Common fixed points for Banach operator pairs from the set of best approximations

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Abstract

The existence of common fixed point results for a Banach operator pair under certain generalized φ-contractions is established. As applications, the corresponding invariant best approximation results are proved. Our results unify and generalize various known results to the more general class of non-commuting mappings.

1 Introduction and preliminaries

Let M be a subset of a normed space $(X, \|.\|)$. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called I-Lipschitz if there exists $k \geq 0$ such that $d(Tx, Ty) \leq kd(Ix, Iy)$ for any $x, y \in M$. If $k < 1$ (respectively, $k = 1$), then T is called an I-contraction (respectively, I-nonexpansive). The mapping T is said to be completely continuous if $\{x_n\}$ converges weakly to $x \in M$ implies that $Tx_n \to Tx$. A point $x \in M$ is a coincidence point (common fixed point) of I and T if $Ix = Tx$ ($x = Ix = Tx$). The set of fixed points of I is denoted by $F(I)$. The set of coincidence points of I and T is denoted by $C(I, T)$. The pair $\{I, T\}$ is called (1) commuting if $TIx = ITx$ for all $x \in M$, (2) R-weakly commuting if for all $x \in M$, there exists $R > 0$ such that $d(ITx, TIx) \leq Rd(Ix, Tx)$. If $R = 1$, then the maps are called weakly commuting; (3) compatible [16] if $\lim_n d(TIx_n, ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M;

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(4) weakly compatible if they commute at their coincidence points, i.e., if $ITx = TIx$ whenever $Ix = Tx$. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called (5) C_q-commuting if $ITx = TIx$ for all $x \in C_q(I,T)$, where $C_q(I,T) = \cup\{C(I,T_k) : 0 \leq k \leq 1\}$ where $T_k x = (1-k)q + kTx$; (6) pointwise R-subweakly commuting [23] on M if for given $x \in M$, there exists a real number $R > 0$ such that $\|ITx - TIx\| \leq R dist(Ix, [q, Tx])$, where $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ and $dist(u, M) = inf\{\|y - u\| : y \in M\}$; (7) R-subweakly commuting on M if for all $x \in M$, there exists a real number $R > 0$ such that $\|ITx - TIx\| \leq R dist(Ix, [q, Tx])$; if $R = 1$, then the maps are called 1-subweakly commuting.
The set $P_M(u) = \{x \in M : \|x - u\| = dist(u, M)\}$ is called the set of best approximants to $u \in X$ out of M. Let $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$. We denote by $\mathbb{N}$ and $cl(M)$ ($wcl(M)$), the set of positive integers and the closure (weak closure) of a set M in X, respectively. A Banach space X satisfies Opial's condition if, for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \neq x$. Every Hilbert space and the space $l_p (1 < p < \infty)$ satisfy Opial's condition. The map $T : M \to X$ is said to be demiclosed at 0 provided for every sequence $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ convergent to $0 \in X$, we have $0 = Tx$.

The Banach Contraction Mapping Principle states that if (X, d) is a complete metric space and $T : X \to X$ satisfies

$$(Tx, Ty) \leq \lambda d\,(x, y)$$

for all $x, y \in X$, where $0 < \lambda < 1$, then T has a unique fixed point, say z in X, and the Picard iterations $\{T^n x\}$ converge to z for all $x \in X$. Jungck extended this principle in the following way;

Theorem 1.1 (Jungck [15]). Let (X, d) be a complete metric space, $T,\ I : X \to X$ satisfy the following contraction-type condition on X,

$$d\,(Tx, Ty) \leq \lambda d(Ix, Iy)$$

where $0 < \lambda < 1$. Suppose that $T,\ \ I$ are commuting maps, I is continuous and $T(X) \subset I(X)$. Then T and I have a unique common fixed point in X.

Jungck [16] coined the idea of compatible maps and extended Theorem 1.1 to compatible maps. Ćirić [6, 7] introduced and studied self-mappings on X satisfying

$$d\,(Tx, Ty) \leq \lambda \max\{d\,(x, y)\,, d\,(x, Tx)\,, d\,(y, Ty)\,, d\,(x, Ty)\,, d\,(y, Tx)\}\,, \qquad (1.1)$$

where $0 < \lambda < 1$.

Theorem 1.2 (Das and Naik [8]). Let (X,d) be a complete metric space, T, $I : X \to X$ satisfy the following contraction-type condition on X,

$$d\,(Tx,Ty) \leq \lambda \max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), d(Ix,Ty), d(Iy,Tx)\}$$

where $0 < \lambda < 1$. Suppose that T, I are commuting maps, I is continuous and $T(X) \subset I(X)$. Then T and I have a unique common fixed point in X.

Further investigations in this direction were carried out by Agarwal et al. [1], Berinde [4], Jungck [17, 18], Jungck and Hussain [19], Hussain et al. [11], Hussain and Rhoades [14], O'Regan and Hussain [23] and many other mathematicians(see [7] and references therein). Applications of the contraction and generalized contraction principle for self mappings are well known (c.f. [7, 25, 26]).

In 1963, Meinardus [22] employed the Schauder fixed point theorem to prove a result regarding invariant approximation. Further generalizations of the result of Meinardus were obtained by Habiniak [9], Jungck and Sessa [20], Sahab et al. [27], Singh [29], Smoluk [31] and Subrahmanyam [32]. Recently, Al-Thagafi [2] extended the work in [27, 29, 31, 32] and proved some results on invariant approximations for commuting maps. Hussain and Jungck [12], Jungck and Hussain [19], O'Regan and Hussain [23], and Pathak et al. [24] extended the work of Al-Thagafi [2] for pointwise R-subweakly commuting, compatible and C_q-commuting maps. Recently, Chen and Li [5] introduced the class of a Banach operator pairs, as a new class of noncommuting maps which is further investigated by Hussain [10] and Pathak and Hussain [25]. The purpose of this paper is to prove common fixed point results for newly defined class of the Banach operator pairs (T,I) where T is generalized I-contraction with respect to a comparison function φ (see [1, 4]). We shall prove our results without the assumptions of linearity or affinity of either T or I and I-nonexpansiveness of T. As application, certain invariant approximation results for this class of maps are also derived. Our results extend, unify and compliment the work of Al-Thagafi [2], Chen and Li [5], Habiniak [9], Pathak and Hussain [25], Jungck and Sessa [20], Khan and Khan [21], Meinardus [22], Sahab, Khan and Sessa [27], Shahzad [28], Singh [29], Smoluk [31], Subrahmanyam [32] and many others.

2 Main Results

The ordered pair (T,I) of two self mappings of a metric space (X,d) is called a Banach operator pair, if the set $F(I)$ is T-invariant, namely $T(F(I)) \subseteq F(I)$. Obviously, commuting pair (T,I) is a Banach operator pair but the converse does not hold, in general ;see [5, 10] and examples below. If (T,I) is a Banach operator pair then, (I,T) need not be a Banach operator pair (cf. Example 1 [5]). If the self mappings T and I of X satisfy

$$d(ITx,Tx) \leq kd(Ix,x),$$

for all $x \in X$ and $k \geq 0$, then (T, I) is a Banach operator pair. In particular , when $I = T$ and X is a normed space, the above inequality can be rewritten as

$$\|T^2x - Tx\| \leq k\|Tx - x\|$$

for all $x \in X$. Such a T is called a Banach operator of type k in [32] (see [9] and [21]).

We begin with the following result.

Theorem 2.1. Let (X, d) be a metric space , K be a nonempty subset of X, $T : K \to K$ be a mapping satisfying the following contraction-type condition:

$$d(Tx, Ty) \leq \max\{\varphi_1(d(x, y)), \varphi_2(d(x, Tx)), \varphi_3(d(y, Ty)), \varphi_4(d(x, Ty)), \varphi_5(d(y, Tx))\} \tag{2.1}$$

for all $x, y \in K$, where $\varphi_j : [0, +\infty) \to [0, +\infty)$ $(j = 1, 2, 3, 4, 5)$ are real functions which are continuous from the right and each has the following properties:
(1) $\varphi_j(t) < t$ for $t > 0$ and
(2) $\varphi_j(t)$ is non-decreasing.
If $cl(T(K))$ is bounded and complete, then T has a unique fixed point in K.

Proof. Let x_0 in K be arbitrary. Define inductively a sequence $\{x_n\}$ in K as follows:

$$x_n = Tx_{n-1} \quad \text{for all} \quad n \geq 1.$$

We shell prove that $\{T^n x\}$ is a Cauchy sequence. Set

$$A_n = \cup_{i=n}^{\infty} Tx_i,$$
$$\alpha_n = diam(A_n)$$

$(n = 0, 1, 2, \ldots)$. Since $\alpha_n \geq \alpha_{n+1}$, it follows that $\{\alpha_n\}$ converges to some $\alpha \geq 0$. We shall show that $\alpha = 0$. Let n be arbitrary and let $T^r x, T^s x \in A_{n+1}$. Then from (2.1),

$$\begin{aligned} d(T^r x, T^s x) &= d(TT^{r-1}x, T^s T^{s-1}x) \\ &\leq \max\{\varphi_1(d(T^{r-1}x, T^{s-1}x), \varphi_2(d(T^{r-1}x, T^r), \varphi_3(d(T^{s-1}x, T^s x), \\ &\varphi_4(d(T^{r-1}x, T^s x), \varphi_5(d(T^{s-1}x, T^r x)\}. \end{aligned} \tag{2.2}$$

Since $T^{r-1}x, T^r x, T^{s-1}x, T^s x \in A_n$ and φ_j are nondecreasing, from (2.2) we get

$$d(T^r x, T^s x) \leq \max\{\varphi_1(\alpha_n), \varphi_2(\alpha_n), \varphi_3(\alpha_n), \varphi_4(\alpha_n), \varphi_5(\alpha_n)\}.$$

Since $T^r x, T^s x \in A_{n+1}$ are arbitrary, we have

$$\alpha_{n+1} \leq \max\{\varphi_1(\alpha_n), \varphi_2(\alpha_n), \varphi_3(\alpha_n), \varphi_4(\alpha_n), \varphi_5(\alpha_n)\}.$$

Hence, as $\alpha \leq \alpha_{n+1}$,

$$\alpha \leq \max\{\varphi_1(\alpha_n), \varphi_2(\alpha_n), \varphi_3(\alpha_n), \varphi_4(\alpha_n), \varphi_5(\alpha_n)\}.$$

Suppose that $\alpha > 0$. Then, by the continuity from the right of φ_j and the property (1), we have

$$\begin{aligned}\alpha &\leq \max\{\lim_{\alpha_n\to\alpha^+}\varphi_1(\alpha_n), \lim_{\alpha_n\to\alpha^+}\varphi_2(\alpha_n), \lim_{\alpha_n\to\alpha^+}\varphi_3(\alpha_n), \lim_{\alpha_n\to\alpha^+}\varphi_4(\alpha_n), \lim_{\alpha_n\to\alpha^+}\varphi_5(\alpha_n)\}\\ &\leq \max\{\varphi_1(\alpha), \varphi_2(\alpha), \varphi_3(\alpha), \varphi_4(\alpha), \varphi_5(\alpha)\}\\ &< \alpha,\end{aligned}$$

a contradiction. Therefore, $\alpha = 0$. Thus, we proved that

$$\lim_{n\to\infty} diam(\{T^n x, T^{n+1}x, \ldots, \}) = 0$$

and consequently, the sequence $\{T^n x\}$ is a Cauchy sequence. Since $cl(T(K))$ is complete, $\{x_n\}$ converges to some point, say z in $cl(T(K))$. We show that $Tz = z$. Suppose, by way of contradiction, that $d(z, Tz) > 0$. Using the triangle inequality and (2.1), we have

$$\begin{aligned}d(z,Tz) &\leq d(z,x_{n+1}) + d(Tx_n, Tz)\\ &\leq d(z,x_{n+1}) + \max\{\varphi_1(d(x_n,z)),\\ &\quad \varphi_2(d(x_n,Tx_n)), \varphi_3(d(z,Tz)), \varphi_4(d(x_n,Tz)), \varphi_5(d(z,Tx_n))\}.\end{aligned}$$

Hence, by the triangle inequality and by the property (2) of φ_j, we get

$$d(z,Tz) \leq d(z,x_{n+1}) + \max\{\varphi_1(r), \varphi_2(r), \varphi_3(r), \varphi_4(r), \varphi_5(r)\} \tag{2.3}$$

where $r = d(x_n, z) + d(x_n, Tx_n) + d(z, Tz)$.

Since $d(x_n, Tx_n) \to 0$ and $d(z, x_n) \to 0$ as $n \to \infty$, it follows that

$$(d(z,Tz) + d(z,x_n) + d(x_n,Tx_n)) \to d(z,Tz)$$

when n tends to infinity. Taking the limit in (2.3) when n tends to infinity and having in mind the continuity from the right of φ_j and the property (1), we get

$$\begin{aligned}d(z,Tz) &\leq \max\{\varphi_1(d(z,Tz)), \varphi_2(d(z,Tz)), \varphi_3(d(z,Tz)), \varphi_4(d(z,Tz)), \varphi_5(d(z,Tz))\}\\ &< d(z,Tz),\end{aligned}$$

a contradiction. Therefore, $d(z, Tz) = 0$. Hence $Tz = z$. Clearly, $z \in K$, as $z = Tz \in T(K) \subseteq K$. The uniqueness of a fixed point is implied by (2.1).

The following result extends and improves Theorems 1.1 and 1.2, Lemma 3.1 of [5] and Theorem 1 in [21].

Lemma 2.2. Let (X, d) be a metric space, K be a nonempty subset of X, T, $I : K \to K$ be mappings satisfying the following contraction-type condition on K,

$$\begin{aligned}d(Tx,Ty) \leq \max\{&\varphi_1(d(Ix,Iy)), \varphi_2(d(Ix,Tx)), \varphi_3(d(Iy,Ty)),\\ &\varphi_4(d(Ix,Ty)), \varphi_5(d(Iy,Tx))\},\end{aligned} \tag{2.4}$$

where $\varphi_j : [0, +\infty) \to [0, +\infty)$ $(j = 1, 2, 3, 4, 5)$ are real functions which are continuous from the right and each has the properties (1)–(2) of Theorem 2.1. Suppose that $F(I)$ is nonempty, $clT(F(I)) \subseteq F(I)$ and $cl(T(K))$ is complete and bounded, then T and I have a unique common fixed point in K.

Proof. $clT(F(I))$ being subset of $clT(M)$ is complete and bounded. Further, for all $x,\ y \in F(I)$, we have by the inequality (2.4),

$$\begin{aligned} d\left(Tx, Ty\right) &\leq \max\left\{\varphi_1\left(d\left(Ix, Iy\right)\right), \varphi_2\left(d\left(Ix, Tx\right)\right), \varphi_3\left(d\left(Iy, Ty\right)\right),\right. \\ &\qquad \left.\varphi_4\left(d\left(Ix, Ty\right)\right), \varphi_5\left(d\left(Iy, Tx\right)\right)\right\} \\ &= \max\left\{\varphi_1\left(d\left(x, y\right)\right), \varphi_2\left(d\left(x, Tx\right)\right), \varphi_3\left(d\left(y, Ty\right)\right),\right. \\ &\qquad \left.\varphi_4\left(d\left(x, Ty\right)\right), \varphi_5\left(d\left(y, Tx\right)\right)\right\} \end{aligned}$$

By Theorem 2.1, T has a unique fixed point z in $F(I)$ and consequently $F(T) \cap F(I)$ is singleton.

Corollary 2.3. Let (X, d) be a metric space, K be a nonempty subset of X, T, $I : K \to K$ be mappings satisfying the contraction-type condition (2.4) on K. Suppose that $cl(T(K))$ is complete and bounded, (T, I) is a Banach operator pair, $F(I)$ is nonempty and closed, then T and I have a unique common fixed point in K.

Proof. By our assumptions, $T(F(I)) \subseteq F(I)$ and $F(I)$ is nonempty and closed. Thus $clT(F(I)) \subseteq clF(I) = F(I)$. The result now follows from Lemma 2.2.

Now we are in position to state our main result.

Theorem 2.4. Let K be a nonempty subset of a normed space X and I and T be self mappings of K. Suppose that $F(I)$ is closed and q-starshaped with $q \in F(I)$. If (T, I) is a Banach operator pair and satisfies, for each $x, y \in K$,

$$\begin{aligned} \|Tx - Ty\| \leq \max\{&\varphi_1(\|Ix - Iy\|), \varphi_2(dist(Ix, [Tx, q])), \varphi_3(dist(Iy, [Ty, q])), \\ &\varphi_4(dist(Ix, [Ty, q])), \varphi_5(dist(Iy, [Tx, q]))\}, \end{aligned} \tag{2.5}$$

where $\varphi_j : [0, +\infty) \to [0, +\infty)$ $(j = 1, 2, 3, 4, 5)$ are real functions which are continuous from the right and each has the following properties:

(1) for each $k \in (0, 1)$, $\varphi_j(t) < \frac{t}{k}$ for $t > 0$,

(2) $\varphi_j(t)$ is non-decreasing,

then $K \cap F(I) \cap F(T) \neq \emptyset$, provided one of the following conditions holds;

(i) $cl(T(K))$ is compact and T is continuous,

(ii) X is complete, $wclT(K)$ is weakly compact, I is weakly continuous and $I - T$ is demiclosed at 0,

(iii) X is complete, $wclT(K)$ is weakly compact, I is weakly continuous and T is completely continuous.

Proof. Define T_n by

$$T_n x = (1 - k_n)q + k_n T x$$

for all $x \in F(I)$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. As (T, I) is a Banach operator pair, for each $x \in F(I)$ we have $Tx \in F(I)$, and hence $T_n x = (1 - k_n)q + k_n T x \in F(I)$ by the fact that $F(I)$ is q-starshaped with $q \in F(I)$. Thus for each $n \geq 1$, (T_n, I) is a Banach operator pair on $F(I)$. Let $\varphi_j^{(n)} := k_n \varphi_j$, $(j = 1, 2, 3, 4, 5)$. Then by (2.5),

$$\begin{aligned}
\|T_n x - T_n y\| &= k_n \|Tx - Ty\| \\
&\leq k_n \Big(\max\{\varphi_1(\|Ix - Iy\|), \varphi_2(dist(Ix, (q, Tx))), \\
&\qquad \varphi_3(dist(Iy, (q, Ty))), \varphi_4(dist(Ix, (q, Ty))), \varphi_5(dist(Iy, (q, Tx)))\} \Big) \\
&\leq max\{k_n\varphi_1(\|Ix - Iy\|), k_n\varphi_2(\|Ix - T_n x\|), k_n\varphi_3(\|Iy - T_n y\|), \\
&\qquad k_n\varphi_4(\|Ix - T_n y\|), k_n\varphi_5(\|Iy - T_n x\|)\} \\
&\leq max\{\varphi_1^{(n)}(\|Ix - Iy\|), \varphi_2^{(n)}(\|Ix - T_n x\|), \varphi_3^{(n)}(\|Iy - T_n y\|), \\
&\qquad \varphi_4^{(n)}(\|Ix - T_n y\|), \varphi_5^{(n)}(\|Iy - T_n x\|)\},
\end{aligned}$$

for each $x, y \in F(I)$.

(i) As $clT(F(I)) \subseteq clT(K)$ is compact, for each $n \in \mathbb{N}$, $clT_n(F(I))$ is compact and hence complete and bounded. By Corollary 2.3, for each $n \geq 1$, there exists $x_n \in K$ such that $x_n = Ix_n = T_n x_n$. The compactness of $cl(T(K))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to z \in cl(T(F(I))) \subseteq F(I)$ as $m \to \infty$. Since $k_m \to 1$, $x_m = T_m x_m = (1 - k_m)q + k_m T x_m \to z$. By the continuity of T, we obtain that, $K \cap F(T) \cap F(I) \neq \emptyset$.

(ii) By weak compactness of $wclT(F(I)) \subseteq wclT(K)$, $wclT_n(F(I))$ is weakly compact[3, 19] and hence complete and bounded for each n. By Corollary 2.3, there exists $x_n \in K$ such that $x_n = Ix_n = T_n x_n$. Since $wclT(K)$ is weakly compact, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ and $y \in K$ such that $x_m \to y$ weakly. The weak continuity of I implies that $y = Iy$. Further, $\|Ix_m - Tx_m\| = \|((1 - k_m)q + k_m T x_m) - Tx_m\| = (1 - k_m)(q - Tx_m)$ converges to 0, as x_m is bounded and $k_m \to 1$. The demiclosedness of $I - T$ at 0 implies that $Iy = Ty$. Thus $K \cap F(I) \cap F(T) \neq \emptyset$.

(iii) As in (ii), we can find a subsequence $\{x_m\}$ of $\{x_n\}$ in $F(I)$ converging weakly to $y \in F(I)$ as $m \to \infty$. Since T is completely continuous, $Tx_m \to Ty$ as $m \to \infty$. Since $k_m \to 1$, $x_m = T_m x_m = k_m f x_m + (1 - k_m)q \to Ty$ as $m \to \infty$. Thus $Tx_m \to T^2 y$ and consequently $T^2 y = Ty$ implies that $Tw = w$, where $w = Ty$. Also, since $y \in F(I)$, we have $Iw = ITy = TIy = Ty = w$. Hence $K \cap F(I) \cap F(T) \neq \emptyset$.

Taking $\varphi_j(t) = t$; $(j = 1, 2, 3, 4, 5)$, in Theorem 2.4, we obtain;

Corollary 2.5 ([25], Theorem 2.2). Let K, X, I and T and $F(I)$ be as in Theorem 2.4. If (T, I) is a Banach operator pair and satisfies, for each $x, y \in K$,

$$\begin{aligned}
\|Tx - Ty\| \leq \max\{&\|Ix - Iy\|, dist(Ix, [Tx, q]), dist(Iy, [Ty, q]), \\
&dist(Ix, [Ty, q]), dist(Iy, [Tx, q])\},
\end{aligned}$$

then $K \cap F(I) \cap F(T) \neq \emptyset$, provided one of the conditions (i)–(iii) in Theorem 2.4 holds.

Theorem 2.4 and Corollary 2.5 extend and improve Theorem 2.2 of [2], Theorems 3.2-3.3 of [5], Theorem 4 in [9] and Theorem 6 of [20]. A comparison of Theorem 2.4 (ii) with Theorem 3.2 in [5] indicates that, the conditions M is q-starshaped, M is weakly compact and I is continuous are dropped and T is not necessarily I-nonexpansive.

For $k \geq 0$, let $D_M^{k,I}(u)$=$P_M(u) \cap G_M^{k,I}(u)$, where $G_M^{k,I}(u)$=$\{x \in M : \|Ix - u\| \leq (2k+1)dist(u, M)\}$.

Theorem 2.6. Let M be subset of a normed space X and $T,\ I : X \to X$ be mappings such that $u \in F(I) \cap F(T)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that $D_M^{k,I}(u) \cap F(I)$ is closed and q-starshaped and $I(D_M^{k,I}(u)) = D_M^{k,I}(u)$. If the pair (T, I) satisfies;

(a) $\|ITx - Tx\| \leq k\|Ix - x\|$ for all $x \in D_M^{k,g}(u)$ and $k \geq 0$

(b) for all $x \in D_M^{k,I}(u) \cup \{u\}$,

$$\|Tx - Ty\| \leq \begin{cases} \|Ix - Iu\| & \text{if } y = u, \\ max\{\varphi_1(\|Ix - Iy\|), \varphi_2(dist(Ix, [Tx, q])), & \\ \quad \varphi_3(dist(Iy, [Ty, q])), \varphi_4(dist(Ix, [Ty, q])), & \\ \quad \varphi_5(dist(Iy, [Tx, q]))\} & \text{if } y \in D_M^{k,I}(u), \end{cases} \tag{2.6}$$

where $\varphi_j : [0, +\infty) \to [0, +\infty)$ $(j = 1, 2, 3, 4, 5)$ are real functions which are continuous from the right and each has the properties (1)-(2) in Theorem 2.4, then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$, provided one of the following conditions is satisfied;

(i) $cl(T(D_M^{k,I}(u)))$ is compact, and T is continuous,

(ii) X is complete, $wcl(T(D_M^{k,I}(u)))$ is weakly compact, I is weakly continuous and $I - T$ is demiclosed at 0,

(iii) X is complete, $cl(T(D_M^{k,I}(u)))$ is weakly compact, I is weakly continuous and T is completely continuous.

Proof. Let $x \in D_M^{I,g}(u)$. Then $x \in P_M(u)$ and hence $\|x - u\| = dist(u, M)$. Note that for any $t \in (0, 1)$,

$$\|tu + (1-t)x - u\| = (1-t)\|x - u\| < dist(u, M).$$

It follows that the line segment $\{tu + (1-t)x : 0 < k < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Also since $Ix \in P_M(u)$, $u \in F(T) \cap F(I)$ and T and I satisfy (2.6) we have

$$\|Tx - u\| = \|Tx - Tu\| \leq \|Ix - Iu\| = \|Ix - u\| = dist(u, M).$$

Thus $Tx \in P_M(u)$. From inequality in (a) and (2.6), it follows that,

$$\begin{aligned}
\|ITx - u\| &= \|ITx - Tx + Tx - u\| \\
&\leq \|ITx - Tx\| + \|Tx - u\| \\
&\leq k\|Ix - x\| + \|Tx - u\| \\
&= k\|Ix - u + u - x\| + \|Tx - u\| \\
&\leq k(\|Ix - u\| + \|x - u\|) + \|Tx - u\| \\
&\leq k(dist(u, M) + dist(u, M)) + dist(u, M) \\
&\leq (2k + 1)dist(u, M).
\end{aligned}$$

Thus $Tx \in G_M^{k,I}(u)$. Consequently, $T(D_M^{k,I}(u)) \subset D_M^{k,I}(u) = I(D_M^{k,I}(u))$. Inequality in (a) also implies that (T, I) is a Banach operator pair. Now by Theorem 2.4 we obtain, $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$ in each of the cases (i)-(iii).

Let $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$. Then $I(P_M(u)) \subset P_M(u)$ implies $P_M(u) \subset C_M^I(u) \subset G_M^{k,I}(u)$ and hence $D_M^{k,I}(u) = P_M(u)$. Consequently, Theorem 2.6 remains valid when $D_M^{k,I}(u) = P_M(u)$ and the pair (T, I) is Banach operator on $P_M(u)$ instead of satisfying (a), which in turn extends the results in [2, 5, 9, 13, 20, 21, 22, 27, 29, 31, 32].

We denote by $\Im_0$ **(resp.** $\Im_0^w$ **)** the class of closed (resp. weakly closed) convex subsets of X containing 0 ([2, 19]). For $M \in \Im_0$, we define $M_u = \{x \in M : \|x\| \leq 2\|u\|\}$. It is clear that $P_M(u) \subset M_u \in \Im_0$ whenever $M \in \Im_0$.

As an application of Theorem 2.4(i), we obtain the following generalization of the corresponding results in [2, 28, 31, 32].

Theorem 2.7. Let I and T be self mappings of a normed space X with $u \in F(T) \cap F(I)$ and $M \in \Im_0$ such that $T(M_u) \subset I(M) = M$. Suppose that $\|Ix - u\| = \|x - u\|$ for all $x \in M$, $\|Tx - u\| \leq \|Ix - u\|$ for all $x \in M_u$, T is continuous on M_u and one of the following two conditions is satisfied;

(a) $clI(M_u)$ is compact,
(b) $clT(M_u)$ is compact.

Then

(i) $P_M(u)$ is nonempty, closed and convex,
(ii) $T(P_M(u)) \subset I(P_M(u)) = P_M(u)$,
(iii) $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$ provided $F(I) \cap P_M(u)$ is closed and q-starshaped with $q \in F(I) \cap P_M(u)$, the pair (T, I) is a Banach operator on $P_M(u)$ and satisfies (2.5) for all $q \in F(I)$ and for all $x, y \in P_M(u)$, where $\varphi_j : [0, +\infty) \to [0, +\infty)$ $(j = 1, 2, 3, 4, 5)$ are real functions which are continuous from the right and each has the properties (1)-(2) in Theorem 2.4.

Proof. (i) We follow the arguments used in [19, 23]. We may assume that $u \notin M$. If $x \in M \setminus M_u$, then $\|x\| > 2\|u\|$. Note that

$$\|x - u\| \geq \|x\| - \|u\| > \|u\| \geq dist(u, M_u).$$

Thus, $dist(u, M_u)$ =dist(u,M)$\leq \|u\|$. Also $\|z - u\| = dist(u, clI(M_u))$ for some $z \in clI(M_u)$. This implies that

$$dist(u, M_u) \leq dist(u, clI(M_u)) \leq dist(u, I(M_u)) \leq \|Ix - u\| = \|x - u\|,$$

for all $x \in M_u$. Hence $\|z - u\| = dist(u, M)$ and so $P_M(u)$ is nonempty. Moreover, this set is closed and convex. The same conclusion holds whenever $clT(M_u)$ is compact where we replace I by T and utilize inequalities $\|Tx - u\| \leq \|Ix - u\|$ and $\|Ix - u\| = \|x - u\|$ to obtain that $P_M(u)$ is nonempty.

(ii) Let $z \in P_M(u)$. Then $\|Iz - u\| = \|Iz - Iu\| \leq \|z - u\| = dist(u, M)$. This implies that $Iz \in P_M(u)$ and so $I(P_M(u)) \subset P_M(u)$. For the converse assume that $y \in P_M(u)$. Then $y \in M = I(M)$. Thus there is some $x \in M$ such that $y = Ix$. Now

$$\|x - u\| = \|Ix - u\| = \|y - u\| = dist(u, M).$$

This implies that $x \in P_M(u)$ and so $I(P_M(u)) = P_M(u)$.
Let $y \in T(P_M(u))$. Since $T(M_u) \subset I(M)$ and $P_M(u) \subset M_u$, there exist $z \in P_M(u)$ and $x_0 \in M$ such that $y = Tz = Ix_0$. Further, we have

$$\|Ix_0 - u\| = \|Tz - Tu\| \leq \|Iz - Iu\| = \|Iz - u\| = \|z - u\| = dist(u, M).$$

Thus, $x_0 \in C^I_M(u) = P_M(u)$ and so (ii) holds.
By our assumption there exists $q \in P_M(u)$ such that $F(I)$ is q-starshaped with $q \in F(I)$. In both of the cases (a) and (b), $clT(P_M(u))$ is compact by (ii). Hence (iii) follows from Theorem 2.4(i).

The following result extends and improves Theorem 4.2 in [2], Theorem 8 in [9], and Theorem 2.1 in [28]

Theorem 2.8. Let I and T be self mappings of a normed space X with $u \in F(I) \cap F(T)$ and $M \in \Im_0$ such that $T(M_u) \subset I(M) \subset M$. Suppose that $\|Ix - u\| \leq \|x - u\|$ and $\|Tx - u\| \leq \|Ix - u\|$ for all $x \in M_u$, T is continuous on M_u and one of the following two conditions is satisfied;

(a) $clI(M_u)$ is compact,
(b) $clT(M_u)$ is compact.

Then

(i) $P_M(u)$ is nonempty, closed and convex,
(ii) $T(P_M(u)) \subset I(P_M(u)) \subset P_M(u)$, provided that $\|Tx - u\| = \|x - u\|$ for all $x \in C^I_M(u)$, and
(iii) $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$ provided that $\|Tx - u\| = \|x - u\|$ for all $x \in C^I_M(u)$, $F(I) \cap P_M(u)$ is closed and q-starshaped with $q \in F(I) \cap P_M(u)$, the pair (T, I) is a Banach operator on $P_M(u)$ and satisfies (2.5) for all $q \in F(I)$ and for all $x, y \in P_M(u)$, where $\varphi_j : [0, +\infty) \to [0, +\infty)$ $(j = 1, 2, 3, 4, 5)$ are real functions which are continuous from the right and each has the properties (1)-(2) in Theorem 2.4.

Proof. (i) and (ii) follow as in Theorem 2.7.
By our assumption there exists $q \in P_M(u)$ such that $F(I)$ is q-starshaped with $q \in F(I)$. In both of the cases (a) and (b), $clT(P_M(u))$ is compact by (ii). Hence (iii) follows from Theorem 2.4(i).

Theorem 2.9. Let I and T be self mappings of a Banach space X with $u \in F(I) \cap F(T)$ and $M \in \Im_0^w$ such that $T(M_u) \subseteq I(M) \subseteq M$. Suppose that $wcl(I(M_u))$ is weakly compact, $\|Ix - u\| = \|x - u\|$ for all $x \in M_u$, I is weakly continuous on M_u, T satisfies $\|Tx - u\| \leq \|Ix - u\|$ for all $x \in M_u$ and $I - T$ is demiclosed at 0. Then

(i) $P_M(u)$ is nonempty, closed and convex,

(ii) $T(P_M(u)) \subseteq I(P_M(u)) \subseteq P_M(u)$, provided that $\|Ix - u\| = \|x - u\|$ for all $x \in C_M^I(u)$, and

(iii) $P_C(u) \cap F(I) \cap F(T) \neq \emptyset$ provided that $\|Ix - u\| = \|x - u\|$ for all $x \in C_M^I(u)$, $F(I) \cap P_M(u)$ is closed and q-starshaped with $q \in F(I) \cap P_M(u)$, (T, I) is a Banach operator pair on $P_M(u)$ and T satisfies (2.5) for all $q \in F(I) \cap P_M(u)$, and $x, y \in P_M(u)$, where $\varphi_j : [0, +\infty) \to [0, +\infty)$ ($j = 1, 2, 3, 4, 5$) are real functions which are continuous from the right and each has the properties (1)-(2) in Theorem 2.4.

Proof. To obtain the result, use an argument similar to that in Theorem 2.7 and apply Theorem 2.4(ii) instead of Theorem 2.4(i). Use Lemma 5.5 in ([30], p. 192) with $f(x) = \|x - u\|$ and $C = wcl(I(M_u))$ to show that there exists a $z \in C$ such that $dist(u, C) = \|z - u\|$.

Following results are the consequences of Theorems 2.1 and 2.2 respectively in [11](see also [1]).

Theorem 2.10. Let K be a subset of a metric space (X, d), and T be a self mapping of K. Assume that $cl(T(K)) \subset K$, $cl(T(K))$ is complete, and there exists a continuous nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) < t$ for $t > 0$ such that

$$d(Tx, Ty) \leq \varphi(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(y, Tx) + d(x, Ty)]\})$$

Then $F(T)$ is a singleton.

Theorem 2.11. Let K, X, and T be as in Theorem 2.10 and there exists a continuous function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) < t$ for $t > 0$ such that

$$d(Tx, Ty) \leq \varphi(\max\{d(x, y), d(x, Tx), d(y, Ty)\})$$

Then $F(T)$ is a singleton.

The proof of the following lemma is similar to that of Corollary 2.3; here instead of applying Theorem 2.1, we apply Theorem 2.10 or Theorem 2.11 to get the conclusion.

Lemma 2.12. Let (X,d) be a metric space, K be a nonempty subset of X, T, $I : K \to K$ be mappings satisfying the following contraction-type condition

$$d(Tx,Ty) \le \varphi(\max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), \frac{1}{2}[d(Iy,Tx)+d(Ix,Ty)]\}) \tag{2.7}$$

$$\big(or \quad d(Tx,Ty) \le \varphi(\max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty)\})\big) \tag{2.8}$$

on K where $\varphi : [0,+\infty) \to [0,+\infty)$ is nondecreasing (or continuous) function satisfying $\phi(t) < t$ for $t > 0$. Suppose that (T,I) is a Banach operator pair, $cl(T(K))$ is complete, $F(I)$ is nonempty and closed, then T and I have a unique common fixed point in K.

As an application of Lemma 2.12, we obtain the following generalizations of the corresponding results in [2, 9, 5, 20, 21, 27, 29, 31, 32].

Theorem 2.13. Let K be a nonempty subset of a normed space X and I and T be self mappings of K. Suppose that $F(I)$ is closed and q-starshaped with $q \in F(I)$. If (T,I) is a Banach operator pair and satisfies, for each $x, y \in K$,

$$\begin{aligned} \|Tx - Ty\| \le \quad & \varphi\Big(\max\{\|Ix - Iy\|, dist(Ix,[q,Tx]), dist(Iy,[q,Ty]), \\ & \frac{1}{2}[dist(Ix,[q,Ty]) + dist(Iy,[q,Tx])\}\Big), \end{aligned} \tag{2.9}$$

where $\varphi : [0,+\infty) \to [0,+\infty)$ is nondecreasing function satisfying for each $k \in (0,1)$, $\phi(t) < \frac{t}{k}$ for $t > 0$, then $K \cap F(I) \cap F(T) \ne \emptyset$, provided one of the following conditions holds;

(i) $cl(T(K))$ is compact and T is continuous,
(ii) X is complete, $wclT(K)$ is weakly compact, I is weakly continuous and $I - T$ is demiclosed at 0,
(iii) X is complete, $wclT(K)$ is weakly compact, I is weakly continuous and T is completely continuous.

Proof. Define T_n as in the proof of Theorem 2.4. As (T,I) is a Banach operator pair, for $x \in F(I)$ we have $Tx \in F(I)$, and hence $T_n x = (1-k_n)q + k_n Tx \in F(I)$ by the fact that $F(I)$ is q-starshaped with $q \in F(I)$. Thus for each $n \ge 1$, (T_n, I) is a Banach operator pair on $F(I)$. Let $\varphi_n := k_n\varphi$. Then by (2.9),

$$\begin{aligned} \|T_n x - T_n y\| &= k_n\|Tx - Ty\| \\ &\le k_n\Big(\varphi(\max\{\|Ix - Iy\|, dist(Ix,[q,Tx]), dist(Iy,[q,Ty]), \\ &\quad \frac{1}{2}[dist(Ix,[q,Ty]) + dist(Iy,[q,Tx])\})\Big) \\ &\le \varphi_n(\max\{\|Ix - Iy\|, \|Ix - T_n x\|, \|Iy - T_n y\|, \\ &\quad \frac{1}{2}[\|Ix - T_n y\| + \|Iy - T_n x\|\}), \end{aligned}$$

for each $x, y \in K$. The analysis in Theorem 2.4 (using Lemma 2.12 above) guarantees that there exists $x_n \in K$ such that $x_n = Ix_n = T_n x_n$. Rest of the proof is similar to that of Theorem 2.4 and so is omitted.

Theorem 2.14. Let I and T be self mappings on a nonempty subset K of a normed space X. Suppose that $F(I)$ is closed and q-starshaped with $q \in F(I)$. If (T, I) is a Banach operator pair and satisfies, for each $x, y \in K$,

$$\|Tx - Ty\| \leq \varphi(\|Ix - Iy\|) \tag{2.10}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is continuous function satisfying, for each $k \in (0, 1)$ $\varphi(t) < \frac{t}{k}$ for $t > 0$, then $K \cap F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds;

(i) $cl(T(K))$ is compact and T is continuous,

(ii) X is complete, $wclT(K)$ is weakly compact, I is weakly continuous and either $I - T$ is demiclosed at 0 or X satisfies Opial's condition

(iii) X is complete, $wclT(K)$ is weakly compact and T is completely continuous.

Proof. Proofs of (i) and (iii) are similar to the proofs of Theorem 2.13(i) and (iii), respectively.

(ii) The analysis in Theorem 2.13, and the completeness of $wcl(T_n(K))$ guarantee that there exists $x_n \in K$ such that $x_n = Ix_n = T_n x_n$. The weak compactness of $wcl(T(K))$ implies that there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y$ weakly as $m \to \infty$. As I is weakly continuous, $Iy = y$. Since $\{x_m\}$ is bounded, $k_m \to 1$, and

$$\begin{aligned} \|x_m - Tx_m\| = \|Ix_m - Tx_m\| &= \|((1 - k_m)q + k_m Tx_m) - Tx_m\| \\ &\leq (1 - k_m)(\|q\| + \|Tx_m\|), \end{aligned}$$

so $\|x_m - Tx_m\| \to 0$ as $m \to \infty$. If $I - T$ is demiclosed at 0, $(I - T)y = 0$ and hence $y = Iy = Ty$.
Suppose that X satisfies Opial's condition. If $y \neq Ty$, then

$$\begin{aligned} \liminf_{m\to\infty} \|x_m - y\| &< \liminf_{m\to\infty} \|x_m - Ty\| \\ &\leq \liminf_{m\to\infty} \|x_m - Tx_m\| + \liminf_{m\to\infty} \|Tx_m - Ty\| \\ &= \liminf_{m\to\infty} \|Tx_m - Ty\| \leq \liminf_{m\to\infty} \varphi(\|Ix_m - Iy\|) \\ &< \liminf_{m\to\infty} \frac{1}{k_m} \|x_m - y\| \\ &= \liminf_{m\to\infty} \|x_m - y\| \end{aligned}$$

which is a contradiction. Thus $Iy = y = Ty$ and hence $K \cap F(I) \cap F(T) \neq \emptyset$.

For $\varphi(t) = t$, $t \in [0, \infty)$, from Theorem 2.14 we obtain:

Corollary 2.15 (see [5], Theorem 3.2-Theorem 3.3). Let I and T be self mappings on a q-starshaped subset K of a normed space X. Assume that (T, I) is a Banach operator pair on K, $F(I)$ is q-starshaped with $q \in F(I)$, I is continuous, T is I-nonexpansive. Then $K \cap F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds;

(i) $cl(T(K))$ is compact,

(ii) X is complete, I is weakly continuous, $wcl(T(K))$ is weakly compact and either $I - T$ is demiclosed at 0 or X satisfies Opial's condition.

Remark 2.16. As an application of Theorems 2.13 and 2.14, the analogue of all the approximation results (Theorem 2.6-Theorem 2.9) can be established for a Banach operator pair (T, I) satisfying inequality (2.9) *or* (2.10) with their respective comparison functions φ.

Remark 2.17.

(i) Theorems 2.7-2.9 represent very strong variants of Theorem 2.4 [2] and Theorem 2.1 [28] in the sense that the commutativity of the maps T and I is replaced by the general hypothesis that (T, I) is a Banach operator pair, I need not be linear or affine and T need not be I-nonexpansive. Further, the comparison of Theorems 2.4-2.9 with the corresponding results in [3, 9, 12, 13, 14, 19, 23, 24], indicates that the concept of a Banach operator pair is more useful for the study of common fixed points in best approximation theory in the sense that here we are able to prove the results without the linearity or affinity of I.

(ii) Banach operator pairs are different from those of weakly compatible, C_q-commuting and R-subweakly commuting maps, so our results are different from those in [3, 12, 13, 14, 19, 23]. Consider $K = \mathbb{R}^2$ with the norm $\|(x, y)\| = |x| + |y|, (x, y) \in K$. Define T and I on K as follows:

$$T(x, y) = \left(x^3 + x - 1, \frac{\sqrt[3]{x^2 + y^3 - 1}}{3}\right),$$

$$I(x, y) = \left(x^3 + x - 1, \sqrt[3]{x^2 + y^3 - 1}\right).$$

Then

$$F(T) = \{(1, 0)\}; \qquad F(I) = \{(1, y) : y \in R^1\};$$

$$C(T, I) = \{(x, y) : y = \sqrt[3]{1 - x^2}, \quad x \in R^1\};$$

$$T(F(I)) = \{T(1, y) : y \in R^1\} = \{(1, \frac{y}{3}) : y \in R^1\} \subseteq \{(1, y) : y \in R^1\} = F(I).$$

Thus, (T, I) is a Banach operator pair. It is easy to see that T is I-contractive (and hence I-nonexpansive) and that T and I do not commute on the set $C(T, I)$. Clearly, I is not affine or linear, $F(I)$ is convex and $(1, 0)$ is a common fixed point of T and I.

(iii) Now we present a simple example which shows that Corollary 2.3 extends and improves Lemma 3.1 of [5], Lemma 2.10 of [10], Theorem 1.1 and Theorem 1 in [21].
Let $K = [0, +\infty)$ be the subset of reals with the usual metric d. Define

$$Tx = \frac{\sqrt{x}}{\sqrt{x}+1} \quad \text{for all} \ \ x \in K,$$
$$Ix = \sqrt{x} \quad \text{for } 0 \le x \le \frac{1}{2}$$
$$Ix = (\sqrt{2})x \quad \text{for } x > \frac{1}{2}$$

and
$$\varphi(t) = \frac{t}{t+1} \quad \text{for all} \quad t \ge 0.$$

Let $x, y \in [0, 0.5]$. Then

$$d(Tx, Ty) = \frac{|\sqrt{x} - \sqrt{y}|}{\sqrt{x} + \sqrt{y} + \sqrt{xy} + 1)} \le \frac{|\sqrt{x} - \sqrt{y}|}{|\sqrt{x} - \sqrt{y}| + 1} = \varphi(|Ix - Iy|).$$

Consider now the case $x > 1/2$ and $y \in [0, 0.5]$. Then $|Ix - Iy| = (\sqrt{2})x - \sqrt{y}$. Thus

$$d(Tx, Ty) = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y} + \sqrt{xy} + 1)} \le \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} - \sqrt{y} + 1} \le \frac{(\sqrt{2})x - \sqrt{y}}{(\sqrt{2})x - \sqrt{y} + 1} = \varphi(|Ix - Iy|).$$

Clearly, $\varphi(t)$ is continuous, increasing and $\varphi(t) < t$ for all $t > 0$. Also, (T, I) is a Banach operator pair. Thus the mappings T and I satisfy all hypotheses in Corollary 2.3 and have a unique common fixed point $z = 0$.

To see that Lemma 3.1 of [5] is not applicable, let $y = 0$ and $0 < x \le 1/2$. Then we have $d(Tx, T0) = Tx = \sqrt{x}/(\sqrt{x} + 1)$,

$$\max\{d(Ix, I0), d(Ix, Tx), d(I0, T0), d(Ix, T0), d(I0, Tx)\} = \sqrt{x}.$$

Thus, for any fixed $\lambda < 1$ and $0 < x < [(1 - \lambda)/\lambda]^2$,

$$d(Tx, T0) = \frac{\sqrt{x}}{\sqrt{x}+1} > \lambda \cdot \sqrt{x} = \lambda \cdot \max\{d(Ix, I0), d(Ix, Tx), d(I0, T0), d(Ix, T0), d(I0, Tx)\}.$$

Therefore, the hypothesis

$$d(Tx, Ty) \le \lambda \cdot \max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\}$$

in Lemma 3.1 of [5] and Theorem 1.1 is not satisfied.

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