda^*)\mu K\alpha - b(\mu K\alpha)^2 - c(\mu K\alpha)^p + (\mu - 1)K \geq 0. \tag{17}$$

Let

$$G(K) = (a - \lambda^*)\mu\alpha - b(\mu\alpha)^2 K - c(\mu\alpha)^p K^{p-1} + (\mu - 1). \tag{18}$$

Then $G(0) = (a - \lambda^*)\mu\alpha + (\mu - 1) > 0$ since $\mu \geq 1$ and $a > \lambda^*$. Also, we have $G'(K) = -b(\mu\alpha)^2 - c(p-1)(\mu\alpha)^p K^{p-2} < 0$. Hence given μ and p there exists a unique $K^* = K^*(a, b, c, \mu, p) > 0$ with $G(K^*) = 0$. Since $G(K) \leq (a - \lambda^*)\mu\alpha -$



$b(\mu\alpha)^2 K + (\mu - 1) = \tilde{G}(K)$ we see that

$$K^* \leq \frac{(a - \lambda^*)\mu\alpha + (\mu - 1)}{b\mu^2\alpha^2} := K_1(a, b, \mu). \tag{19}$$

Note that $K_1(a,b,\mu)$ is bounded for $\mu \in [1,\infty)$. Hence K^* is bounded for $\mu \in [1,\infty)$. Let $K_0(a,b,c,p) = \sup_{\mu\geq 1} K^*(a,b,c,\mu,p)$. Now let $\tilde{K} < K_0(a,b,c,p)$. By definition there will exist a $\tilde{\mu} \geq 1$ such that $\tilde{K} < K^*(a,b,c,\tilde{\mu},p) < K_0(a,b,c,p)$. Choose $\psi = \tilde{\mu}\tilde{K}z$. With $\mu = \tilde{\mu}$ we have $G(\tilde{K}) \geq 0$ and hence

$$(a-\lambda^*)\tilde{\mu}\tilde{K}\alpha - b(\tilde{\mu}\tilde{K}\alpha)^2 - c(\tilde{\mu}\tilde{K}\alpha)^p + (\tilde{\mu}-1)\tilde{K} \geq 0. \tag{20}$$

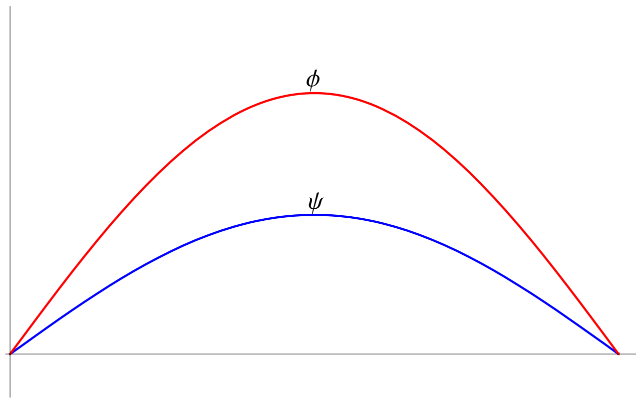
Thus ψ is a subsolution to (3).

Now for a supersolution choose $\phi = Me$, where

$$\begin{cases} -\Delta e = 1, & x \in \Omega \\ \quad e = 0, & x \in \partial\Omega, \end{cases} \tag{21}$$

and $M > 0$ is such that $f(u) = au - bu^2 - c\dfrac{u^p}{1+u^p} - K \leq M$ for all $u \geq 0$. Clearly $-\Delta\phi = M \geq f(\phi)$ and ϕ is a supersolution.

Since by the Hopf maximum principle $\dfrac{\partial e}{\partial \nu} < 0$ on $\partial\Omega$ (where ν is the outer unit normal to Ω), we can choose $M >> 1$ so that $\phi = Me \geq \psi$. Hence by Lemma 1.1



the problem has a positive solution for all $K < K_0(a,b,c,p)$.

Corollary 2.1. *Let $p = 2$ then (3) has a solution for all $K < \tilde{K}_0(a,b,c)$, where $\tilde{K}_0(a,b,c) = \sup_{\mu\geq 1} \dfrac{(a-\lambda^*)\mu\alpha + \mu - 1}{(b+c)(\mu\alpha)^2}$*

Proof. In this case

$$G(K) = (a-\lambda^*)\mu\alpha - b(\mu\alpha)^2K - c(\mu\alpha)^2K + (\mu-1).$$

and $K^* = K^*(a,b,c,\mu) = \dfrac{(a-\lambda^*)\mu\alpha + \mu - 1}{(b+c)(\mu\alpha)^2}$. Hence (3) has a solution for all $K < \tilde{K}_0(a,b,c) = \sup_{\mu\geq 1} \dfrac{(a-\lambda^*)\mu\alpha + \mu - 1}{(b+c)(\mu\alpha)^2}$. ∎

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