On the singular locus of Grassmann secant varieties

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Abstract

Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety. If the (h, k)-Grassmann secant variety $G_{h,k}(X)$ of X is not the whole Grassmannian $\mathbb{G}(h, N)$, we have that the singular locus of $G_{h,k}(X)$ contains $G_{h,k-1}(X)$. Moreover, if X is a smooth curve without $(2k+2)$-secant $2k$-space divisors, we obtain the equality $\mathrm{Sing}(G_{h,k}(X)) = G_{h,k-1}(X)$.

1 Introduction

Let $X \subset \mathbb{P}^N$ be a projective irreducible non-degenerate variety and let h and k be integers such that $0 \leq h \leq k \leq N$. Denote by $G_{h,k}(X) \subset \mathbb{G}(h, N)$ the (h, k)-Grassmann secant variety of X, i.e. the closure of the set of h-dimensional linear subspaces contained in the span of $k + 1$ independent points of X.

In case $h = 0$, the variety $G_{h,k}(X)$ coincides with kth secant variety $S^k(X)$ of X. This case has been intensively studied (see for example [Zak]). The study of the case $h > 0$ is more recent (see for example [ChCo]).

Grassmann secant varieties are interesting objects, since they are in relation with projections of varieties into lower dimensional projective spaces. They are also in connection with Waring problems for homogeneous forms and tensors (see for example [CaCh] and [Fon]).

In this paper, we will study the singular locus of Grassmann secant varieties. If $G_{h,k}(X) \neq \mathbb{G}(h, N)$, we will prove that the singular locus of $G_{h,k}(X)$ contains $G_{h,k-1}(X)$ (Proposition 3.1). Moreover, if X is a smooth curve such that every effective divisor of length $2k+2$ on X spans a $(2k+1)$-dimensional linear subspace of $\mathbb{P}^N$, we are able to prove that the singular locus of $G_{h,k}(X)$ is equal to $G_{h,k-1}(X)$ (Theorem 3.3). Note that these results generalize the results established in [Cop].

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2 Preliminaries

We start this section with some conventions.

We denote by $\mathbb{P}^N$ the projective space of dimension N over the field $\mathbb{C}$ of complex numbers. We say that a variety $X \subset \mathbb{P}^N$ is non-degenerate if it is not contained in any hyperplane of $\mathbb{P}^N$.

The linear span $\langle Y \rangle$ of a closed subscheme Y of $\mathbb{P}^N$ is the intersection of all hyperplanes H containing Y as a closed subscheme. If $P_0, \ldots, P_k$ are different points of $\mathbb{P}^N$, we write $\langle P_0, \ldots, P_k \rangle$ to denote the linear span of the reduced subscheme of $\mathbb{P}^N$ supported by those points. If $P \in Y$, we denote by $\mathbb{T}_P(Y)$ the embedded tangent space of Y at P.

Definition 2.1 (Plücker embedding of Grassmannians)**.** The Grassmannian $\mathbb{G}(h, N)$, parameterizing h-dimensional linear subspaces in $\mathbb{P}^N$, can be embedded in a large projective space as follows. Let H be an element of $\mathbb{G}(h, N)$ spanned by points $Q_0, \ldots, Q_h \in \mathbb{P}^N$ with $Q_i = (q_{i0} : \ldots : q_{iN})$ for all $i \in \{0, \ldots, h\}$. Let $\mathcal{S}$ be the set of subsets $D \subset \{0, \ldots, N\}$ of length $h+1$. Take $D \in \mathcal{S}$ and write $D = \{j_0, \ldots, j_h\}$ with $j_0 < \ldots < j_h$. Denote by $p_D(H)$ the determinant of the matrix $[q_{ij_k}]_{i,k \in \{0,\ldots,h\}}$. Consider the map $p : \mathbb{G}(h, N) \to \mathbb{P}^M$ with $M = \binom{N+1}{h+1} - 1$ sending H to $(p_D(H))_{D \in \mathcal{S}}$. Note that p is well-defined since the image of H is independent of the choice of its generators $Q_0, \ldots, Q_h$. We call p the Plücker embedding of $\mathbb{G}(h, N)$. By considering affine subsets of $\mathbb{G}(h, N)$, one can show that $\mathbb{G}(h, N)$ is smooth of dimension $(h+1)(N-h)$.

Lemma 2.2. *Let $P, P_1, \ldots, P_r \in \mathbb{P}^N$ such that $P \in \langle P_1, \ldots, P_r \rangle$ and let $G \in \mathbb{G}(h-1, N)$ be a linear subspace not containing any of the points $P, P_1, \ldots, P_r$. Let p be the Plücker embedding of $\mathbb{G}(h, N)$ in $\mathbb{P}^M$. Then we have $p(\langle G, P \rangle) \in \langle p(\langle G, P_1 \rangle), \ldots, p(\langle G, P_r \rangle) \rangle$.*

Proof. Fixing projective coordinates in $\mathbb{P}^N$, we can write P as a linear combination $a_1.P_1 + \ldots + a_r.P_r$ with $a_1, \ldots, a_r \in \mathbb{C}$, since $P \in \langle P_1, \ldots, P_r \rangle$. By expanding the determinant $p_D(\langle G, P \rangle)$ along the last row (i.e. the row corresponding to the point P), we get

$$p_D(\langle G, P \rangle) = a_1.p_D(\langle G, P_1 \rangle) + \ldots + a_r.p_D(\langle G, P_r \rangle))$$

for every subset D of length $h+1$ of $\{0, \ldots, N\}$. We conclude

$$p(\langle G, P \rangle) = a_1.p(\langle G, P_1 \rangle) + \ldots + a_r.p(\langle G, P_r \rangle),$$

hence $p(\langle G, P \rangle) \in \langle p(\langle G, P_1 \rangle), \ldots, p(\langle G, P_r \rangle) \rangle$. ∎

Definition 2.3 (Grassmann secant varieties)**.** Let $X \subset \mathbb{P}^N$ be a projective irreducible non-degenerate variety. If $k \leq N$ is an integer, denote by

$$i : X^{k+1} \dashrightarrow \mathbb{G}(k, N)$$

the rational map sending $(P_0, \ldots, P_k)$ to $\langle P_0, \ldots, P_k \rangle$. An element of the image is called a $(k+1)$-secant k-space of X.

Consider for all integers $h \leq k$ the diagram

$$\begin{tikzcd}
 & & I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X^{k+1} \arrow[r, dashed, "i"] & \mathbb{G}(k,N) & & \mathbb{G}(h,N)
\end{tikzcd}$$

where $I = \{(G,H) | \, G \supset H\} \subset \mathbb{G}(k,N) \times \mathbb{G}(h,N)$ and p_1, p_2 the projections to the first and second factor, respectively. We define the (h,k)-Grassmann secant variety $G_{h,k}(X)$ of X as the subvariety $p_2(p_1^{-1}(\overline{\mathrm{Im}(i)}))$ of $\mathbb{G}(h,N)$.

Remark 2.4. Using the Plučker embedding of $G(h,N)$ in $\mathbb{P}^M$, we can consider the Grassmann secant variety $G_{h,k}(X)$ as a subvariety of $\mathbb{P}^M$.

3 Singular locus of Grassmann secant varieties

Proposition 3.1. *If $X \subset \mathbb{P}^N$ is a non-degenerate variety and $0 \leq h < k$ are integers such that*

$$G_{h,k}(X) \subsetneq \mathbb{G}(h,N) \subset \mathbb{P}^M,$$

we have $G_{h,k-1}(X) \subset Sing(G_{h,k}(X))$.

Proof. Let H be a general element of $G_{h,k-1}(X)$, hence $H \subset \langle P_0, \ldots, P_{k-1} \rangle$ for some $P_0, \ldots, P_{k-1} \in X$. Since $\mathrm{Sing}(G_{h,k}(X))$ is a Zariski closed subset of $G_{h,k}(X)$, we only need to show that $H \in \mathrm{Sing}(G_{h,k}(X))$. Denote by $\mathbb{T} := \mathbb{T}_H(G_{h,k}(X)) \subset \mathbb{P}^M$ the embedded tangent space of $G_{h,k}(X) \subset \mathbb{P}^M$ at H.

We write H as $\langle G, Q \rangle$ with $G \in \mathbb{G}(h-1,N)$ and $Q \in H \subset \mathbb{P}^N$. Let $P \in X \setminus (H \cup \{P_0, \ldots, P_{k-1}\})$.

Let $R \in \langle Q, P \rangle$. If $R \in G$, we have $R \neq Q$ and

$$P \in \langle Q, R \rangle \subset \langle Q, G \rangle \subset \langle P_0, \ldots, P_{k-1} \rangle.$$

Since $X \cap \langle P_0, \ldots, P_{k-1} \rangle = \{P_0, \ldots, P_{k-1}\}$ as a scheme, this gives us a contradiction. We get that $\langle G, R \rangle$ is h-dimensional.

Write $L_{G,Q,P}$ to denote the subset $\{\langle G, R \rangle \,|\, R \in \langle Q, P \rangle\} \subset \mathbb{G}(h,N)$. Note that $L_{G,Q,P} \subset G_{h,k}(X)$, since $R \in \langle Q, P \rangle$ implies

$$\langle G, R \rangle \subset \langle G, Q, P \rangle = \langle H, P \rangle \subset \langle P_0, \ldots, P_{k-1}, P \rangle.$$

On the other hand, Lemma 2.2 implies $p(L_{G,Q,P})$ is a line in $\mathbb{P}^M$. This gives us $p(L_{G,Q,P}) \subset \mathbb{T}$ and a fortiori $p(\langle G, P \rangle) \in \mathbb{T}$ for all $P \in X \setminus (H \cup \{P_0, \ldots, P_k\})$. Since $\mathbb{T}$ is linear, we even get $p(\langle G, P \rangle) \in \mathbb{T}$ for all $P \in X \setminus H$.

We claim that $p(\widetilde{H}) \in \mathbb{T}$ if $\dim(H \cap \widetilde{H}) \geq h-1$. Indeed, take $G = H \cap \widetilde{H}$ and $Q \in \widetilde{H} \setminus G$, thus $\widetilde{H} = \langle G, Q \rangle$. Since $X \subset \mathbb{P}^N$ is non-degenerate, we can find points $P'_0, \ldots, P'_N \in X \setminus H$ such that $\langle P'_0, \ldots, P'_N \rangle = \langle X \rangle = \mathbb{P}^N$. Now we can apply Lemma 2.2 because $p(\langle G, P'_i \rangle) \in \mathbb{T}$. This gives us $p(H) = p(\langle G, Q \rangle) \in \mathbb{T}$ since $\mathbb{T}$ is linear.

To finish the proof of this theorem, it is enough to show that the dimension of the span of $\{p(\widetilde{H}) \,|\, \dim(H \cap \widetilde{H}) \geq h-1\}$ is equal to $\dim(\mathbb{G}(h,N)) =$

$(h+1)(N-h)$. Take projective coordinates on $\mathbb{P}^N$ such that H is the linear subspace $\langle E_0,\ldots,E_h\rangle$, where $E_i = (0 : \ldots : 0 : 1 : 0 \ldots : 0)$ with one on the ith coordinate. Denote for each $i \in \{0,\ldots,h\}$ and $j \in \{h+1,\ldots,N\}$, the subspace $\langle E_0,\ldots,E_{i-1},E_{i+1},\ldots,E_h,E_j\rangle$ by $H_{i,j}$. The above claim implies that $p(H_{i,j}) \in \mathbb{T}$ since $\dim(H \cap H_{i,j}) \geq h-1$. It is easy to see that the set of points $p(H), p(H_{0,h+1})$, $\ldots, p(H_{h,N})$ is independent, hence $\dim(\mathbb{T}) \geq (h+1)(N-h)$. Of course, we have $\dim(\mathbb{T}) \leq (h+1)(N-h)$, since $G_{h,k}(X) \subset \mathbb{G}(h,N)$ and $\mathbb{G}(h,N)$ is smooth. ∎

In order to state Theorem 3.3, we need the following definition.

Definition 3.2. Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate curve. If D is an effective divisor of degree d such that $\dim\langle D\rangle = e$, we say that D is an d-secant e-space divisor.

Theorem 3.3. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate curve and let $0 \leq h < k$ be integers such that $G_{h,k}(X) \neq \mathbb{G}(h,N)$. If X has no $(2k+2)$-secant $2k$-space divisors, we have $Sing(G_{h,k}(X)) = G_{h,k-1}(X)$.*

Proof. Since X has no $(k+2)$-secant k-space divisors, [Cop] implies that the map $i : X^{k+1} \to \mathbb{G}(k,N)$ is an embedding. Hence we may identify X^{k+1} and $i(X^{k+1})$.

Denote $J := p_1^{-1}(i(X^{k+1})) \subset I$ and $q : J \to \mathbb{G}(h,N)$. Since the restriction of p_1 to J is a $\mathbb{P}^{(h+1)(k-h)}$-bundle above $i(X^{k+1})$, we see Y is smooth. Also note that $q(J) = G_{h,k}(X)$.

Let $(D,H) \in J$ with $H \notin G_{h,k-1}(X)$. We are going to prove that q locally defines an embedding of J in $\mathbb{G}(h,N)$ at (D,H). In particular, we will show that the tangent map

$$d_{(D,H)}(q) : \mathbb{T}_{(D,H)}(J) \to \mathbb{T}_H(\mathbb{G}(h,N))$$

is injective.

Assume $d_{(D,H)}(q)$ is not injective, so there exists a tangent vector $(\alpha,\beta) \in \mathbb{T}_{(D,H)}(J)$ with $\beta = 0$ but $\alpha \neq 0$. Take a holomorphic arc $D(t)$ in $i(X^{k+1})$ with $D(0) = D$ corresponding to α. The arc $D(t)$ gives rise to holomorphic arcs $P_0(t),\ldots,P_k(t)$ on X with $P_i(0) = P_i$, such that $D(t) = \langle P_0(t),\ldots,P_k(t)\rangle$. Let $\widehat{P}_0,\ldots,\widehat{P}_k,\widehat{P}_0(t),\ldots,\widehat{P}_k(t),\widehat{D}(t),\widehat{H}$ be corresponding objects in the affine cone $\mathbb{C}^{N+1}$ above $\mathbb{P}^N$ of respectively $P_0,\ldots,P_k,P_0(t),\ldots,P_k(t),D(t),H$.

Using the description of tangent spaces of Grassmannians in [Har, Lecture 16], the tangent vector $(\alpha,\beta) \in \mathbb{T}_{(D,H)}(J) \subset \mathbb{T}_{(D,H)}(I)$ gives rise to a commutative diagram

$$\begin{array}{ccc} \widehat{H} & \xrightarrow{\quad\beta\quad} & \mathbb{C}^{N+1}/\widehat{H} \\ \downarrow & & \downarrow \\ \widehat{D} & \xrightarrow{\quad\alpha\quad} & \mathbb{C}^{N+1}/\widehat{D} \end{array}$$

where $\beta \in \mathrm{Hom}(\widehat{H},\mathbb{C}^{N+1}/\widehat{H})$ and $\alpha \in \mathrm{Hom}(\widehat{D},\mathbb{C}^{N+1}/\widehat{D})$. Since $\beta \equiv 0$, we have $\alpha|_{\widehat{H}} \equiv 0$.

Let $\widehat{P} \in \widehat{D}$, so $\widehat{P} = a_0.\widehat{P}_0 + \ldots + a_k.\widehat{P}_k$ for some $a_0,\ldots,a_k \in \mathbb{C}$. If $\widehat{P}(t)$ is an arc satisfying $\widehat{P}(t) \in \widehat{D}(t)$ and $\widehat{P}(0) = \widehat{P}$, the map α sends $\widehat{P}$ to $\widehat{v} + \widehat{D}$, with $\widehat{v} = \frac{d\widehat{P}(t)}{dt}(0)$. For example, we can take

$$\widehat{P}(t) = a_0.\widehat{P}_0(t) + \ldots + a_k.\widehat{P}_k(t),$$

hence $\alpha(\widehat{P}) = a_0.\widehat{v_0} + \ldots + a_k.\widehat{v_k} + \widehat{D}$ with $\widehat{v_i} = \frac{d\widehat{P_i}(t)}{dt}(0)$. We conclude that α is the map sending $a_0.\widehat{P_0} + \ldots + a_k.\widehat{P_k}$ to $a_0.\widehat{v_0} + \ldots + a_k.\widehat{v_k} + \widehat{D}$.

If $\widehat{P} \in \widehat{H} \subset \widehat{D}$ is general, we have $\widehat{P} = a_0.\widehat{P_0} + \ldots + a_k.\widehat{P_k}$ with $a_0, \ldots, a_k$ all different from zero. Since $\alpha|_{\widehat{H}} \equiv 0$, we get

$$a_0.\widehat{v_0} + \ldots + a_k.\widehat{v_k} \in \widehat{D}.$$

This is only possible if $\widehat{v_0} = \ldots = \widehat{v_k} = 0$, since $2D = 2P_0 + \ldots + 2P_k$ is a $(2k+2)$-secant $(2k+1)$-space divisor of X. However, this implies $\alpha \equiv 0$, a contradiction.

We have proven that q is locally an embedding of J in $\mathbb{G}(h, N)$ around (D, H) if $H \notin G_{h,k-1}(X)$. To finish this theorem, we only need to show that J is injective outside $q^{-1}(G_{h,k-1}(X))$, since $\dim(J) = \dim(G_{h,k}(X))$ (see [ChCi]).

Let $H \in G_{h,k}(X) \setminus G_{h,k-1}(X)$ and assume that (D_1, H) and (D_2, H) are two different points of J above H. Consider D_1 and D_2 as divisors on X. Let E be the scheme theoretical intersection of D_1 and D_2. If $\deg(E) = e$, we have $e < k+1$ since $D_1 \neq D_2$. Since $H \notin G_{h,k-1}(X)$, we see $H \not\subset \langle E \rangle$, hence

$$\dim\langle\langle D_1\rangle \cap \langle D_2\rangle\rangle \geq \dim\langle\langle E\rangle, H\rangle \geq e.$$

So we have

$$d := \dim\langle\langle D_1\rangle, \langle D_2\rangle\rangle = \dim\langle D_1\rangle + \dim\langle D_2\rangle - \dim\langle\langle D_1\rangle \cap \langle D_2\rangle\rangle \leq 2k - e.$$

Since $d = \dim\langle D_1 + D_2 - E\rangle$, we get that $D_1 + D_2 - E$ is a $(2k+2-e)$-secant d-space divisor on X with $d \leq 2k - e$, a contradiction. ∎

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