On the global solvability of the Cauchy problem for damped Kirchhoff equations

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Abstract

We study the Cauchy problem for the damped Kirchhoff equation in the phase space $H^r \times H^{r-1}$, with $r \geq 3/2$. We prove global solvability and decay of solutions when the initial data belong to an open, dense subset B of the phase space such that $B + B = H^r \times H^{r-1}$.

1 Introduction

We consider here the Cauchy problem for the damped Kirchhoff equation:

$$u_{tt} - m\left(\int |\nabla u|^2\,dx\right)\Delta u + 2\gamma\, u_t = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where

$$\begin{cases} m \in C^1[0,\infty)\,, \\ m(r) \geq \delta > 0, \quad \forall\, r \in [0,\infty)\,, \\ \gamma > 0\,. \end{cases} \tag{1.3}$$

Global solvability and asymptotic behavior of solutions were studied by Yamada [9] and Yamazaki [11] for small initial data. Roughly speaking, in these papers the authors assumed that $\|u_0\|_{H^r} + \|u_1\|_{H^{r-1}} \leq \varepsilon$, for some $r \geq 3/2$, with $\varepsilon > 0$ a constant depending only on $m(\cdot)$ and γ. Without smallness assumptions, equation

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(1.1) was investigated by Nishihara [6] for the initial boundary value problem with Dirichlet condition. Namely, considering equation (1.1) in $\Omega \times [0, \infty)$, with $\Omega \subset \mathbb{R}^n$ a bounded analytic domain, Nishihara [6] proved global existence and decay of solutions when the initial data belong to some (*quasi-analytic*) function space lying between the analytic class and $\bigcap_{s>1} G_s$, where G_s is the Gevrey class of order s (see also [5] for more details).

The main interest of the present paper is to investigate global existence and decay of solutions of the Cauchy problem (1.1)–(1.2) in the phase space $H^r \times H^{r-1}$, with $r \geq \frac{3}{2}$, when no smallness condition is assumed on the initial data. To this purpose, we will consider special classes of initial data defined as follows:

Definition 1.1. *Given $u_0, u_1 \in L^2$, we say that $(u_0, u_1) \in \tilde{B}^1_\Delta$ if $\forall N \geq 0$ there exist positive numbers $\tilde{\rho}_j = \tilde{\rho}_j(N)$, for $j \geq 1$, such that $\tilde{\rho}_j \to \infty$ as $j \to \infty$ and, denoting with $\hat{u}_i$ $(i = 0, 1)$ the Fourier transform of u_i,*

$$\sup_j \; e^{N\tilde{\rho}_j} \int_{|\xi| > \tilde{\rho}_j} \Big[|\xi|^3 \, |\hat{u}_0(\xi)|^2 + |\xi| \, |\hat{u}_1(\xi)|^2 \Big] \, d\xi < \infty \,. \tag{1.4}$$

Besides, for $k \geq 1$, we say that $(u_0, u_1) \in B^k_\Delta$ if there exist $\eta > 0$ and a sequence of positive numbers $\{\rho_j\}_{j \geq 1}$, $\rho_j \to \infty$, such that

$$\sup_j \int_{|\xi| > \rho_j} \Big[|\xi|^{k+2} \, |\hat{u}_0(\xi)|^2 + |\xi|^k \, |\hat{u}_1(\xi)|^2 \Big] \, \frac{e^{\eta \, \rho_j^k / |\xi|^{k-1}}}{\rho_j^k} \, d\xi < \infty \,. \tag{1.5}$$

Given $r \geq 0$, we set

$$E_r(u;t) \stackrel{\text{def}}{=} |u(\cdot,t)|^2_{\frac{r}{2}+1} + |u_t(\cdot,t)|^2_{\frac{r}{2}} \,, \tag{1.6}$$

where, for $h \geq 0$, $|\cdot|_h$ is the semi-norm

$$|f(\cdot)|_h \stackrel{\text{def}}{=} \| \, |\xi|^h \hat{f}(\cdot) \, \|_{L^2} \,.$$

Theorem 1.2 (Global Solvability)**.** *Given $k \geq 1$ integer, assume that $m \in C^k$ and $(u_0, u_1) \in \tilde{B}^1_\Delta$ (resp. B^k_Δ) if $k = 1$ (resp. $k > 1$). Then* (1.1)–(1.2) *has a unique global solution $u \in C^j([0, \infty); H^{1+\frac{k}{2}-j}(\mathbb{R}^n))$ $(j = 0, 1, 2)$ which satisfies*

$$\sup_{t \geq 0} \left(t \, E_r(u;t) + \int_0^t E_r(u;\tau) \, d\tau \right) < \infty \,, \tag{1.7}$$

for $0 \leq r \leq k$.

Here will prove in details Theorem 1.2 only for $k = 1, 2$. For $k \geq 3$ we will sketch the proof in § 7, using some results obtained in [4]. The solutions of (1.1)–(1.2) satisfy stronger decay properties than (1.7). Namely, assuming (1.3), we have:

Theorem 1.3 (Decay Estimates). *Let* $u \in C^j([0,\infty); H^{1+\frac{k}{2}-j}(\mathbb{R}^n))$ $(j = 0,1,2)$, $k \geq 1$ *integer, be a global solution of* (1.1)–(1.2). *Suppose further that*

$$\sup_{t\geq 0} |u(t)|_2 < \infty \quad \textit{if} \quad k \geq 2\,; \quad \sup_{t\geq 0} t\, E_1(u;t) < \infty \quad \textit{if} \quad k = 1\,. \tag{1.8}$$

Then, for $0 \leq r \leq k/2$, u *satisfies*

$$\sup_{t>0} \left\{ t\,\theta^r \left(|u|^2_{r+1} + |u_t|^2_r \right) + \int_0^t \theta^r \left(|u|^2_{r+1} + \tau\,|u_t|^2_r \right) d\tau \right\} < \infty\,, \tag{1.9}$$

where $\theta = t$ *if* $k \geq 2$; $\theta = t^\sigma$, *with* σ *any real number in* $[0,1)$, *if* $k = 1$.

For arbitrary data $(\tilde{u}_0, \tilde{u}_1)$ in the phase space $H^r \times H^{r-1}$, $r \geq \frac{3}{2}$, the problem of global solvability remains open. However, using Theorem 1.3 and a stability argument developed by Nishihara in [7], it is easy to prove the following:

Corollary 1.4 (Stability). *Let* u *be a fixed solution of* (1.1)–(1.2) *satisfying the assumptions of Theorem 1.3 for some integer* $k \geq 1$. *Given* $(\tilde{u}_0, \tilde{u}_1) \in H^{1+\frac{k}{2}} \times H^{\frac{k}{2}}$, *if* $\tilde{u}_0 - u_0$ *and* $\tilde{u}_1 - u_1$ *are sufficiently small in the sense that*

$$|\tilde{u}_0 - u_0|_{\frac{k}{2}+1} + |\tilde{u}_1 - u_1|_{\frac{k}{2}} \leq \varepsilon \;\; \textit{for some} \;\; \varepsilon > 0\,, \tag{1.10}$$

then there exists a unique $\tilde{u} \in C^j([0,\infty); H^{1+\frac{k}{2}-j}(\mathbb{R}^n))$ $(j = 0,1,2)$ *solution of* (1.1) *such that* $\tilde{u}(x,0) = \tilde{u}_0(x)$, $\tilde{u}_t(x,0) = \tilde{u}_1(x)$. *Moreover,* $\tilde{u}$ *satisfies* (1.9).

We will not give the proof of this stability result, because it is a straightforward consequence of Theorem 1.3, the argument of [7], known results of local/global solvability and continuous dependence upon initial data proved in [1], [9], [10]. We only observe that, by Theorem 1.2 and Corollary 1.4, global solvability and decay of solutions are assured for initial data (u_0, u_1) in an open, dense subset of $H^{\frac{3}{2}} \times H^{\frac{1}{2}}$, say B, such that

$$B + B = H^{\frac{3}{2}} \times H^{\frac{1}{2}}\,. \tag{1.11}$$

This fact will be clear from the remarks below.

1.1 Some properties of $\tilde{B}^1_\Delta$, B^k_Δ $(k \geq 1)$

It is clear that $\tilde{B}^1_\Delta \subset B^1_\Delta \subset H^{\frac{3}{2}} \times H^{\frac{1}{2}}$, with strict inclusions. It is also easy to prove that

1) $\tilde{B}^1_\Delta + \tilde{B}^1_\Delta = H^{\frac{3}{2}} \times H^{\frac{1}{2}}$,

2) $\mathcal{A}_{L^2} \times \mathcal{A}_{L^2} \not\subset \tilde{B}^1_\Delta$,

where $\mathcal{A}_{L^2} = \left\{ f \in L^2(\mathbb{R}^n) : \int e^{\rho|\xi|}\,|\hat{f}|^2\,d\xi < \infty \text{ for some } \rho > 0 \right\}$. For $k \geq 1$ we have $B^k_\Delta \subset H^{1+\frac{k}{2}} \times H^{\frac{k}{2}}$. Moreover, see [3], the following properties hold:

3) $B^k_\Delta + B^k_\Delta = H^{1+\frac{k}{2}} \times H^{\frac{k}{2}}$,

4) $B^k_\Delta \cap \left(H^{1+\frac{k'}{2}} \times H^{\frac{k'}{2}}\right) \subset B^{k'}_\Delta$ for all $k' > k$,

5) $\mathcal{A}_{L^2} \times \mathcal{A}_{L^2} \subset B^k_\Delta$,

with strict inclusions in 4) and 5). Using a result of Paley and Wiener [8], it is also possible to show (see [2]) that $\tilde{B}^1_\Delta$, B^k_Δ do not contain compactly supported functions. Let us show, for instance, that $\tilde{B}^1_\Delta + \tilde{B}^1_\Delta = H^{\frac{3}{2}} \times H^{\frac{1}{2}}$:

Proof. Given $(u_0, u_1) \in H^{\frac{3}{2}} \times H^{\frac{1}{2}}$, we set $\rho_1 = 1$ and then, for $j \geq 1$, we inductively select $\rho_{j+1} \geq \rho_j + 1$ such that

$$e^{j\rho_j} \int_{|\xi|>\rho_{j+1}} \left[|\xi|^3 \, |\hat{u}_0(\xi)|^2 + |\xi| \, |\hat{u}_1(\xi)|^2 \right] d\xi \, \leq \, 1 \,. \tag{1.12}$$

Then, considering the characteristic function

$$\chi(\xi) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if} \quad \rho_{2j} \leq |\xi| \leq \rho_{2j+1} \quad \text{for some} \quad j \geq 1\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{1.13}$$

we define $v_i(x)$, $w_i(x)$ by setting

$$\hat{v}_i(\xi) = \chi(\xi)\, \hat{u}_i(\xi)\,, \qquad \hat{w}_i(\xi) = \big(1 - \chi(\xi)\big)\, \hat{u}_i(\xi) \tag{1.14}$$

for $i = 0, 1$. Hence, $(v_0, v_1) + (w_0, w_1) = (u_0, u_1)$. Then, using (1.12)–(1.13), it is easy to see that (v_0, v_1) satisfies condition (1.4) of Definition 1.1 for all $N \geq 0$ if we define $\tilde{\rho}_j(N) \stackrel{\text{def}}{=} \rho_{2j+1}$ for $j \geq 1$; (w_0, w_1) satisfies condition (1.4), for all $N \geq 0$, taking the sequence $\tilde{\rho}_j(N) \stackrel{\text{def}}{=} \rho_{2j}$ for $j \geq 1$. ∎

1.2 Main notation

We close this section introducing some notations which will be used in what follows.

• For $z \in \mathbb{C}$, we indicate with $\mathrm{Re}(z)$ the real part of z.

• We use $\|\cdot\|$ and $(\cdot\,,\cdot)_{L^2}$ as L^2 norm and L^2 scalar product over $\mathbb{R}^n$, i.e.

$$\|f\| = \left(\int_{\mathbb{R}^n} |f|^2 \, dx \right)^{\frac{1}{2}}, \quad (f, g)_{L^2} = \int_{\mathbb{R}^n} f \, \bar{g} \, dx \,. \tag{1.15}$$

• Given $f(x,t) : \mathbb{R}^n_x \times [0,T) \to \mathbb{C}$, we indicate with $\hat{f}(\xi, t) : \mathbb{R}^n_\xi \times [0,T) \to \mathbb{C}$ the partial Fourier transform in space variables:

$$\hat{f}(\xi, t) = (2\,\pi)^{-n/2} \int e^{-i\,x\cdot\xi} f(x,t)\, dx \,. \tag{1.16}$$

• Finally, we often denote by C (or $C_1, C_2, \dots$) various positive constants independent of $t \geq 0$, but possibly depending on γ, $m(\cdot)$, $m^{(i)}(\cdot)$ $(1 \leq i \leq k)$ and some norms of the initial data of problem (1.1)–(1.2).

2 A-priori estimates for $||u||$, $||u_t||$, $||\nabla u||$.

We recall here some known a-priori estimates for $||u(t)||$, $||u_t(t)||$ and $||\nabla u(t)||$, when u is a sufficiently regular solution of (1.1) in $\mathbb{R}^n \times [0,T)$, with $T > 0$.

As usual, we introduce the Hamiltonian function

$$\mathcal{H}(t) \stackrel{\text{def}}{=} \frac{1}{2}\,\|u_t(t)\|^2 + \frac{1}{2}\,M\big(\|\nabla u(t)\|^2\big)\,, \tag{2.1}$$

where

$$M(r) \stackrel{\text{def}}{=} \int_0^r m(\nu)\,d\nu\,. \tag{2.2}$$

For simplicity, we also set

$$s(t) \stackrel{\text{def}}{=} \|\nabla u(t)\|^2\,. \tag{2.3}$$

Besides, assuming that $m \in C^k$, for some $k \geq 1$, we introduce the constants

$$\mu_i \stackrel{\text{def}}{=} \max_{[0,2\mathcal{H}(0)/\delta]} |m^{(i)}(r)|\,, \tag{2.4}$$

for $i = 0, \ldots, k$.

Definition 2.1. *We say that u is a strong solution if $u \in C^j\big([0,T); H^{2-j}(\mathbb{R}^n)\big)$ for $j = 0,1,2$. When $T = +\infty$, we say that u is a global strong solution.*

Lemma 2.2. *Let u be a strong solution of* (1.1) *in $\mathbb{R}^n \times [0,T)$ for some $T > 0$. Then, for all $t \in [0,T)$ we have*

(i) $\mathcal{H}(t) + 2\gamma \int_0^t \|u_t\|^2\,d\tau = \mathcal{H}(0)\,.$

(ii) $\frac{\gamma}{2}\,\|u\|^2 + \int_0^t m\big(\|\nabla u\|^2\big)\,\|\nabla u\|^2\,d\tau \leq \frac{3}{2}\,\frac{\mathcal{H}(0)}{\gamma} + \big[\mathrm{Re}\,(u_t,u)_{L^2} + \gamma\,\|u\|^2\big]_{t=0}\,,$

(iii) $\int_0^t \mathcal{H}(\tau)\,d\tau \leq C\big(\mathcal{H}(0) + \|u(0)\|^2\big),$

(iv) $t\,\mathcal{H}(t) + 2\gamma \int_0^t \tau\,\|u_t\|^2\,d\tau \leq C\big(\mathcal{H}(0) + \|u(0)\|^2\big).$

where $C = C(\delta,\gamma,\mu_0) > 0$ is a suitable constant independent of T. In particular, $s(t) \leq 2\mathcal{H}(0)/\delta$ for all $t \in [0,T)$.

Proof. Since u is a strong solution, $t \to ||u_t||^2$ and $t \to ||\nabla u||^2$ are C^1 functions on $[0,T)$. Then, multiplying (1.1) by $\bar{u}_t$, $\bar{u}$ and integrating over $\mathbb{R}^n$, we find

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}M\big(\|\nabla u\|^2\big) + 2\gamma\,\|u_t\|^2 = 0\,, \tag{2.5}$$

$$\frac{d}{dt}\big[\mathrm{Re}\,(u_t,u)_{L^2} + \gamma\,\|u\|^2\big] + m(\|\nabla u\|^2)\,\|\nabla u\|^2 = \|u_t\|^2\,, \tag{2.6}$$

respectively. Integrating (2.5) over $[0,t)$ we get

$$\mathcal{H}(t) + 2\gamma \int_0^t \|u_t\|^2\,d\tau = \mathcal{H}(0)\,. \tag{2.7}$$

Hence we have (i). While, integrating (2.6), we obtain

$$\begin{aligned} \operatorname{Re}(u_t,u)_{L^2}+\gamma\,\|u\|^2+&\int_0^t m(\|\nabla u\|^2)\,\|\nabla u\|^2\,d\tau \\ &=\int_0^t \|u_t\|^2\,d\tau+\left[\operatorname{Re}(u_t,u)_{L^2}+\gamma\,\|u\|^2\right]_{t=0}. \end{aligned} \tag{2.8}$$

Since $\gamma>0$, we have $|(u_t,u)|\le \frac{1}{2\gamma}\,\|u_t\|^2+\frac{\gamma}{2}\,\|u\|^2$. Therefore, using (i), we find

$$\frac{\gamma}{2}\,\|u\|^2+\int_0^t m\big(\|\nabla u\|^2\big)\,\|\nabla u\|^2\,d\tau\le \frac{3}{2}\,\frac{\mathcal{H}(0)}{\gamma}+\left[\operatorname{Re}(u_t,u)_{L^2}+\gamma\,\|u\|^2\right]_{t=0}. \tag{2.9}$$

This completes the proof of (ii). To prove (iii), we observe that (i) gives

$$\|\nabla u(t)\|^2\le \frac{2\,\mathcal{H}(0)}{\delta}\qquad \forall\, t\in[0,T), \tag{2.10}$$

since $m(r)\ge\delta$. This implies that $\delta\,\|\nabla u\|^2\le M(\|\nabla u\|^2)\le \mu_0\,\|\nabla u\|^2$, where μ_0 is the constant defined in (2.4) for $i=0$. Hence we find

$$\mathcal{H}(t)\le \frac{1}{2}\,\|u_t(t)\|^2+\frac{\mu_0}{2}\,\|\nabla u(t)\|^2\,, \tag{2.11}$$

for all $t\in[0,T)$. Applying (i) and (ii), it follows that $\int_0^t\mathcal{H}(\tau)\,d\tau\le C\big(\mathcal{H}(0)+\|u(0)\|^2\big)$, for a suitable $C=C(\delta,\gamma,\mu_0)\ge 0$, for all $t\in[0,T)$. Thus (iii) holds. Finally, multiplying (2.5) by t^j with $j\ge 1$, we find

$$\frac{d}{dt}\big(t^j\,\mathcal{H}(t)\big)+2\gamma\,t^j\,\|u_t\|^2=j\,t^{j-1}\,\mathcal{H}(t)\,. \tag{2.12}$$

Then, setting $j=1$, we immediately deduce that

$$t\,\mathcal{H}(t)+2\gamma\int_0^t\tau\,\|u_t\|^2\,d\tau=\int_0^t\mathcal{H}(\tau)\,d\tau. \tag{2.13}$$

Hence (iv) follows from (iii). ∎

A close inspection of the proof of Lemma 2.2 reveals that (i)–(iv) above also remain valid under slight weaker hypotheses on the regularity of u.

Lemma 2.3. *The statements* (i), (ii), (iii), (iv) *continue to hold for a solution u of* (1.1) *such that $u\in C^j\big([0,T);H^{r-j}(\mathbb{R}^n)\big)$ $(j=0,1,2\,)$, for some $r\ge 3/2$.*

Proof. It is sufficient to prove that (i)–(iv) are valid when $u\in C^j\big([0,T);H^{\frac{3}{2}-j}(\mathbb{R}^n)\big)$ $(j=0,1,2\,)$. Then, we consider the Hilbert triple

$$H^{\frac{1}{2}}\hookrightarrow L^2\hookrightarrow H^{-\frac{1}{2}}\,, \tag{2.14}$$

denoting with $\langle\cdot,\cdot\rangle\overset{\text{def}}{=}\langle\cdot,\cdot\rangle_{H^{-1/2},H^{1/2}}$ the duality between $H^{-\frac{1}{2}}$ and $H^{\frac{1}{2}}$ which extends the scalar product in L^2. Then, since $(\cdot,\cdot)_{L^2}=\langle\cdot,\cdot\rangle$ on $L^2\times H^{\frac{1}{2}}$, it is easy to verify that the following identities hold: $\langle u_t, u_t\rangle = \|u_t\|^2$, $-\langle \Delta u, u\rangle = \|\nabla u\|^2$ and

$$\frac{d}{dt}\|u_t\|^2 = 2\,\mathrm{Re}\,\langle u_{tt}, u_t\rangle\,,$$
$$\frac{d}{dt}\|\nabla u\|^2 = -2\,\mathrm{Re}\,\langle \Delta u, u_t\rangle\,,$$
$$\frac{d}{dt}(u, u_t)_{L^2} = \|u_t\|^2 + \langle u_{tt}, u\rangle\,,$$
$$\frac{d}{dt}||u||^2 = 2\,\mathrm{Re}\,(u_t, u) = 2\,\mathrm{Re}\,\langle u_t, u\rangle\,.$$

Therefore, the identities (2.5) and (2.6) continue to hold even if we merely suppose $u \in C^j\big([0,T); H^{\frac{3}{2}-j}(\mathbb{R}^n)\big)$ $(j = 0,1,2\,)$. Hence we can derive (i)–(iv) as above. ∎

Remark 2.4. *If u satisfies the assumptions of Lemma 2.2 or 2.3, recalling definition* (1.6), *we have*

$$2\min\{1, \mu_0^{-1}\}\,\mathcal{H}(t) \le E_0(u;t) \le 2\max\{1, \delta^{-1}\}\,\mathcal{H}(t)\,, \tag{2.15}$$

for all $t \in [0,T)$. *Hence* $\mathcal{H}(t) \approx E_0(u;t)$.

When u is a global solution of (1.1), it follows from Lemmas 2.2, 2.3 that $\|\nabla u(t)\|$, $\|u_t(t)\| \to 0$ as $t \to \infty$. Applying (ii)–(iv), we can also prove:

Proposition 2.5. *Let* $u \in C^j\big([0,\infty); H^{r-j}(\mathbb{R}^n)\big)$ $(j = 0,1,2\,)$, $r \ge 3/2$, *be a global solution of equation* (1.1). *Then* $\|u(t)\| \to 0$ *as* $t \to \infty$.

Proof. By Lemmas 2.2, 2.3, we know that $\|\nabla u(t)\| \to 0$ as $t \to \infty$. Therefore

$$\lim_{t\to\infty}\int_{|\xi|\ge\varepsilon} |\hat{u}(\xi,t)|^2\,d\xi = 0\,, \tag{2.16}$$

for all $\varepsilon > 0$. Hence it remains to show that

$$\int_{|\xi|\le\rho} |\hat{u}(\xi,t)|^2\,d\xi \to 0 \quad \text{as } t \to \infty\,, \tag{2.17}$$

for some $\rho > 0$. To this end, writing

$$m(0) = a_o\,, \quad b(t) = m(\|\nabla u(t)\|^2) - m(0)\,, \tag{2.18}$$

we note that $\hat{u}(\xi,\cdot)$ satisfies the ordinary problem

$$\hat{u}_{tt} + (a_o + b(t))\,|\xi|^2\,\hat{u} + 2\gamma\,\hat{u}_t = 0\,, \qquad t \ge 0\,, \tag{2.19}$$
$$\hat{u}(\xi,0) = \hat{u}_0(\xi)\,, \quad \hat{u}_t(\xi,0) = \hat{u}_1(\xi)\,, \tag{2.20}$$

with a parameter $\xi \in \mathbb{R}^n$. Now, by condition (1.3) and (ii) of Lemma 2.2, we have $a_o > 0$, $\gamma > 0$ and

$$\int_0^\infty |b(t)|\,dt \,\le\, \mu_1\int_0^\infty \|\nabla u(t)\|^2\,dt < \infty\,, \tag{2.21}$$

where μ_1 is defined in (2.4). Then, to prove (2.17), it suffices to apply Lemma 10.1 of APPENDIX II and the Lebesgue dominated-convergence theorem. ∎

2.1 Application of Lemmas 2.2, 2.3 to the global solvability

As it is well-known (see [1], [9], [11]) when (1.3) holds problem (1.1)–(1.2) is well posed in $H^r \times H^{r-1}$, for $r \geq 3/2$. More precisely, given $(u_0, u_1) \in H^r \times H^{r-1}$, with $r \geq 3/2$, there exists a unique local solution $u \in C^j([0,T); H^{r-j}(\mathbb{R}^n))$ $(j = 0,1,2)$ for some $T > 0$. Besides, if T is maximal, then $T = +\infty$ or

$$\limsup_{t \to T^-} \Big(\|u(\cdot,t)\|_{H^r} + \|u_t(\cdot,t)\|_{H^{r-1}} \Big) = +\infty . \tag{2.22}$$

Since $\tilde{B}^1_\Delta \subset H^{\frac{3}{2}} \times H^{\frac{1}{2}}$ and $B^k_\Delta \subset H^{\frac{k}{2}+1} \times H^{\frac{k}{2}}$ for $k \geq 1$, by Lemmas 2.2, 2.3, to prove the global solvability of problem (1.1)–(1.2) for $\tilde{B}^1_\Delta$ (resp. B^k_Δ) initial data we only need to show that, independently of $T \in (0, \infty)$,

$$\sup_{t \in [0,T)} E_1(u;t) < +\infty \quad (\text{resp.} \quad \sup_{t \in [0,T)} E_k(u,t) < \infty) . \tag{2.23}$$

3 Global existence and decay for $\tilde{B}^1_\Delta$ data

By Fourier transform in space variables, (1.1) is equivalent to the following infinite system of second order equations:

$$\hat{u}_{tt} + m \left(\int |\xi|^2 |\hat{u}(\xi,t)|^2 \, d\xi \right) |\xi|^2 \, \hat{u} + 2\gamma \, \hat{u}_t = 0 , \quad t \geq 0 , \tag{3.1}$$

depending on $\xi \in \mathbb{R}^n$. As remarked in § 2.1, to prove the global solvability when $(u_0, u_1) \in \tilde{B}^1_\Delta$ we need only to show that $E_1(u;t)$ cannot blow-up in finite time. To this end, we begin by considering the quadratic form

$$q(\xi,t) = e^{2\tilde{\gamma}t} \Big(|\xi| \, |\hat{u}_t|^2 + m \, |\xi|^3 |\hat{u}|^2 + \alpha \, |\xi| \, \mathrm{Re}(\bar{\hat{u}} \, \hat{u}_t) \Big) , \tag{3.2}$$

where $\tilde{\gamma}$, $\alpha \in \mathbb{R}$ are constants that we shall choose in the following. Deriving $q(\xi,t)$ with respect to t, we easily find the expression

$$\begin{aligned} \frac{dq}{dt} &= (2\tilde{\gamma} + \alpha - 4\gamma) \, e^{2\tilde{\gamma}t} \, |\xi| \, |\hat{u}_t|^2 \\ &\quad + (2\tilde{\gamma} - \alpha) \, e^{2\tilde{\gamma}t} \, m \, |\xi|^3 |\hat{u}|^2 \\ &\quad + (2\tilde{\gamma} - 2\gamma) \, \alpha \, e^{2\tilde{\gamma}t} \, |\xi| \, \mathrm{Re}(\bar{\hat{u}} \, \hat{u}_t) \\ &\quad + e^{2\tilde{\gamma}t} \, m' \, s' \, |\xi|^3 \, |\hat{u}|^2 , \end{aligned} \tag{3.3}$$

where $m' = m'(s(t))$. Then we select $\tilde{\gamma}$, α such that

$$\begin{cases} q \geq \frac{1}{2} e^{2\tilde{\gamma}t} \left(|\xi| \, |\hat{u}_t|^2 + m \, |\xi|^3 |\hat{u}|^2 \right) , \\ q' \leq e^{2\tilde{\gamma}t} \, m' \, s' \, |\xi|^3 \, |\hat{u}|^2 , \end{cases} \tag{3.4}$$

for $|\xi|$ sufficiently large. A simple choice of $\tilde{\gamma}$, α is the following:

$$\tilde{\gamma} = \gamma , \quad \alpha = 2\gamma . \tag{3.5}$$

In fact, we obtain the identity $q' = e^{2\gamma t}\, m'\, s'\, |\xi|^3\, |\hat{u}|^2$ for all $\xi \in \mathbb{R}^n$. Besides, having $m(r) \geq \delta > 0$, the first condition of (3.4) is certainly verified as soon as

$$|\xi| \geq \frac{2\gamma}{\sqrt{\delta}}. \tag{3.6}$$

Definition 3.1. *Let* $u \in C^j\big([0,T); H^{\frac{3}{2}-j}(\mathbb{R}^n)\big)$ $(j = 0,1,2)$ *be a solution of* (1.1) *in* $\mathbb{R}^n \times [0,T)$ *for some* $T > 0$. *We define*

$$\tilde{\mathcal{E}}(\xi,t) \stackrel{\text{def}}{=} e^{2\gamma t}\Big(|\xi|\, |\hat{u}_t|^2 + m\, |\xi|^3 |\hat{u}|^2 + 2\gamma\, |\xi|\, \mathrm{Re}(\bar{\hat{u}}\, \hat{u}_t)\Big). \tag{3.7}$$

Thus, for $|\xi| \geq \frac{2\gamma}{\sqrt{\delta}}$, both the conditions of (3.4) are verified with $q = \tilde{\mathcal{E}}$; moreover, in the second one the equality holds.

Proof of Theorem 1.2 for k = 1

Let (u_0, u_1) be a given initial data in $\tilde{B}^1_\Delta$ and let $u \in C^j\big([0,T); H^{\frac{3}{2}-j}(\mathbb{R}^n)\big)$ $(j = 0,1,2)$ be the corresponding unique solution of problem (1.1)–(1.2) in $\mathbb{R}^n \times [0,T)$, for some $T > 0$. Without loss of generality we may suppose T maximal. Taking account of Lemmas 2.2 and 2.3, we may select $N > 0$ so large that, independently of T, one has

$$N \geq \frac{4\,\mu_1}{\delta^{3/2}} \int_0^T \big[\mathcal{H}(t) + e^{-2\gamma t}\big]\, dt. \tag{3.8}$$

Besides, by Definition 1.1 of $\tilde{B}^1_\Delta$ and Definition 3.1 of $\tilde{\mathcal{E}}(\xi,t)$, there exists a sequence of positive numbers

$$\tilde{\rho}_j = \tilde{\rho}_j(N) \quad \text{for} \quad j \geq 1, \tag{3.9}$$

such that $\tilde{\rho}_j \to +\infty$ and

$$\sup_{j\geq 1}\; e^{N\tilde{\rho}_j} \int_{|\xi|>\rho_j} \tilde{\mathcal{E}}(\xi,0)\, d\xi < \infty. \tag{3.10}$$

Now, from (3.4)–(3.6), we have $\tilde{\mathcal{E}}\,' = e^{2\gamma t}\, m'\, s'\, |\xi|^3\, |\hat{u}|^2$ and

$$|\tilde{\mathcal{E}}\,'(\xi,t)| \leq \frac{2\,\mu_1}{\delta}\, |s'(t)|\; \tilde{\mathcal{E}}(\xi,t), \tag{3.11}$$

for $|\xi| \geq \frac{2\gamma}{\sqrt{\delta}}$. Furthermore, see also (4.7) below, we easily have

$$|s'(t)| \leq \frac{2\,\rho\,\mathcal{H}(t)}{\sqrt{\delta}} + \frac{2\,e^{-2\gamma t}}{\sqrt{\delta}} \int_{|\xi|>\rho} \tilde{\mathcal{E}}(\xi,t)\, d\xi, \tag{3.12}$$

for all $\rho \geq \frac{2\gamma}{\sqrt{\delta}}$. Hence, for $t \in [0,T)$, we obtain

$$|\tilde{\mathcal{E}}\,'| \leq \frac{4\,\rho\,\mu_1}{\delta^{3/2}} \Big(\mathcal{H}(t) + e^{-2\gamma t}\, \frac{\tilde{\mathcal{E}}^\rho}{\rho}\Big)\, \tilde{\mathcal{E}} \quad \text{for} \quad \rho,\, |\xi| \geq \frac{2\gamma}{\sqrt{\delta}}, \tag{3.13}$$

where

$$\tilde{\mathcal{E}}^{\rho}(t) \stackrel{\text{def}}{=} \int_{|\xi|>\rho} \tilde{\mathcal{E}}(\xi,t)\,d\xi\,. \tag{3.14}$$

Now, for $\rho \geq \frac{2\gamma}{\sqrt{\delta}}$, we define

$$\tilde{T}(\rho) \stackrel{\text{def}}{=} \sup\Big\{\tau:\ 0 \leq \tau < T\,,\ \tilde{\mathcal{E}}^{\rho}(t) \leq \rho \quad \forall\, t \in [0,\tau)\Big\}. \tag{3.15}$$

It is clear that $\tilde{T}(\rho) > 0$ provided ρ is large enough. Moreover, recalling (3.8), we derive the a-priori estimate

$$\tilde{\mathcal{E}}(\xi,t) \leq \tilde{\mathcal{E}}(\xi,0)\; e^{N\rho}\,, \tag{3.16}$$

for all $t \in [0, \tilde{T}(\rho))$ and $|\xi| \geq \frac{2\gamma}{\sqrt{\delta}}$. From this we obtain

$$\frac{\tilde{\mathcal{E}}^{\rho}(t)}{\rho} \;\leq\; \frac{e^{N\rho}}{\rho} \int_{|\xi|>\rho} \tilde{\mathcal{E}}(\xi,0)\,d\xi \quad \text{in} \quad [0,\tilde{T}(\rho))\,, \tag{3.17}$$

for all $\rho \geq \frac{2\gamma}{\sqrt{\delta}}$ large enough. Finally, by (3.9)–(3.10), there exists an integer $j_0 \geq 1$ such that: $\tilde{\rho}_j \geq \frac{2\gamma}{\sqrt{\delta}}$ and

$$\frac{e^{N\tilde{\rho}_j}}{\tilde{\rho}_j} \int_{|\xi|>\tilde{\rho}_j} \tilde{\mathcal{E}}(\xi,0)\,d\xi \;\leq\; \frac{1}{2}\,, \tag{3.18}$$

for all $j \geq j_0$. This means that, taking $\rho = \tilde{\rho}_j$ with $j \geq j_0$, we have

$$\tilde{\mathcal{E}}^{\tilde{\rho}_j}(t) \;\leq\; \frac{1}{2}\,\tilde{\rho}_j\,, \qquad \forall\, t \in \big[0,\, \tilde{T}(\tilde{\rho}_j)\big)\,. \tag{3.19}$$

By the definition (3.15) of $\tilde{T}(\rho)$, it follows that $\tilde{T}(\tilde{\rho}_j) = T$, when $j \geq j_0$, and that $E_1(u;t)$ is uniformly bounded in $[0,T)$ because (3.19) implies

$$E_1(u;t) \leq\; C\,\tilde{\rho}_j\,\Big(\mathcal{H}(t) + e^{-2\gamma t}\Big) \quad \text{in} \quad [0,T)\,, \tag{3.20}$$

for all $j \geq j_0$, with $C = 2\max\{1,\delta^{-1}\}$. Since we are assuming T maximal, it follows that

$$T = \infty \tag{3.21}$$

and, consequently, that $u \in C^j\big([0,\infty); H^{\frac{3}{2}-j}(\mathbb{R}^n)\big)$ $(j = 0,1,2)$ is a global solution of (1.1). Finally, using Lemmas 2.2, 2.3, we can easily deduce (1.7) in the case $k = 1$. In fact, by (3.19), we have

$$E_r(u;t) \leq\; C\,(\tilde{\rho}_{j_0})^r\,\big[\mathcal{H}(t) + e^{-2\gamma t}\big] \tag{3.22}$$

for $0 \leq r \leq 1$, where C is the same constant of (3.20). Thus

$$tE_r(u;t) + \int_0^t E_r(u;\tau)\,d\tau \leq\; C\,(\tilde{\rho}_{j_0})^r\left(\frac{1}{\gamma} + t\mathcal{H} + \int_0^t \mathcal{H}\,d\tau\right)\,, \tag{3.23}$$

for all $t \geq 0$ and for all $r \in [0,1]$. ∎

4 Second order form for strong solutions

Let u be a strong solution of (1.1) in $\mathbb{R}^n \times [0, T)$ for some $T > 0$. Assuming (1.3) with $m \in C^2[0, \infty)$ and taking account of the results of APPENDIX I on damped linear wave equations, we introduce the following quadratic forms.

For $\xi \in \mathbb{R}^n$ and $t \in [0, T)$, we set

$$\begin{aligned} \mathcal{Q}(\xi, t) &\stackrel{\text{def}}{=} e^{2\gamma t} \Big(\frac{\sqrt{m}}{2} |\xi|^4 |\hat{u}|^2 + \frac{1}{2\sqrt{m}} |\xi|^2 |\hat{u}_t|^2 \Big) \\ &\qquad + e^{2\gamma t} \Big(\frac{\gamma}{\sqrt{m}} + \frac{m' s'}{4 m^{3/2}} \Big) |\xi|^2 \operatorname{Re}(\bar{\hat{u}}\, \hat{u}_t), \end{aligned} \tag{4.1}$$

$$\mathcal{E}(\xi, t) \stackrel{\text{def}}{=} e^{2\gamma t} \Big(\frac{\sqrt{m}}{2} |\xi|^4 |\hat{u}|^2 + \frac{1}{2\sqrt{m}} |\xi|^2 |\hat{u}_t|^2 \Big), \tag{4.2}$$

where $m = m(s(t))$ and $m' = m'(s(t))$. Deriving $\mathcal{Q}$ with respect to t, from (3.1) (or (9.8)–(9.9), with obvious substitutions) we easily obtain the identity

$$\mathcal{Q}' = e^{2\gamma t} \Big(\frac{\gamma}{\sqrt{m}} + \frac{m' s'}{4 m^{3/2}} \Big)' |\xi|^2 \operatorname{Re}(\bar{\hat{u}}\, \hat{u}_t). \tag{4.3}$$

By (i) of Lemma 2.2, $\mathcal{H}(t) \le \mathcal{H}(0)$ for all $t \in [0, T)$. Moreover, we have:

$$\int |\hat{u}_t|^2 \, d\xi + \delta \int |\xi|^2 |\hat{u}|^2 \, d\xi \le 2\, \mathcal{H}(t). \tag{4.4}$$

Lemma 4.1. *Let u be a strong solution of* (1.1) *in $\mathbb{R}^n \times [0, T)$. Then $s(t) \in C^2$ and for all $\rho > 0$ the following estimates holds:*

$$|s'(t)| \le \frac{2\rho\, \mathcal{H}(t)}{\sqrt{\delta}} + 2 e^{-2\gamma t} \int_{|\xi| > \rho} \frac{\mathcal{E}(\xi, t)}{|\xi|} \, d\xi, \tag{4.5}$$

$$\begin{aligned} |s''(t)| &\le 4\rho^2 \Big(1 + \frac{\mu_0}{\delta} + \frac{\gamma}{\rho \sqrt{\delta}} \Big) \mathcal{H}(t) \\ &\qquad + 4 e^{-2\gamma t} \int_{|\xi| > \rho} \mathcal{E}(\xi, t) \Big(\sqrt{\mu_0} + \frac{\gamma}{|\xi|} \Big) d\xi. \end{aligned} \tag{4.6}$$

Proof. It is immediate that $s(t) \in C^2$, when u is a strong solution. Now, for any $\rho > 0$, we have:

$$\begin{aligned} |s'(t)| &= 2 \left| \int |\xi|^2 \operatorname{Re}(\bar{\hat{u}}\, \hat{u}_t) \, d\xi \right| \\ &\le 2 \int_{|\xi| \le \rho} |\xi|^2 |\hat{u}| \, |\hat{u}_t| \, d\xi + 2 \int_{|\xi| > \rho} |\xi|^2 |\hat{u}| \, |\hat{u}_t| \, d\xi \\ &\le \frac{1}{\sqrt{\delta}} \int_{|\xi| \le \rho} \Big(|\xi| \, |\hat{u}_t|^2 + \delta |\xi|^3 |\hat{u}|^2 \Big) d\xi \\ &\quad + 2 \int_{|\xi| > \rho} \Big(\frac{\sqrt{m}}{2} |\xi|^3 |\hat{u}|^2 + \frac{1}{2\sqrt{m}} |\xi| \, |\hat{u}_t|^2 \Big) d\xi \\ &\le \frac{2\rho\, \mathcal{H}(t)}{\sqrt{\delta}} + 2 e^{-2\gamma t} \int_{|\xi| > \rho} \frac{\mathcal{E}(\xi, t)}{|\xi|} \, d\xi. \end{aligned} \tag{4.7}$$

To estimate $|s''(t)|$, we observe that (3.1) gives the identity

$$s''(t) = 2\int |\xi|^2 \Big(|\hat{u}_t|^2 - m(s(t))\,|\xi|^2\,|\hat{u}|^2 - 2\gamma\,\mathrm{Re}(\bar{\hat{u}}\,\hat{u}_t)\Big)\,d\xi. \tag{4.8}$$

Applying the same reasoning as above, we find

$$\int |\xi|^2\,|\hat{u}_t|^2\,d\xi \le 2\rho^2\,\mathcal{H}(t) + 2\sqrt{\mu_0}\,e^{-2\gamma t}\int_{|\xi|>\rho} \mathcal{E}(\xi,t)\,d\xi\,, \tag{4.9}$$

$$m(s(t))\int |\xi|^4\,|\hat{u}|^2\,d\xi \le \frac{2\,\mu_0}{\delta}\,\rho^2\,\mathcal{H}(t) + 2\sqrt{\mu_0}\,e^{-2\gamma t}\int_{|\xi|>\rho} \mathcal{E}(\xi,t)\,d\xi\,. \tag{4.10}$$

Thus, having (4.7) and the inequalities (4.9)–(4.10), we easily get (4.6). ∎

For simplicity of notation, we introduce the quantities:

Definition 4.2. *For $t \in [0,T)$, we set*

$$\mathcal{E}^\rho(t) \stackrel{\text{def}}{=} \int_{|\xi|>\rho} \mathcal{E}(\xi,t)\,d\xi\,, \tag{4.11}$$

$$J_\rho(t) \stackrel{\text{def}}{=} \mathcal{H}(t) + e^{-2\gamma t}\,\rho^{-2}\,\mathcal{E}^\rho(t) \quad (\rho>0)\,. \tag{4.12}$$

Corollary 4.3. *Let u be a strong solution of* (1.1). *Then there exist constants $C_1 = C_1(\delta)$ and $C_2 = C_2(\delta,\gamma,\mu_0)$ such that for $t \in [0,T)$ one has*

$$|s'(t)| \le C_1\,\rho\,J_\rho(t)\,, \quad \forall\,\rho>0\,, \tag{4.13}$$

$$|s''(t)| \le C_2\,\rho^2\,J_\rho(t)\,, \quad \forall\rho\ge 1\,. \tag{4.14}$$

Proof. Inequality (4.13) is an immediate consequence of (4.5) and Definition 4.2. To verify (4.14), it is enough to observe that, when $\rho \ge 1$, (4.6) gives

$$\begin{aligned} |s''| \le\ & 4\rho^2\Big(1 + \frac{\mu_0}{\delta} + \frac{\gamma}{\sqrt{\delta}}\Big)\,\mathcal{H} \\ & + 4\,e^{-2\gamma t}\,(\sqrt{\mu_0}+\gamma)\int_{|\xi|>\rho} \mathcal{E}\,d\xi\,. \end{aligned} \tag{4.15}$$

Then (4.14) follows from (4.15) and Definition 4.2. ∎

Now we can easily estimate $\mathcal{Q}-\mathcal{E}$, $\mathcal{Q}'$ and $\mathcal{E}'$. In fact, with the same assumptions of the Lemma 4.1 and Corollary 4.3, we have:

Lemma 4.4. *There exist positive constants $C_3(\delta,\mu_1)$, $C_4(\delta,\gamma,\mu_1,\mu_2)$, $C_5(\delta,\gamma,\mu_1)$ such that for $|\xi|>0$ and $t\in[0,T)$*

$$|(\mathcal{Q}-\mathcal{E})| \le C_3\,\rho\,\Big(\frac{\gamma}{\rho} + J_\rho\Big)\,\frac{\mathcal{E}}{|\xi|}\,, \quad \forall\rho>0\,, \tag{4.16}$$

$$|\mathcal{Q}'| \le C_4\,\rho^2\,J_\rho\,(1+J_\rho)\,\frac{\mathcal{E}}{|\xi|}\,, \quad \forall\,\rho\ge 1\,, \tag{4.17}$$

$$|\,\mathcal{E}'| \le C_5\,\rho\,\Big(\frac{\gamma}{\rho} + J_\rho\Big)\,\mathcal{E}\,, \quad \forall\,\rho>0\,. \tag{4.18}$$

Proof. From (4.1)–(4.2) and (4.5), for $|\xi| > 0$ one has

$$
\begin{aligned}
|\mathcal{Q} - \mathcal{E}| &\le \left| \frac{\gamma}{\sqrt{m}} + \frac{m'(s)\, s'}{4\, m^{3/2}} \right| \frac{\mathcal{E}}{|\xi|} \\
&\le \rho \left(\frac{\gamma}{\rho\sqrt{\delta}} + \frac{\mu_1}{2\,\delta^2}\,\mathcal{H} + \frac{\mu_1}{2\,\delta^{3/2}}\, e^{-2\gamma t}\, \frac{\mathcal{E}\rho}{\rho^2} \right) \frac{\mathcal{E}}{|\xi|}\,,
\end{aligned}
\tag{4.19}
$$

for all $\rho > 0$. Hence (4.16) is verified. In the same way, we can show that (4.17) holds. In fact, from (4.3) and Corollary 4.3, for $|\xi| > 0$ and $\rho \ge 1$ we have

$$
\begin{aligned}
|\mathcal{Q}'| \;&\le \left| -\frac{\gamma\, m'\, s'}{2\, m^{3/2}} + \frac{m'(s)\, s'' + m''\, s'^2}{4\, m^{3/2}} - \frac{3}{8}\frac{m'^2\, s'^2}{m^{5/2}} \right| \frac{\mathcal{E}}{|\xi|} \\
&\le C\Big(\rho\, J_\rho + \rho^2\, J_\rho + \rho^2\, J_\rho^2 \Big) \frac{\mathcal{E}}{|\xi|} \\
&\le C\, \rho^2 \left(J_\rho + J_\rho^2 \right) \frac{\mathcal{E}}{|\xi|}\,.
\end{aligned}
\tag{4.20}
$$

Hence (4.17) is proved. Finally, deriving $\mathcal{E}$ with respect to t, we find

$$
\begin{aligned}
\mathcal{E}' \;=\; & 2\gamma\, \mathcal{E} + e^{2\gamma t} \left(\frac{m'\, s'}{4\sqrt{m}}\, |\xi|^4\, |\hat{u}|^2 - \frac{m'\, s'}{4\, m^{3/2}}\, |\xi|^2\, |\hat{u}_t|^2 \right) \\
& - 2\,\gamma\, e^{2\gamma t}\, \frac{1}{\sqrt{m}} |\xi|^2\, |\hat{u}_t|^2
\end{aligned}
\tag{4.21}
$$

Hence we have

$$
|\,\mathcal{E}'\,| \;\le\; \Big(2\gamma + \frac{\mu_1}{2\delta}\, |s'| \Big)\, \mathcal{E}\,. \tag{4.22}
$$

Then, using (4.13), we immediately obtain (4.18). ∎

5 Global existence for B^2_Δ data

We will prove here the first part of Theorem 1.2 for $k = 2$; namely, the global existence of the strong solution u, when $m \in C^2[0,\infty)$ and $(u_0, u_1) \in B^2_\Delta$.

As observed in § 2.1, it is sufficient to show that $E_2(u;t)$ remains bounded in every finite interval $[0, T)$. To begin with, since $(u_0, u_1) \in B^2_\Delta$, there exist $\eta > 0$ and a sequence $\{\rho_j\}_{j \ge 1}$ such that $\rho_j > 0$, $\lim\limits_j \rho_j = +\infty$ and

$$
\Upsilon \stackrel{\text{def}}{=} \sup_j \int_{|\xi| > \rho_j} \left(\frac{\sqrt{\mu_0}}{2}\, |\xi|^4\, |\hat{u}_0(\xi)|^2 + \frac{|\xi|^2\, |\hat{u}_1(\xi)|^2}{2\sqrt{\delta}} \right) \frac{e^{\eta\, \rho_j^2 / |\xi|}}{\rho_j^2}\, d\xi \;<\; \infty\,. \tag{5.1}
$$

Further, we consider u in the stripe $\mathbb{R}^n \times [T - \varepsilon, T)$, where $\varepsilon \in (0, T]$ is a parameter that we will fix in the following. For $T - \varepsilon \le t < T$, we have

$$
\mathcal{E}(\xi, t) \;=\; \mathcal{E}(\xi, T - \varepsilon) - \Big[(\mathcal{Q} - \mathcal{E})(\xi, \tau) \Big]_{T-\varepsilon}^{t} + \int_{T-\varepsilon}^{t} \mathcal{Q}'(\xi, \tau)\, d\tau\,. \tag{5.2}
$$

To estimate the right-hand side of (5.2), we apply the following simplified form of the inequalities of Lemma 4.4. For $\rho \geq 1$ and $|\xi| > 0$, we have

$$|\mathcal{Q} - \mathcal{E}| \leq C_3 \left[\Gamma + \frac{\mathcal{E}^\rho}{\rho^2} \right] \frac{\rho}{|\xi|} \mathcal{E}\,, \tag{5.3}$$

$$|\mathcal{Q}'| \leq C_4\, p_2\Big(\Gamma + \frac{\mathcal{E}^\rho}{\rho^2} \Big) \frac{\rho^2}{|\xi|} \mathcal{E}\,, \tag{5.4}$$

$$|\,\mathcal{E}'| \leq C_5\, \rho \left[\Gamma + \frac{\mathcal{E}^\rho}{\rho^2} \right] \mathcal{E} \tag{5.5}$$

where $p_2(r) \overset{\text{def}}{=} r + r^2$ and

$$\Gamma \overset{\text{def}}{=} \gamma + \mathcal{H}(0)\,. \tag{5.6}$$

Since $\mathcal{H}(0) \geq 0$ and $\gamma > 0$, it is clear that $\Gamma > 0$ (note also that $\mathcal{H}(0) = 0$ implies $u \equiv 0$). Next, we set

$$\lambda = 4\, C_3\, \Gamma + 1\,, \tag{5.7}$$

and then we write

$$\mathcal{E}^\rho(t) = \mathcal{E}_\rho^{\lambda\rho}(t) + \mathcal{E}^{\lambda\rho}(t)\,, \tag{5.8}$$

where, obviously,

$$\mathcal{E}_\rho^{\lambda\rho}(t) \overset{\text{def}}{=} \int_{\rho < |\xi| \leq \lambda\,\rho} \mathcal{E}(\xi, t)\, d\xi\,. \tag{5.9}$$

Assuming from now on $\rho \geq 1$, from (5.3) and (5.7), we have

$$|(\mathcal{Q} - \mathcal{E})(\xi, t)| < \frac{\mathcal{E}(\xi, t)}{2} \quad \text{for} \quad |\xi| \geq \lambda\, \rho \tag{5.10}$$

and $t \in [T - \varepsilon, T)$ such that the quantities $\mathcal{E}_\rho^{\lambda\rho}(t)$, $\mathcal{E}_{\lambda\rho}(t)$ satisfy

$$\text{(a)}\ \frac{\mathcal{E}_\rho^{\lambda\rho}(t)}{\rho^2} < \frac{\Gamma}{2}\,, \qquad \text{(b)}\ \frac{\mathcal{E}^{\lambda\rho}(t)}{\rho^2} < \frac{\Gamma}{2}\,. \tag{5.11}$$

Now we choose $\varepsilon > 0$ and $\tilde{\rho} \geq 1$ such that

$$2\, C_5\, \rho\, \Gamma\, \varepsilon + \ln\Big(\frac{4\,\Upsilon + 1}{\Gamma} \Big) \leq \frac{\eta\, \rho}{\lambda} \quad \forall\, \rho \geq \tilde{\rho}\,, \tag{5.12}$$

$$2\, C_4\, p_2(2\,\Gamma)\, \varepsilon \leq \frac{\eta}{2}\,, \tag{5.13}$$

where Υ, η are the constants introduced in (5.1). Besides, noting that $\mathcal{E}^\rho(T - \varepsilon) \to 0$ as $\rho \to +\infty$, we may also suppose that

$$\frac{\mathcal{E}_\rho^{\lambda\rho}(T - \varepsilon)}{\rho^2} \leq \frac{\Gamma}{4}\,, \quad \frac{\mathcal{E}^{\lambda\rho}(T - \varepsilon)}{\rho^2} \leq \frac{\Gamma}{4} \quad \forall\, \rho \geq \tilde{\rho}\,. \tag{5.14}$$

Thanks to (5.14), for every $\rho \geq \tilde{\rho}$ conditions (a), (b) of (5.11) are both verified in some maximal right neighborhood of $T - \varepsilon$, say

$$\big[T - \varepsilon, \hat{T} \big)\,, \quad \text{where} \quad \hat{T} = \hat{T}(\rho) \tag{5.15}$$

is maximal and, clearly, $T - \varepsilon < \hat{T}(\rho) \leq T$. In the sequel we will prove that $\hat{T}(\rho_j) = T$, provided ρ_j is a sufficiently large element of the sequence $\{\rho_j\}_{j \geq 1}$.

Estimate of $\mathcal{E}_{\rho}^{\lambda\rho}(t)$

Since $\mathcal{E}'$ satisfies inequality (5.5), taking $\varrho(T-\varepsilon) \geq 1$ according to Lemma 9.2 of APPENDIX I, i.e. such that

$$\mathcal{E}(\xi, T-\varepsilon) \leq 2\,\mathcal{E}(\xi, 0) \quad \text{for} \quad |\xi| \geq \varrho(T-\varepsilon)\,, \tag{5.16}$$

we have

$$\mathcal{E}(\xi, t) \leq 2\,\mathcal{E}(\xi, 0)\, \exp\Big\{C_5\,\rho \int_{T-\varepsilon}^{t} \Big[\Gamma + \frac{\mathcal{E}^{\rho}}{\rho^2}\Big]\, d\tau\Big\}\,, \tag{5.17}$$

for all $|\xi| \geq \varrho(T-\varepsilon)$ and $t \in [T-\varepsilon, T)$. Besides, by definition, $\dfrac{\mathcal{E}^{\rho}(t)}{\rho^2} < \Gamma$ in the interval $[T-\varepsilon, \hat{T}(\rho))$ when $\rho \geq \tilde{\rho}$. Hence we find

$$\int_{T-\varepsilon}^{t} \Big[\Gamma + \frac{\mathcal{E}^{\rho}}{\rho^2}\Big]\, d\tau \leq 2\,\Gamma\,\varepsilon\,, \tag{5.18}$$

for $t \in [T-\varepsilon, \hat{T}(\rho))$, provided $\rho \geq \tilde{\rho}$. Thus, for $\rho = \rho_j$ with $\rho_j \geq \max\{\tilde{\rho}, \varrho(T-\varepsilon)\}$, from (5.12), (5.17)-(5.18) and the definition of Υ, we have

$$\begin{aligned} \frac{\mathcal{E}_{\rho_j}^{\lambda\rho_j}(t)}{\rho_j^2} &\leq \int_{\rho_j<|\xi|\leq\lambda\rho_j} \frac{2\,\mathcal{E}(\xi,0)}{\rho_j^2}\, \exp\Big\{2\,C_5\,\rho_j\,\Gamma\,\varepsilon\Big\}\, d\xi \\ &\leq \frac{\Gamma}{4\,\Upsilon+1} \int_{\rho_j<|\xi|\leq\lambda\rho_j} \frac{2\,\mathcal{E}(\xi,0)}{\rho_j^2}\, \exp\Big\{\frac{\eta\rho_j^2}{|\xi|}\Big\}\, d\xi \\ &\leq \Gamma\,\frac{2\,\Upsilon}{4\,\Upsilon+1} < \frac{\Gamma}{2}\,, \end{aligned} \tag{5.19}$$

for all $t \in [T-\varepsilon, \hat{T}(\rho_j))$. This means that, for $\rho = \rho_j$ with $\rho_j \geq \max\{\tilde{\rho}, \varrho(T-\varepsilon)\}$, condition (a) in (5.11) is always verified as long as condition (b) holds and, in conclusion, it only remains to prove that for $\rho = \rho_j$, with j large enough, (5.11) (b) holds for all $t \in [T-\varepsilon, T)$.

Estimate of $\mathcal{E}^{\lambda\rho}(t)$

From (5.2)–(5.4) and (5.10), for every fixed $\rho \geq \tilde{\rho}$ and for all $t \in [T-\varepsilon, \hat{T}(\rho))$ we have the inequality

$$\mathcal{E}(\xi, t) \leq 3\,\mathcal{E}(\xi, T-\varepsilon) + 2\,C_4\,\frac{p_2(2\,\Gamma)\,\rho^2}{|\xi|} \int_{T-\varepsilon}^{t} \mathcal{E}(\xi,\tau)\, d\tau\,, \tag{5.20}$$

for all $\xi \in \mathbb{R}^n$ such that $|\xi| \geq \lambda\,\rho$. Now, using (5.13), (5.16) and applying Gronwall's lemma to (5.20), for $\rho \geq \max\{\tilde{\rho}, \varrho(T-\varepsilon)\}$ and $t \in \big[T-\varepsilon, \hat{T}(\rho)\big)$ we find

$$\begin{aligned} \mathcal{E}(\xi, t) &\leq 3\,\mathcal{E}(\xi, T-\varepsilon)\, \exp\Big\{\frac{\eta\,\rho^2}{2\,|\xi|}\,\frac{t-T+\varepsilon}{\varepsilon}\Big\} \\ &\leq 6\,\mathcal{E}(\xi, 0)\, \exp\Big\{\frac{\eta\,\rho^2}{2\,|\xi|}\,\frac{t-T+\varepsilon}{\varepsilon}\Big\} \\ &\leq 6\,\mathcal{E}(\xi, 0)\, \exp\Big\{\frac{\eta\,\rho^2}{2\,|\xi|}\Big\}\,, \end{aligned} \tag{5.21}$$

when $|\xi| \geq \lambda\,\rho$. Thus, in order to verify that (5.11) (b) holds for all $t \in [T-\varepsilon, T)$, whenever $\rho = \rho_j$ with j large enough, it will be sufficient to observe that

$$\lim_{j\to\infty} \int_{|\xi|>\lambda\rho_j} \mathcal{E}(\xi,0)\, \frac{e^{\eta\,\rho_j^2/2\,|\xi|}}{\rho_j^2}\, d\xi \;=\; 0\,. \tag{5.22}$$

To this end, we demonstrate the following:

Lemma 5.1. *Assume that* (5.1) *holds. Then, for all* $\eta' < \eta$ *we have*

$$\lim_{j} \int_{|\xi|>\rho_j} \mathcal{E}(\xi,0)\, \frac{e^{\eta'\rho_j^2/|\xi|}}{\rho_j^2}\, d\xi \;=\; 0\,. \tag{5.23}$$

Proof. Given $\Lambda > 0$, for all $j \geq 1$ we define the sets

$$\begin{aligned} A_j &= \{\xi : |\xi| > \rho_j\,,\ \rho_j^2 \geq \Lambda|\xi|\,\}\,,\\ B_j &= \{\xi : |\xi| > \rho_j\,,\ \rho_j^2 < \Lambda|\xi|\,\}\,. \end{aligned} \tag{5.24}$$

Integrating, we find

$$\begin{aligned} &\int_{|\xi|>\rho_j} \mathcal{E}(\xi,0)\, \frac{e^{\eta'\rho_j^2/|\xi|}}{\rho_j^2}\, d\xi \\ &\quad= \int_{A_j} \mathcal{E}(\xi,0)\, \frac{e^{\eta'\rho_j^2/|\xi|}}{\rho_j^2}\, d\xi + \int_{B_j} \mathcal{E}(\xi,0)\, \frac{e^{\eta'\rho_j^2/|\xi|}}{\rho_j^2}\, d\xi \\ &\quad\leq e^{-\Lambda(\eta-\eta')} \int_{A_j} \mathcal{E}(\xi,0)\, \frac{e^{\eta\,\rho_j^2/|\xi|}}{\rho_j^2}\, d\xi \;+\; \int_{B_j} \mathcal{E}(\xi,0)\, \frac{e^{\Lambda\eta'}}{\rho_j^2}\, d\xi \\ &\quad\leq\; e^{-\Lambda(\eta-\eta')}\,\Upsilon \;+\; \int_{|\xi|>\rho_j} \mathcal{E}(\xi,0)\, \frac{e^{\Lambda\eta'}}{\rho_j^2}\, d\xi\,. \end{aligned} \tag{5.25}$$

Since $\eta \,>\, \eta'$, the term $e^{-\Lambda(\eta-\eta')}\,\Upsilon$ can be made arbitrarily small taking $\Lambda > 0$ sufficiently large. Besides, the last integral in (5.25) tends to 0 as $\rho_j \to +\infty$, because $B^2_\Delta \subset H^2 \times H^1$. From these facts we immediately have (5.23). ∎

Conclusion

It follows that $\hat{T}(\rho_j) = T$ for all integer j large enough, i.e. both the conditions of (5.11) are verified in $[T-\varepsilon, T)$. In particular, fixed j_0 sufficiently large, by (5.8) we have $\mathcal{E}^{\rho_{j_0}}(t) \;\leq\; \rho_{j_0}^2\,\Gamma$ in $[T-\varepsilon, T)$. Finally, using also the a-priori estimate (4.4), we obtain

$$E_2(u;t) \;\leq\; C\,\rho_{j_0}^2 \left(\mathcal{H}(t) \;+ e^{-2\gamma t}\,\Gamma\right), \tag{5.26}$$

for $t \in [T-\varepsilon, T)$, with $C = 2\,\max\{1, \delta^{-1}, \mu_0^{\frac{1}{2}}\}$. Thus $E_2(u;t)$ is uniformly bounded in $[0, T)$. Since T is an arbitrary positive number, this in turn implies the global existence of the solution of (1.1)–(1.2).

6 boundedness and decay for B^2_Δ data

Here we shall prove that the global solution obtained in § 5 satisfies (1.7). As in the previous section we consider the case $k = 2$. We may, therefore, suppose that:

(a) condition (5.1) holds for fixed $\eta > 0$ and $\{\rho_j\}_{j\geq 1}$;

(b) u is the unique global strong solution of (1.1)–(1.2).

Given $\mathcal{T} \geq 0$, that we will fix in (6.4)-(6.5) below, we consider the solution in $\mathbb{R} \times [\mathcal{T}, \infty)$. Then, for $t \geq \mathcal{T}$, we have the identity

$$\mathcal{E}(\xi,t) \;=\; \mathcal{E}(\xi,\mathcal{T}) - \Big[(\mathcal{Q}-\mathcal{E})(\xi,\tau)\Big]_{\mathcal{T}}^{t} \;+\; \int_{\mathcal{T}}^{t} \mathcal{Q}'(\xi,\tau)\,d\tau\,. \tag{6.1}$$

To estimate the terms $\mathcal{Q}-\mathcal{E}$ and $\mathcal{Q}'$ in formula (6.1), we apply the inequalities of Lemma 4.4. Precisely, for $\rho \geq 1$ and $|\xi| > 0$ we have:

$$|\mathcal{Q}-\mathcal{E}| \;\leq\; C_3 \left[\frac{\gamma}{\rho} + \mathcal{H}(t) + e^{-2\gamma t}\,\frac{\mathcal{E}^{\rho}}{\rho^2}\right] \frac{\rho}{|\xi|}\;\mathcal{E}\,, \tag{6.2}$$

$$|\mathcal{Q}'| \;\leq\; C_4\; p_2\Big(\mathcal{H}(t) + e^{-2\gamma t}\,\frac{\mathcal{E}^{\rho}}{\rho^2}\Big)\,\frac{\rho^2}{|\xi|}\;\mathcal{E}\,, \tag{6.3}$$

where, as above, $p_2(r) \stackrel{\text{def}}{=} r + r^2$. Since $\gamma > 0$, by using (iii) and (iv) of Lemma 2.2, we can select $\mathcal{T} \geq 0$ so large that

$$C_3\,\mathcal{H}(t) \leq \frac{1}{8} \quad \text{for} \quad t \geq \mathcal{T}\,, \tag{6.4}$$

$$2\,C_4 \int_{\mathcal{T}}^{\infty} p_2\Big(\,\mathcal{H}(t) + \frac{e^{-2\gamma t}}{4\,C_3}\,\Big)\,dt \;\leq\; \frac{\eta}{2}\,. \tag{6.5}$$

Besides, we can take $\bar\rho \geq 1$ such that

$$\begin{cases} C_3\,\dfrac{\gamma}{\rho} \;\leq\; \dfrac{1}{8} \\[2mm] C_3\,\dfrac{\mathcal{E}^{\rho}(\mathcal{T})}{\rho^2} \;\leq\; \dfrac{1}{8} \end{cases} \qquad \text{for} \quad \forall\,\rho \geq \bar\rho\,. \tag{6.6}$$

Therefore, by (6.2), for every fixed $\rho \geq \bar\rho$ we have

$$|(\mathcal{Q}-\mathcal{E})(\xi,t)| \;\leq\; \frac{1}{2}\,\mathcal{E}(\xi,t) \quad \text{for all} \quad |\xi| \geq \rho \tag{6.7}$$

whenever $t \geq \mathcal{T}$ and

$$C_3\,\frac{\mathcal{E}^{\rho}(t)}{\rho^2} \;<\; \frac{1}{4}\,. \tag{6.8}$$

Then, for $\rho \geq \bar\rho$, we define

$$\hat{\mathcal{T}}(\rho) \stackrel{\text{def}}{=} \sup\,\{\tau \geq \mathcal{T} \;:\; \text{(6.8) holds for all } t \in [\mathcal{T},\,\tau)\}\,. \tag{6.9}$$

Since $\mathcal{T}$, $\bar{\rho}$ verify (6.4)–(6.6), it is clear that $\hat{\mathcal{T}}(\rho) > \mathcal{T}$ for all $\rho \geq \bar{\rho}$. In the following, we shall prove that $\hat{\mathcal{T}}(\rho_j) = \infty$ if ρ_j is any sufficiently large element of the sequence $\{\rho_j\}_{j\geq 1}$, i.e. (6.8) is verified for all $t \in [\mathcal{T}, \infty)$ when $\rho = \rho_j$ with ρ_j large enough. Now, by (6.1) and (6.7), for every fixed $\rho \geq \bar{\rho}$ we have

$$\mathcal{E}(\xi, t) \leq 3\,\mathcal{E}(\xi, \mathcal{T}) + 2 \int_{\mathcal{T}}^{t} \mathcal{Q}'(\xi, \tau)\, d\tau\,, \tag{6.10}$$

for all $|\xi| \geq \rho$ and $t \in [\mathcal{T}, \hat{\mathcal{T}}(\rho))$. From this, using the estimate (6.3), condition (6.5) and applying Gronwall's lemma, we easily get

$$\mathcal{E}(\xi, t) \leq 3\,\mathcal{E}(\xi, \mathcal{T})\, e^{\eta\, \rho^2 / 2|\xi|}\,, \tag{6.11}$$

for $|\xi| \geq \rho$ and $t \in [\mathcal{T}, \hat{\mathcal{T}}(\rho))$. Then, taking $\varrho(\mathcal{T})$ according to Lemma 9.2, i.e. such that $\mathcal{E}(\xi, \mathcal{T}) \leq 2\,\mathcal{E}(\xi, 0)$ for $|\xi| \geq \varrho(\mathcal{T})$, from (6.11) we derive further the inequality

$$\mathcal{E}(\xi, t) \leq 6\,\mathcal{E}(\xi, 0)\, e^{\eta\, \rho^2 / 2|\xi|}\,, \tag{6.12}$$

for $\rho \geq \max\{\bar{\rho}, \varrho(\mathcal{T})\}$, $|\xi| \geq \rho$ and $t \in [\mathcal{T}, \hat{\mathcal{T}}(\rho))$. This means that for $\rho \geq \max\{\bar{\rho}, \varrho(\mathcal{T})\}$ we can estimate $\mathcal{E}^{\rho}(t)/\rho^2$ as follows:

$$\frac{\mathcal{E}^{\rho}(t)}{\rho^2} \leq 6 \int_{|\xi|>\rho} \mathcal{E}(\xi, 0)\, \frac{e^{\eta\, \rho^2 / 2|\xi|}}{\rho^2}\, d\xi\,, \tag{6.13}$$

for all $t \in [\mathcal{T}, \hat{\mathcal{T}}(\rho))$. Thus, in order to conclude that $\hat{\mathcal{T}}(\rho_j) = +\infty$ if j is large enough, it is sufficient to apply Lemma 5.1. In particular, there exists an integer $j_1 \geq 1$ large such that

$$\mathcal{E}^{\rho_{j_1}}(t) \leq \frac{\rho_{j_1}^2}{4\, C_3} \quad \forall\, t \in [\mathcal{T}, \infty)\,. \tag{6.14}$$

Now we can easily derive the decay estimate (1.7) for $E_r(u;t)$, when $0 \leq r \leq 2$. In fact, from (4.4) and (6.14), we readily have the estimate

$$E_r(u;t) \leq C\, (\rho_{j_1})^r \left(\mathcal{H}(t) + \frac{e^{-2\gamma t}}{4\, C_3} \right), \tag{6.15}$$

for $t \in [\mathcal{T}, \infty)$, with $C = 2\max\{1, \delta^{-1}, \mu_0^{\frac{1}{2}}\}$. Having $\gamma > 0$, combining (6.15) with (iii) and (iv) of Lemma 2.2, (1.7) follows.

7 Sketch of the proof of theorem 1.2 for $k \geq 2$

Having proved in details Theorem 1.2 for $k = 1, 2$, we now sketch the idea of the proof for a generic integer $k \geq 2$. To do this we apply the results of [4]. We divide the proof in three steps.

7.1 Quadratic forms for the linearized equation

Let us consider the infinite system of linear oscillating equations with dissipative term

$$w_{tt} + a(t)\,|\xi|^2\,w + 2\gamma\,w_t = 0 \quad \text{for} \quad t \in [0,T)\,, \quad \xi \in \mathbb{R}^n\,, \tag{7.1}$$

where $0 < T \le \infty$, $a(t) \in C^k[0,T)$, $a(t) \ge \delta > 0$ and $\gamma > 0$. Setting

$$z(\xi,t) = e^{\gamma t}\,w(\xi,t)\,, \tag{7.2}$$

it follows that, for $|\xi| > 0$,

$$z_{tt} + a_*(|\xi|,t)\,|\xi|^2\,z = 0\,, \tag{7.3}$$

where

$$a_*(|\xi|,t) = a(t) - \gamma^2|\xi|^{-2}\,. \tag{7.4}$$

Since our arguments require that $a_* \ge c > 0$, from now on we will assume

$$|\xi| \ge \frac{2\gamma}{\sqrt{\delta}}\,, \tag{7.5}$$

thus $a_*(|\xi|,t) \ge \frac{3}{4}\,\delta$. Then, for $|\xi| \ge \frac{2\gamma}{\sqrt{\delta}}$ and $z(\xi,t)$ a complex-valued solution of (7.3), we consider the quadratic form:

$$\begin{aligned} Q_k^*(z,z_t) &\stackrel{\text{def}}{=} \sum_{0\le i\le[\frac{k}{2}]-1} \alpha_i^*(t)\,|\xi|^{-2i}\,\big(a_*(t)\,|\xi|^2\,|z|^2 + |z_t|^2\big) \\ &\quad + \sum_{0\le i\le[\frac{k}{2}]-1} \beta_i^*(t)\,|\xi|^{-2i}\,\mathrm{Re}\big(\bar z\,z_t\big) + \sum_{0\le i<\frac{k}{2}-1} \gamma_i^*(t)\,|\xi|^{-2i-2}\,|z_t|^2\,, \end{aligned} \tag{7.6}$$

where α_i^*, β_i^*, γ_i^* are real-valued functions on $[0,T)$ satisfying the system

$$\gamma_{-1}^* \equiv 0\,, \quad \begin{cases} (a_*\,\alpha_i^*)' - a_*\beta_i^* = 0 \\ \alpha_i^{*\prime} + \beta_i^* = -\gamma_{i-1}^{*}{}' \\ \beta_i^{*\prime} - 2\,a_*\gamma_i^* = 0 \end{cases} \qquad (0 \le i \le [k/2]-1)\,. \tag{7.7}$$

By the result of [4], system (7.7) is solvable and α_i^*, β_i^*, γ_i^* are polynomials in

$$\omega_* \stackrel{\text{def}}{=} \frac{1}{2\sqrt{a_*}} \tag{7.8}$$

and its derivatives of orders not greater than, respectively, $2i$, $2i+1$, $2i+2$. More precisely, computing the solutions system (7.7), we may select the coefficients of integration such that $\omega_*^{-1}\,\alpha_i^*$, β_i^*, $\omega_*^{-1}\,\gamma_i^*$ are homogeneous in the sense that

$$\omega_*^{-1}\,\alpha_i^* = \sum c_{\eta_0,\dots\eta_{2i}}\,\big(\omega_*\big)^{\eta_0}\,\big(\omega_*^{(1)}\big)^{\eta_1}\,\cdots\,\big(\omega_*^{(2i)}\big)^{\eta_{2i}} \tag{7.9}$$

for $0 \le i \le [\frac{k}{2}]-1$, with $c_{\eta_0,\dots,\eta_{2i}} \in \mathbb{R}$ and $\eta_0,\dots,\eta_{2i} \ge 0$ integers such that

$$\sum_{0\le h\le 2i} \eta_h = 2i \quad \text{and} \quad \sum_{0\le h\le 2i} h\,\eta_h = 2i\,;$$

while β_i^*, $\omega_*^{-1}\gamma_i^*$ have analogous expansions on replacing $2i$ by $2i+1$ and $2i+2$. In particular, we have

$$\alpha_0^* = c_0\,\omega_*\,, \quad \beta_0^* = -c_0\,\omega_*'\,, \quad \gamma_0^* = -2c_0\,\omega_*^2\,\omega_*''\,, \tag{7.10}$$

where c_0 is an arbitrary real constant. Thus, setting $c_0 = 1$, the first term of $\mathcal{Q}_k^*(z, z_t)$ is the energy function

$$\mathcal{E}^*(z, z_t) \stackrel{\text{def}}{=} \frac{\sqrt{a_*(t)}}{2}\,|\xi|^2\,|z|^2 + \frac{|z_t|^2}{2\,\sqrt{a_*(t)}}\,. \tag{7.11}$$

Finally, let us recall that

$$\frac{d}{dt}\,\mathcal{Q}_k^*(z, z_t) \;=\; \begin{cases} (\beta^*_{[\frac{k}{2}]-1})'\;|\xi|^{-k+2}\;\mathrm{Re}\big(\bar{z}\,z_t\big) & \text{for } k \geq 2 \text{ even}, \\ (\gamma^*_{[\frac{k}{2}]-1})'\;|\xi|^{-k+1}\,|z_t|^2 & \text{for } k \geq 3 \text{ odd}, \end{cases} \tag{7.12}$$

for every complex-valued solution of (7.3). See Theorems 1.1 and 1.2 of [4].

7.2 Quadratic forms for the damped Kirchhoff equation

Let us suppose that $u \in C^j([0, T]; H^{1+\frac{k}{2}-j}(\mathbb{R}^n))$ $(j = 0, 1, 2)$ be a solution of (1.1). Since we suppose $m \in C^k$, it follows that $s(t) = \|\,|\xi|\hat{u}(t)\,\|^2$ is of class C^k, which in turn implies that $m(s(t)) \in C^k[0, T]$. On a account of the arguments developed in § 7.1, by Fourier transform in space variables, we consider the equivalent equation

$$\hat{u}_{tt} + m(s(t))\,|\xi|^2\,\hat{u} + 2\gamma\,\hat{u}_t = 0\,, \quad \xi \in \mathbb{R}^n,\; t \geq 0\,. \tag{7.13}$$

Then, setting

$$z(\xi, t) = e^{\gamma t}\,\hat{u}(\xi, t)\,, \tag{7.14}$$

$$a_*(|\xi|, t) = m(s(t)) - \gamma^2|\xi|^{-2}\,, \tag{7.15}$$

$$\omega_* \stackrel{\text{def}}{=} \frac{1}{2\sqrt{a_*}} = \frac{1}{2\sqrt{m(s(t)) - \gamma^2|\xi|^{-2}}}\,, \tag{7.16}$$

for $|\xi| \geq \frac{2\gamma}{\sqrt{\delta}}$ we introduce the quadratic form $\mathcal{Q}_k^*$ with coefficients α_i^*, β_i^*, γ_i^* polynomials in ω_* and satisfying (7.9). Finally, we also take $\alpha_0^* = \omega_*$ in order that $\mathcal{Q}_k^*$ begin with the energy $\mathcal{E}^*$ defined in (7.11). Then, since $z_t = e^{\gamma t}(\gamma\hat{u} + \hat{u}_t)$, we define:

Definition 7.1. *For* $|\xi| \geq \frac{2\gamma}{\sqrt{\delta}}$, *we set*

$$\mathcal{Q}_k \stackrel{\text{def}}{=} e^{2\gamma t}\,|\xi|^k\,\mathcal{Q}_k^*(\hat{u}, \gamma\hat{u} + \hat{u}_t)\,, \tag{7.17}$$

$$\mathcal{E}_k \stackrel{\text{def}}{=} e^{2\gamma t}\,|\xi|^k\,\mathcal{E}^*(\hat{u}, \gamma\hat{u} + \hat{u}_t)\,. \tag{7.18}$$

Since $\mathcal{Q}_k = |\xi|^k\, \mathcal{Q}_k^*(e^{\gamma t}\,\hat{u},\,(e^{\gamma t}\hat{u})_t)$, from (7.12) we immediately obtain

$$\frac{d}{dt}\,\mathcal{Q}_k = \begin{cases} e^{2\gamma t}\,(\beta^*_{[\frac{k}{2}]-1})'\,|\xi|^2\,\operatorname{Re}\big(\bar{\hat{u}}\,(\gamma\hat{u}+\hat{u}_t)\big) & \text{for } k\geq 2 \text{ even}\,,\\ e^{2\gamma t}\,(\gamma^*_{[\frac{k}{2}]-1})'\,|\xi|\,|\gamma\hat{u}+\hat{u}_t|^2 & \text{for } k\geq 3 \text{ odd}\,, \end{cases} \tag{7.19}$$

Besides, $\mathcal{E}_k = |\xi|^k\,\mathcal{E}^*\big(e^{\gamma t}\,\hat{u},\,(e^{\gamma t}\hat{u})_t\big)$ and

$$\mathcal{E}_k \geq \frac{e^{2\gamma t}}{4}\left(\frac{\sqrt{m}}{2}\,|\xi|^{k+2}\,|\hat{u}|^2 + \frac{|\xi|^k\,|\hat{u}_t|^2}{2\sqrt{m}}\right), \tag{7.20}$$

when $|\xi| \geq \frac{2\gamma}{\sqrt{\delta}}$. Now we can proceed estimating $\mathcal{Q}_k(\hat{u},\hat{u}_t)'$ and the remainder term

$$\mathcal{R}_k \stackrel{\text{def}}{=} \mathcal{Q}_k - \mathcal{E}_k\,, \tag{7.21}$$

as in § 4. Setting

$$\mathcal{E}_k^\rho(t) \stackrel{\text{def}}{=} \int_{|\xi|>\rho} \mathcal{E}_k(\xi,t)\,d\xi\,, \tag{7.22}$$

after some calculations, similar to those of [4], for every $\rho \geq \frac{2\gamma}{\sqrt{\delta}}$ we have

$$\int |\xi|^l|\hat{u}||\hat{u}_t|\,d\xi \leq C\rho^{l-1}\left(\mathcal{H}+\frac{\mathcal{E}_k^\rho}{\rho^k}\right), \quad 1\leq l\leq k+1, \tag{7.23}$$

$$\int |\xi|^l|\hat{u}_t|^2\,d\xi \leq C\rho^{l}\left(\mathcal{H}+\frac{\mathcal{E}_k^\rho}{\rho^k}\right), \quad 0\leq l\leq k, \tag{7.24}$$

$$\int |\xi|^l|\hat{u}|^2\,d\xi \leq C\rho^{l-2}\left(\mathcal{H}+\frac{\mathcal{E}_k^\rho}{\rho^k}\right), \quad 2\leq l\leq k+2, \tag{7.25}$$

where $C = C(\delta,\mu_0) > 0$ is a suitable constant. Using these a priori bounds and the expansions (7.9) to estimate α_i^*, β_i^*, γ_i^*, we finally obtain that:

$$|\mathcal{R}_k| \leq C_k\,p_{k-1}\!\left(\mathcal{H}+e^{-2\gamma t}\,\frac{\mathcal{E}_k^\rho}{\rho^k}\right)\,\frac{\rho}{|\xi|}\;\mathcal{E}_k\,, \tag{7.26}$$

$$|\mathcal{Q}_k'| \leq C_k\,p_k\!\left(\mathcal{H}+e^{-2\gamma t}\,\frac{\mathcal{E}_k^\rho}{\rho^k}\right)\,\frac{\rho^k}{|\xi|^{k-1}}\;\mathcal{E}_k\,, \tag{7.27}$$

for all $|\xi| \geq \rho \geq \max\{1, \frac{2\gamma}{\sqrt{\delta}}\}$, with $C_k = C_k(\delta,\gamma,\mu_0,\ldots,\mu_k)$ a positive constant and $p_j(r) = r + r^j$, for $j\geq 1$.

7.3 Global solvability and decay estimates

Having the a-priori estimates (7.26)–(7.27) it is now easy to complete the proof of Theorem 1.2 for $k\geq 2$. We can follow almost the same reasoning of § 5 and § 6.

To begin with, assuming $(u_0, u_1) \in B^k_\Delta$, there exist $\eta > 0$ and a sequence $\{\rho_j\}_{j\geq 1}$ such that $\rho_j > 0$, $\lim_j \rho_j = +\infty$ and

$$\Upsilon_k \stackrel{\text{def}}{=} \sup_j \int_{|\xi|>\rho_j} \Big(\frac{\sqrt{\mu_0}}{2} |\xi|^{k+2} |\hat{u}_0(\xi)|^2 + \frac{|\xi|^2 |\hat{u}_1(\xi)|^k}{2\sqrt{\delta}} \Big) \frac{e^{\eta \rho_j^k / |\xi|^{k-1}}}{\rho_j^k} \, d\xi < \infty . \quad (7.28)$$

Global solvability. We argue by contradiction: let $u \in C^j([0,T); H^{1+\frac{k}{2}-j}(\mathbb{R}^n))$ $(j = 0,1,2)$ be a solution of (1.1)–(1.2) in $\mathbb{R}^n \times [0,T)$ with $0 < T < \infty$ maximal. Then we consider u in the stripe $\mathbb{R}^n \times [T-\varepsilon, T)$, where $\varepsilon \in (0,T]$ is a parameter that we shall fix imposing condition similar to those of (5.12)–(5.14). For $T - \varepsilon \leq t < T$ and $|\xi| \geq \frac{2\gamma}{\sqrt{\delta}}$, we have

$$\mathcal{E}_k(\xi,t) \;=\; \mathcal{E}_k(\xi, T-\varepsilon) - \Big[\mathcal{R}_k(\xi,\tau)\Big]_{T-\varepsilon}^{t} + \int_{T-\varepsilon}^{t} \mathcal{Q}'_k(\xi,\tau)\, d\tau . \quad (7.29)$$

Using (7.26), (7.27), (4.18) and Lemma 9.2 of Appendix I, we can proceed in the estimate the right-hand side of (7.29) exactly as in § 5. After some calculations, this leads us to conclude that

$$\sup_{t\in[0,T)} E_k(u,t) < \infty , \quad (7.30)$$

proving that T cannot be maximal. See also the proof of Theorem 1.4 of [4].

Decay Estimate (1.7). Let $u \in C^j([0,\infty); H^{1+\frac{k}{2}-j}(\mathbb{R}^n))$ $(j = 0,1,2)$ be a global solution of (1.1)–(1.2). We take $\Gamma_k > 0$ such that

$$C_k \, p_{k-1}(2\,\Gamma_k) = \frac{1}{2}, \quad (7.31)$$

then we select $\mathcal{T}_k \geq 0$ so large that

$$\mathcal{H}(t) \leq \Gamma_k \quad \text{for} \quad t \geq \mathcal{T}_k , \quad (7.32)$$

$$2\,C_k \int_{\mathcal{T}_k}^{\infty} p_k\Big(\mathcal{H}(t) + e^{-2\gamma t}\,\Gamma_k \Big)\, dt \;\leq\; \frac{\eta}{2} . \quad (7.33)$$

Finally, we take $\rho_k \geq \max\{1, \frac{2\gamma}{\sqrt{\delta}}\}$ such that

$$\frac{\mathcal{E}_k^{\rho}(\mathcal{T}_k)}{\rho^k} \;\leq\; \frac{\Gamma_k}{2} \quad \text{for} \quad \forall\, \rho \geq \rho_k . \quad (7.34)$$

Then, using (7.26)–(7.27), we can proceed on the estimation of $\mathcal{E}_k^{\rho}(t)$ for $t \geq \mathcal{T}_k$ by repeating almost the same proof of § 6.

8 Proof of theorem 1.3

The idea of the proof is essentially due to Yamada [9]. Given an integer $k \geq 1$, let $u \in C^j([0,\infty); H^{1+\frac{k}{2}-j}(\mathbb{R}^n))$ $(j = 0,1,2)$ be a global solution of equation (1.1). By Lemmas 2.2 and 2.3, we know that

$$\sup_{t\geq 0} \Big\{ t\big(|u|_1^2 + |u_t|_0^2 \big) + \int_0^t \big(|u|_1^2 + \tau |u_t|_0^2 \big)\, d\tau \Big\} < \infty . \quad (8.1)$$

Hence (1.9) holds with $r = 0$. Then, to prove (1.9) for $0 < r \leq k/2$, we proceed as follows. By partial Fourier transform in space variables, we consider the equivalent equation

$$\hat{u}_{tt} + m(s(t))\,|\xi|^2\,\hat{u} + 2\gamma\,\hat{u}_t = 0\,, \qquad \xi \in \mathbb{R}^n,\ t \geq 0\,, \tag{8.2}$$

where, as usual, $s(t) = \|\,|\xi|\hat{u}(t)\,\|^2$. Besides, we introduce the auxiliary functions

$$\chi(\xi)\,\phi(t) \quad \text{and} \quad \omega(\xi)\,\psi(t)\,, \tag{8.3}$$

where $\phi(t)$, $\psi(t) \in C^2[0,\infty)$ and $\chi(\xi)$, $\omega(\xi)$ are suitable weights that we shall fix in the following. Multiplying equation (8.2) by $\chi(\xi)\,\phi(t)\,\bar{\hat{u}}_t$ and integrating other $\mathbb{R}^n_\xi \times [\bar{t}, t)$, with $0 \leq \bar{t} < t$, we get

$$\begin{aligned}
\phi(t) \int \chi(\xi)\,\big(|\hat{u}_t|^2 + m|\xi|^2|\hat{u}|^2\big)\,d\xi\Big|_t &+ 4\,\gamma \int_{\bar{t}}^t \int \phi\,\chi(\xi)|\hat{u}_t|^2\,d\xi\,d\tau \\
= \phi(\bar{t}) &\int \chi(\xi)\,\big(|\hat{u}_t|^2 + m|\xi|^2|\hat{u}|^2\big)\,d\xi\Big|_{\bar{t}} \\
&+ \int_{\bar{t}}^t \int \chi(\xi)\Big[\phi'|\hat{u}_t|^2 + \phi'\,m\,|\xi|^2\,|\hat{u}|^2 + \phi\,m'\,s'\,|\xi|^2|\hat{u}|^2\Big]d\xi\,d\tau\,,
\end{aligned} \tag{8.4}$$

where, as usual, $m = m(s(t))$, $m' = m'(s(t))$. Similarly, multiplying (8.2) by $\omega(\xi)\,\psi(t)\,\bar{\hat{u}}$ and integrating over $\mathbb{R}^n_\xi \times [\bar{t}, t)$, we have

$$\begin{aligned}
2\psi(t) &\int \omega(\xi)\,\mathrm{Re}(\hat{u}\bar{\hat{u}}_t)\,d\xi\Big|_t + \big(2\gamma\psi(t) - \psi'(t)\big) \int \omega(\xi)\,|\hat{u}|^2\,d\xi\Big|_t \\
&+ \int_{\bar{t}}^t \int \omega(\xi)\Big[2\,\psi\,m\,|\xi|^2\,|\hat{u}|^2 + \big(\psi'' - 2\gamma\psi'\big)|\hat{u}|^2\Big]\,d\xi\,d\tau \\
&= \int \Big(2\psi\,\omega(\xi)\,\mathrm{Re}(\hat{u}\bar{\hat{u}}_t) + \big(2\gamma\psi - \psi'\big)\,\omega(\xi)\,|\hat{u}|^2\Big)\,d\xi\Big|_{\bar{t}} \\
&\quad + 2\int_{\bar{t}}^t \int \psi\,\omega(\xi)\,|\hat{u}_t|^2\,d\xi\,d\tau\,.
\end{aligned} \tag{8.5}$$

Now, adding (8.4) and (8.5) with

$$\omega = \chi\,,$$

some elementary calculations give

$$\begin{aligned}
&\int \phi\,m\,\chi(\xi)\,|\xi|^2\,|\hat{u}|^2\,d\xi\Big|_t \\
&+ \int \Big[\phi\,|\hat{u}_t|^2 + 2\psi\,\mathrm{Re}(\hat{u}\bar{\hat{u}}_t) + \big(2\gamma\psi - \psi'\big)\,|\hat{u}|^2\Big]\,\chi(\xi)\,d\xi\Big|_t \\
&+ \int_{\bar{t}}^t \int \Big[4\gamma\phi - 2\psi - \phi'\Big]\,\chi(\xi)\,|\hat{u}_t|^2\,d\xi\,d\tau \\
&+ \int_{\bar{t}}^t \int \Big[2\psi m - \phi' m - \phi\,m's'\Big]\,\chi(\xi)\,|\xi|^2\,|\hat{u}|^2\,d\xi\,d\tau \\
&+ \int_{\bar{t}}^t \int \big(\psi'' - 2\gamma\psi'\big)\,\chi(\xi)\,|\hat{u}|^2\,d\xi\,d\tau \\
&= \int \Big[\phi\,|\hat{u}_t|^2 + 2\psi\,\mathrm{Re}(\hat{u}\bar{\hat{u}}_t) + \big(m\phi\,|\xi|^2 + 2\gamma\psi - \psi'\big)\,|\hat{u}|^2\Big]\,\chi(\xi)\,d\xi\Big|_{\bar{t}}\,.
\end{aligned} \tag{8.6}$$

In order to proceed in the proof of (1.9), it is convenient to deal with the cases $k \geq 2$ even, $k \geq 3$ odd and $k = 1$ separately.

Case $k \geq 2$ even

First of all, we apply the identity (8.6) with

$$\chi(\xi) = |\xi|^2, \quad \phi(t) = t, \quad \psi(t) = \frac{\gamma}{4}\, t. \tag{8.7}$$

Considering the terms in (8.6), we immediately see that

$$m\,\phi \geq \delta t \quad \text{and} \quad \psi'' - 2\gamma\psi' = -\gamma^2/2, \quad \forall\, t \geq 0. \tag{8.8}$$

It is also easy to verify that, when $t \geq 2/\gamma$,

$$\phi|\zeta_1|^2 + 2\psi\,\mathrm{Re}(\zeta_1\bar{\zeta}_2) + (2\gamma\psi - \psi')|\zeta_2|^2 \geq \frac{t}{2}\,|\zeta_1|^2 + \frac{\gamma^2\, t}{4}\,|\zeta_2|^2, \quad \forall\, \zeta_1, \zeta_2 \in \mathbb{C}, \tag{8.9}$$

$$4\gamma\phi - 2\psi - \phi' \geq 3\gamma t. \tag{8.10}$$

Furthermore, since we are assuming the first of (1.8) holds and, by (i)–(iii) of Lemma 2.2, $\sup_{t\geq 0}(1+t)^{\frac{1}{2}}\,\|u_t\| < \infty$, it follows that $\sup_{t\geq 0}(1+t)^{\frac{1}{2}}\,|s'(t)| < \infty$. Consequently, we deduce the inequality

$$2\psi m - \phi' m - \phi\, m' s' \geq \left(\frac{\gamma t}{2} - 1\right)\delta - C\,t^{\frac{1}{2}}, \quad \forall\, t \geq 2/\gamma, \tag{8.11}$$

with a suitable constant $C > 0$. Then, taking account of (8.8)–(8.11) and recalling that $\int_0^\infty |u|_1^2\, dt < \infty$, we apply the identity (8.6) with $\bar{t} = t_0 \geq 0$ large enough. It readily follows that

$$\sup_{t\geq 0}\,\left\{t\,|u|_2^2 + t\,(|u|_1^2 + |u_t|_1^2) + \int_0^t \tau(|u|_2^2 + |u_t|_1^2)\, d\tau\right\} < \infty, \tag{8.12}$$

which in turn implies that

$$|s'(t)| \leq 2\,|u(t)|_1\,|u_t(t)|_1 \leq C\,(1+t)^{-1}, \quad \forall\, t \geq 0, \tag{8.13}$$

for a suitable constant $C > 0$. To continue, for $j \geq 1$, we set

$$\chi_j(\xi) = |\xi|^{2j}, \quad \phi(t) = t^{j+1}, \quad \psi(t) = \lambda_j\, t^j, \tag{8.14}$$

where λ_j are suitable positive parameters. More precisely, taking account of (8.13), it is not difficult to see that we can fix $\lambda_j > 0$ such that, for $t \geq t_j$ (with $t_j \geq 0$ large enough), the following hold:

$$m\,\phi \geq \delta t^{j+1}, \tag{8.15}$$

$$\phi|\zeta_1|^2 + 2\psi\,\mathrm{Re}(\zeta_1\bar{\zeta}_2) + (2\gamma\psi - \psi')|\zeta_2|^2 \geq \frac{t^{j+1}}{2}\,|\zeta_1|^2 + \gamma\lambda_j\, t^j\,|\zeta_2|^2, \tag{8.16}$$

$$4\gamma\phi - 2\psi - \phi' \geq 3\gamma t^{j+1}, \tag{8.17}$$

$$2\psi m - \phi' m - \phi\, m' s' \geq \lambda_j\, t^j, \tag{8.18}$$

$$|\psi'' - 2\gamma\psi'| \leq 2\gamma\,\lambda_j\, t^{j-1}. \tag{8.19}$$

Then, fixed $\lambda_j > 0$, for $j \geq 1$, such that (8.15)-(8.19) hold, we proceed to prove by induction (1.9) for $r = 0, 1, \ldots, \frac{k}{2}$.

As already remarked, see (8.1) above, (1.9) holds for $r = 0$. Suppose that it holds when $r = j - 1$, for some integer j with $1 \leq j \leq \frac{k}{2}$. In particular, we have

$$\int_0^\infty t^{j-1} \, |u(t)|_j^2 \, dt < \infty \,. \tag{8.20}$$

Then, setting $\chi = \chi_j$, $\phi = \phi_j$ and $\psi = \psi_j$ as in (8.14), we apply the identity (8.6) with $\bar{t} = t_j$. Taking account of (8.15)–(8.19) and (8.20), we derive that

$$t^{j+1} \, |u|_{j+1}^2 + t^{j+1} |u_t|_j^2 + t^j |u|_j^2 + \int_0^t \left(\tau^j |u|_{j+1}^2 + \tau^{j+1} |u_t|_j^2 \right) d\tau \leq K_j \,, \tag{8.21}$$

for all $t \geq 0$, for a suitable constant $K_j > 0$. By induction, this proves that (1.9) holds for all integers r, $0 \leq r \leq \frac{k}{2}$. Finally, using a standard interpolation argument, we obtain (1.9) for all real values r, $0 \leq r \leq \frac{k}{2}$.

Case $k \geq 3$ **odd**

We set $\tilde{k} = k - 1$. Then $u \in C^j([0, \infty); H^{1+\frac{\tilde{k}}{2}-j}(\mathbb{R}^n))$ $(j = 0, 1, 2)$ where $\tilde{k} \geq 2$ in an even integer. By the previous case, this implies that (8.13) holds and that the statement (1.9) is verified for $0 \leq r \leq \frac{\tilde{k}}{2}$. In particular, setting $r = \frac{\tilde{k}}{2} - \frac{1}{2} = \frac{k}{2} - 1$, it follows that

$$\int_0^\infty t^{\frac{k}{2}-1} \, |u(t)|_{\frac{k}{2}}^2 \, dt < \infty \,. \tag{8.22}$$

Now, we define

$$\tilde{\chi}(\xi) = |\xi|^k \,, \quad \tilde{\phi}(t) = t^{\frac{k}{2}+1} \,, \quad \tilde{\psi}(t) = \tilde{\lambda} \, t^{\frac{k}{2}} \,, \tag{8.23}$$

where $\tilde{\lambda} > 0$ is a suitable parameter such that, by replacing j with $\frac{k}{2}$ and λ_j with $\tilde{\lambda}$, conditions like (8.15)–(8.19) are verified for $t \geq \tilde{t}$ (with $\tilde{t} \geq 0$ large enough). Then, we put $\chi = \tilde{\chi}$, $\phi = \tilde{\phi}$, $\psi = \tilde{\psi}$ and $\bar{t} = \tilde{t}$ in (8.6). Using condition (8.22), we obtain that (1.9) holds also for $r = \frac{k}{2}$. Finally, by interpolation, we deduce that (1.9) holds for all real values r, $0 \leq r \leq \frac{k}{2}$.

Case $k = 1$

Since

$$|s'(t)| \leq 2 \, |u(t)|_{\frac{3}{2}} \, |u_t(t)|_{\frac{1}{2}} \,, \tag{8.24}$$

by the second assumption of (1.8) it follows that (8.13) holds. Since $\sup_{t \geq 0} \; |u|_0^2 < \infty$ (see Proposition 2.5) and $\sup_{t \geq 0} (1 + t) \, |u|_1^2 < \infty$, by interpolation we get

$$\sup_{t \geq 0} \, (1 + t)^{\frac{1}{2}} \, |u|_{\frac{1}{2}}^2 < \infty \,. \tag{8.25}$$

This implies that

$$\int_0^\infty (1+\tau)^\beta \, |u|^2_{\frac{1}{2}} \, d\tau < \infty \quad \text{if} \quad \beta < -1/2 \,. \tag{8.26}$$

Then, given a real number α, $0 \le \alpha < 1/2$, we define

$$\chi^*(\xi) = |\xi| \,, \quad \phi^*(t) = t^{\alpha+1} \,, \quad \psi^*(t) = \lambda^* \, t^\alpha \,, \tag{8.27}$$

where $\lambda^* > 0$ is a suitable parameter such that, by replacing j with α and λ_j with λ^*, conditions like (8.15)–(8.19) are verified for $t \ge t^*$ (with $t^* \ge 0$ large enough). Then, we put $\chi = \chi^*$, $\phi = \phi^*$, $\psi = \psi^*$ and $\bar{t} = t^*$ in identity (8.6). Using condition (8.26), we readily obtain

$$t^{\alpha+1} \, |u|^2_{\frac{3}{2}} + t^{\alpha+1} |u_t|^2_{\frac{1}{2}} + \int_0^t \big(\tau^\alpha |u|^2_{\frac{3}{2}} + \tau^{\alpha+1} |u_t|^2_{\frac{1}{2}} \big) \, d\tau \le K^* \,, \tag{8.28}$$

for all $t \ge 0$, with $K^* > 0$ a suitable constant. Finally, by interpolation, we derive (1.9) for $0 \le r \le \frac{1}{2}$.

9 Appendix I

In this appendix we shall derive a suitable quadratic form for the solutions of the damped wave equation

$$u_{tt} - a(t)\Delta u + 2\gamma \, u_t = 0 \quad \text{in} \quad \mathbb{R}^n \times [0, T), \tag{9.1}$$

where $0 < T \le +\infty$ and

$$a(t) \in C^2[0,\, T) \,, \quad a(t) > 0 \,, \quad \gamma > 0 \,. \tag{9.2}$$

By Fourier transform in the space variables, we are led to consider the infinite system of linear oscillating equations with dissipative terms

$$w_{tt} + a(t) \, |\xi|^2 \, w + 2\gamma \, w_t = 0 \quad \text{for} \quad t \in [0, T) \,, \quad \xi \in \mathbb{R}^n \,. \tag{9.3}$$

For the solutions of (9.3) we introduce the quadratic form

$$q(\xi, t) \;=\; \frac{1}{2} a_1(t) \, a(t) \, |\xi|^4 \, |w|^2 + \frac{1}{2} a_1(t) \, |\xi|^2 \, |w_t|^2 + a_2(t) \, |\xi|^2 \, \mathrm{Re}(\bar{w} \, w_t) \,, \tag{9.4}$$

where we suppose $a_1(t)$, $a_2(t) \in C^1[0, T)$. Deriving with respect to t, using (9.3) and collecting like terms, we obtain

$$\begin{aligned} \frac{d}{dt} q(\xi, t) \;=\; & \Big[\frac{1}{2} (a_1 \, a)' - a_2 \, a \Big] \, |\xi|^4 \, |w|^2 \\ & + \Big[\frac{1}{2} a_1' - 2 \, \gamma \, a_1 + a_2 \Big] \, |\xi|^2 \, |w_t|^2 \\ & + \big[a_2' - 2 \, \gamma \, a_2 \big] \, |\xi|^2 \, \mathrm{Re}(\bar{w} \, w_t)^2 \,. \end{aligned} \tag{9.5}$$

Now, considering (9.5), we require that the coefficients of $|\xi|^4|w|^2$ and $|\xi|^2|w_t|^2$ vanish. Namely we search $a_1(t)$, $a_2(t)$ satisfying the conditions:

$$\begin{cases} \frac{1}{2}\,(a_1\,a)' - a_2\,a \;=\; 0 \\ \frac{1}{2}\,a_1' - 2\,\gamma\,a_1 + a_2 \;=\; 0 \end{cases} \tag{9.6}$$

An easy computation shows that $a_1 = c\,\frac{e^{2\gamma t}}{\sqrt{a}}$ and $a_2 = c\,e^{2\gamma t}\left(\frac{\gamma}{\sqrt{a}} + \frac{a'}{4\,a^{3/2}}\right)$, where $c \in \mathbb{R}$ is an arbitrary constant. Taking $c = 1$, from now on we set:

$$a_1 \stackrel{\text{def}}{=} \frac{e^{2\gamma t}}{\sqrt{a}}\,, \quad a_2 \stackrel{\text{def}}{=} e^{2\gamma t}\left(\frac{\gamma}{\sqrt{a}} + \frac{a'}{4\,a^{3/2}}\right). \tag{9.7}$$

Taking account of the previous calculations, we define

Definition 9.1. *Let $w(\xi, t)$ be a solution of equation* (9.3), *we set*

$$\begin{aligned} \mathcal{Q}(\xi,t) \stackrel{\text{def}}{=} & e^{2\gamma t}\left(\frac{\sqrt{a}}{2}\,|\xi|^4\,|w|^2 + \frac{1}{2\sqrt{a}}\,|\xi|^2\,|w_t|^2\right) \\ & + e^{2\gamma t}\Big(\frac{\gamma}{\sqrt{a}} + \frac{a'}{4\,a^{3/2}}\Big)\,|\xi|^2\,\mathrm{Re}(\bar{w}\,w_t)\,. \end{aligned} \tag{9.8}$$

From (9.5)–(9.7), it is immediate that

$$\frac{d}{dt}\,\mathcal{Q}(\xi,t) \;=\; e^{2\gamma t}\left(\frac{\gamma}{\sqrt{a}} + \frac{a'}{4\,a^{3/2}}\right)'\,|\xi|^2\,\mathrm{Re}(\bar{w}\,w_t)\,. \tag{9.9}$$

Besides, introducing the energy

$$\mathcal{E}(\xi,t) \stackrel{\text{def}}{=} e^{2\gamma t}\left(\frac{\sqrt{a}}{2}\,|\xi|^4\,|w|^2 + \frac{1}{2\sqrt{a}}\,|\xi|^2\,|w_t|^2\right), \tag{9.10}$$

we easily have the estimates:

$$\big|(\mathcal{Q}-\mathcal{E})(\xi,t)\big| \;\le\; \Big|\frac{\gamma}{\sqrt{a}} + \frac{a'}{4\,a^{3/2}}\Big|\;\frac{\mathcal{E}(\xi,t)}{|\xi|}\,, \tag{9.11}$$

$$\big|Q'(\xi,t)\big| \;\le\; \Big|\Big(\frac{\gamma}{\sqrt{a}} + \frac{a'}{4\,a^{3/2}}\Big)'\Big|\;\frac{\mathcal{E}(\xi,t)}{|\xi|}\,, \tag{9.12}$$

for all $t \in [0, T)$ and $|\xi| > 0$. Finally, applying these inequalities, we prove:

Lemma 9.2. *Assume that* (9.2) *holds. Besides, let $w(\xi, t)$ be a solution of* (9.3). *Then for all $\bar{T} \in (0, T)$ there exists $\varrho = \varrho(\bar{T}) > 0$ such that*

$$|\xi| \ge \varrho \quad \Rightarrow \quad \mathcal{E}(\xi,t) \le 2\,\mathcal{E}(\xi,0) \quad \textit{for} \quad t \in [0,\bar{T}]\,. \tag{9.13}$$

Proof. In the interval $I = [0, \bar{T}]$ we have

$$\begin{gathered} \inf_I a(t) > 0\,, \\ \sup_I |a'(t)| < \infty\,, \quad \sup_I |a''(t)| < \infty\,. \end{gathered} \tag{9.14}$$

Hence, by (9.11) and (9.12), there exists $C = C(\bar{T}) > 0$ such that:

$$|\mathcal{Q} - \mathcal{E}| \leq \frac{C}{|\xi|}\,\mathcal{E}\,, \quad |\mathcal{Q}'| \leq \frac{C}{|\xi|}\,\mathcal{E}\,, \tag{9.15}$$

for all $t \in [0, \bar{T}]$ and $|\xi| > 0$. Then, using (9.9) and (9.15), it is easy to derive the estimate (9.13). In fact, integrating (9.9) with respect to t, for $|\xi| > 0$ and $0 \leq t \leq \bar{T}$, we find the inequality

$$\mathcal{E}(\xi, t)\left(1 - \frac{C}{|\xi|}\right) \leq \mathcal{E}(\xi, 0)\left(1 + \frac{C}{|\xi|}\right) + \frac{C}{|\xi|}\int_0^t \mathcal{E}(\xi, \tau)\,d\tau\,. \tag{9.16}$$

From this, applying Gronwall's lemma, we can estimate $\mathcal{E}(\xi, t)$ if $|\xi| > C$. In particular, taking $|\xi|$ large enough, we obtain (9.13). ∎

10 Appendix II

Let us consider the ordinary problem

$$w'' + (a_o + b(t))\,|\xi|^2\,w + 2\gamma\,w' = 0\,, \quad t \geq 0\,, \tag{10.1}$$

$$w(\xi, 0) = w_0(\xi)\,, \quad w_t(\xi, 0) = w_1(\xi)\,, \tag{10.2}$$

with a parameter $\xi \in \mathbb{R}^n$ and coefficients a_o, γ, $b(t)$ such that

$$a_o,\, \gamma > 0, \quad b(t) \in L^1_{loc}[0, \infty)\,. \tag{10.3}$$

Here, we will estimate $|w(\xi, t)|$ for $|\xi|$ small enough.

Lemma 10.1. *For the solution $w(t, \xi)$ of* (10.1)-(10.2) *for all $t \geq 0$ there holds*

$$|w(t, \xi)| \leq W(\xi)\,\exp\left\{-\frac{a_o\,|\xi|^2\,t}{2\,\gamma} + \frac{|\xi|^2}{\gamma}\int_0^t |b(\tau)|\,d\tau\right\}\,, \tag{10.4}$$

for all $|\xi| \leq \sqrt{\frac{3}{4\,a_o}}\,\gamma$, where $W(\xi) = \Big[2\,|w_0(\xi)| + \gamma^{-1}\,|w_1(\xi)|\Big]$.

Proof. By putting

$$w(\xi, t) \overset{\text{def}}{=} e^{-\gamma t}\,z(\xi, t)\,, \tag{10.5}$$

the Cauchy problem (10.1)-(10.2) is transformed to

$$z'' + \Big(a_o + b(t)\Big)\,|\xi|^2 z - \gamma^2 z = 0\,, \tag{10.6}$$

$$z(0, \xi) = w_0(\xi) \quad z'(0, \xi) = \gamma\,w_0(\xi) + w_1(\xi)\,. \tag{10.7}$$

To estimate $z(\xi, t)$, we rewrite (10.6) in the form

$$z'' - \lambda^2 z = -b(t)|\xi|^2 z\,, \tag{10.8}$$

with $\lambda \in \mathbb{C}$ such that

$$\lambda^2 = \gamma^2 - a_o|\xi|^2\,. \tag{10.9}$$

Assuming $\lambda \neq 0$, by the Lagrange's method of variation of parameters, we easily obtain that $z(\xi, t)$ satisfies the relation

$$\begin{aligned} z(\xi,t) &= \frac{w_0(\xi)}{2}\left(e^{\lambda t}+e^{-\lambda t}\right)+\frac{\gamma\, w_0(\xi)+w_1(\xi)}{2\lambda}\left(e^{\lambda t}-e^{-\lambda t}\right) \\ &\quad -\frac{|\xi|^2}{2\lambda}\int_0^t\left(e^{\lambda(t-\tau)}-e^{-\lambda(t-\tau)}\right) b(t)\, z(\xi,\tau)\, d\tau\,. \end{aligned} \tag{10.10}$$

Besides, setting

$$\phi_\lambda(s) \stackrel{\text{def}}{=} 1-e^{-2\lambda s} \quad \text{for} \quad s\in\mathbb{R}\,, \tag{10.11}$$

we have also

$$\begin{aligned} e^{-\lambda t} z(\xi,t) &= w_0(\xi)\,\frac{1+e^{-2\lambda t}}{2}+\Big[\gamma\, w_0(\xi)+w_1(\xi)\Big]\,\frac{\phi_\lambda(t)}{2\lambda} \\ &\quad -\frac{|\xi|^2}{2\lambda}\int_0^t \phi_\lambda(t-\tau)\, b(\tau)\; e^{-\lambda\tau} z(\xi,\tau)\, d\tau\,. \end{aligned} \tag{10.12}$$

Now let $|\xi| \leq \sqrt{\frac{3}{4a_o}}\,\gamma$ as in the statement above, thus $\gamma^2 - a_o|\xi|^2 \geq \gamma^2/4$. Choosing λ as the positive square root of the right side of (10.9), we easily see that

$$\frac{\gamma}{2} \leq \lambda \leq \gamma - \frac{a_o\,|\xi|^2}{2\,\gamma}\,. \tag{10.13}$$

Furthermore, having $\lambda > 0$, $\phi_\lambda(s)$ is increasing and $0 \leq \phi_\lambda(s) \leq 1$ for $s \in [0,\infty)$. Hence, applying Gronwall's lemma to (10.12), we get

$$\begin{aligned} e^{-\lambda t}|z(\xi,t)| &\leq \Big[\,|w_0(\xi)|+|\gamma\, w_0(\xi)+w_1(\xi)|\,\frac{\phi_\lambda(t)}{2\lambda}\,\Big] \\ &\qquad \cdot\; \exp\left\{\frac{|\xi|^2}{2\lambda}\int_0^t \phi_\lambda(t-\tau)\,|b(\tau)|\, d\tau\right\}\,. \end{aligned} \tag{10.14}$$

Finally, taking account of (10.11) and (10.13), for $w(\xi, t)$ we have

$$\begin{aligned} |w(\xi,t)| &= e^{-(\gamma-\lambda)t} e^{-\lambda t}|z(\xi,t)| \\ &\leq\; W(\xi)\; \exp\left\{-\frac{a_o\,|\xi|^2\, t}{2\,\gamma}\;+\;\frac{|\xi|^2}{\gamma}\int_0^t |b(\tau)|\, d\tau\right\}\,, \end{aligned} \tag{10.15}$$

for $|\xi| \leq \sqrt{\frac{3}{4a_o}}\,\gamma$ and $t \geq 0$, with $W(\xi)$ defined as in the statement above. ■

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