Neighborhoods of a Certain Family of Multivalent Functions Defined by Using a Fractional Derivative Operator

M. K. Aouf

Abstract

Making use of a fractional derivative operator, we introduce and investigate two new classes $K_j(p, \lambda, b, \beta)$ and $L_j(p, \lambda, b, \beta, \mu)$ of multivalently analytic functions of complex order. In this paper we obtain the coefficient estimates and inclusion relationships involving the $(j, \delta)-$ neighborhood of various subclasses of multivalently analytic functions of complex order.

1 Introduction

Let $T(j, p)$ denote the class of functions of the form :

$$f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k \quad (a_k \geq 0; p, j \in N = \{1, 2, ...\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. In terms of the fractional derivative operator D_z^{λ} of order λ, defined below, with

$$D_z^0 f(z) = f(z) \quad \text{and} \quad D_z^1 f(z) = f^{'}(z), \tag{1.2}$$

Srivastava and Aouf [15] defined and studied the operator :

Received by the editors September 2007.

Communicated by F. Brackx.

2000 *Mathematics Subject Classification* : 30C45.

Key words and phrases : Multivalent analytic functions, fractional derivative operator, complex order, neighborhood.

$$\Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \quad (0 \le \lambda \le 1; p \in N). \tag{1.3}$$

For each $f(z) \in T(p,j)$, we have

$$(i)\ \Omega_z^{(\lambda,p)} f(z) = z^p - \sum_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k z^k \tag{1.4}$$

and

$$(ii)\ \Omega_z^{(0,p)} f(z) = f(z) \ \text{ and } \ \Omega_z^{(1,p)} f(z) = \frac{z f'(z)}{p}. \tag{1.5}$$

Now, making use of the operator $\Omega_z^{(\lambda,p)} f(z)$ given by (1.3) and (1.4), we introduce a new subclass $K_j(p,\lambda,b,\beta)$ of the p-valently analytic function class $T(j,p)$ which consist of functions $f(z) \in T(j,p)$ satisfying the following inequality :

$$\left| \frac{1}{b} \left(\frac{z(\Omega_z^{(\lambda,p)} f(z))'}{\Omega_z^{(\lambda,p)} f(z)} - p \right) \right| < \beta$$

$$(z \in U; p,j \in N; 0 \le \lambda \le 1; b \in C \backslash \{0\}; 0 < \beta \le 1). \tag{1.6}$$

We note that :
(i) $K_j(p,0,b,\beta) = S_j(p,b,\beta)$

$$= \{f(z) \in T(j,p) : \ \left| \frac{1}{b} (\frac{z f'(z)}{f(z)} - p) \right| < \beta$$

$$(z \in U; p,j \in N; b \in C \backslash \{0\}; 0 < \beta \le 1)\ \}; \tag{1.7}$$

(ii) $K_j(p,1,b,\beta) = C_j(p,b,\beta)$

$$= \{\ f(z) \in T(j,p) : \ \left| \frac{1}{b} (1 + \frac{z f''(z)}{f'(z)} - p) \right| < \beta$$

$$(z \in U; p,j \in N; b \in C \backslash \{0\}; 0 < \beta \le 1)\ \}. \tag{1.8}$$

Now, following the earlier investigations by Goodman [6], Ruscheweyh [14], and others including Altintas and Owa [1], Altintas et al. ([2] and [3]), Murugusundaramoorthy and Srivastava [8], Raina and Srivastava [13] and Srivastava and Orhan [16] (see also [9] , [10] and [18]), we define the $(j,\delta)-$Neighborhood of a function $f(z) \in T(j,p)$ by (see, for example, [3, p-1668])

$$N_{j,\delta}(f) = \left\{ g : g \in T(j,p), g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k \ \text{ and } \ \sum_{k=j+p}^{\infty} k \left| a_k - b_k \right| \le \delta \right\}. \tag{1.9}$$

In particular, if

$$h(z) = z^p \ \ (p \in N), \tag{1.10}$$

we immediately have

$$N_{j,\delta}(h) = \left\{ g : g \in T(j,p), g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k \ \text{ and } \sum_{k=j+p}^{\infty} k\,|b_k| \le \delta \right\}. \tag{1.11}$$

Also, let $L_j(p,\lambda,b,\beta,\mu)$ denote the subclass of $T(j,p)$ consisting of functions $f(z) \in T(j,p)$ which satisfy the inequality :

$$\left| \frac{1}{b} \left\{ \left[(1-\mu)\frac{\Omega_z^{(\lambda,p)} f(z)}{z^p} + \mu \frac{(\Omega_z^{\lambda,p} f(z))'}{pz^{p-1}} \right] - 1 \right\} \right| < \beta$$

$$(z \in U; p, j \in N; 0 \le \lambda \le 1; b \in C\backslash\{0\}; 0 < \beta \le 1; \mu \ge 0). \tag{1.12}$$

We note that :

(i) $L_j(p,0,b,\beta,\mu) = P_j(p,b,\beta,\mu)$

$$= \{ f(z) \in T(j,p) : \ \left| \frac{1}{b} \left\{ \left[(1-\mu)\frac{f(z)}{z^p} + \mu \frac{f'(z)}{pz^{p-1}} - 1 \right] \right\} \right| < \beta$$

$$(z \in U; p, j \in N; b \in C\backslash\{0\}; 0 < \beta \le 1; \mu \ge 0) \ \}; \tag{1.13}$$

(ii) $L_j(p,1,b,\beta,\mu) = R_j(p,b,\beta,\mu)$

$$= \{ f(z) \in T(j,p) : \ \left| \frac{1}{b} \left\{ \left[(p+\mu(1-p))\frac{f'(z)}{pz^{p-1}} + \mu \frac{f''(z)}{pz^{p-2}} \right] - p \right\} \right| < p\beta$$

$$(z \in U; p, j \in N; b \in C\backslash\{0\}; 0 < \beta \le 1; \mu \ge 0) \ \}; \tag{1.14}$$

(iii) $L_j(1,1,b,\beta,\mu) = R_j(b,\beta,\mu)$ (Altintas et al. [3])

$$= \{ f(z) \in T(j,p) : \ \left| \frac{1}{b} \left\{ f'(z) + \mu z f''(z) - 1 \right\} \right| < \beta$$

$$(z \in U; j \in N; b \in C\backslash\{0\}; 0 < \beta \le 1; \mu \ge 0) \ \}. \tag{1.15}$$

Various operators of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature rather extensively (cf., e.g., [5], [11], [17] and [18]). For our present investigation, we recall the following definitions.

Definition 1. *(Fractional Integral Operator). The fractional integral operator of order λ is defined, for a function $f(z)$, by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \ \ (\lambda > 0), \tag{1.16}$$

where $f(z)$ is an analytic function in a simply- connected region of the z - plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring log $(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 2. *(Fractional Derivative Operator). The fractional derivative of order λ is defined, for a function $f(z)$, by*

$$D_z^{\lambda} f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta \quad (0\leq\lambda<1), \tag{1.17}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1.

Definition 3. *(Extended Fractional Derivative Operator). Under the hypotheses of Definition 2, the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$, by*

$$D_z^{n+\lambda} f(z)=\frac{d^n}{dz^n}D_z^{\lambda} f(z) \quad (0\leq\lambda<1; n\in N_0=N\cup\{0\}). \tag{1.18}$$

2 Neighborhoods for the classes $K_j(p,\lambda,b,\beta)$ and $L_j(p,\lambda,b,\beta,\mu)$

In our investigation of the inclusion relations involving $N_{j,\delta}(h)$, we shall require Lemmas 1 and 2 below.

Lemma 1. *Let the function $f(z)\in T(j,p)$ be defined by* (1.1). *Then $f(z)\in K_j(p,\lambda,b,\beta)$ if and only if*

$$\sum_{k=j+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\ (k+\beta\,|b|-p)a_k\leq\beta\,|b|\,. \tag{2.1}$$

Proof. Let a function $f(z)$ of the form (1.1) belong to the class $K_j(p,\lambda,b,\beta)$.Then, in view of (1.4) and (1.6), we obtain the following inequality

$$Re\left\{\frac{z(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}-p\right\}>-\beta\,|b| \quad (z\in U), \tag{2.2}$$

or, equivalently,

$$Re\left\{\frac{-\sum\limits_{k=j+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\ (k-p)a_k z^{k-p}}{1-\sum\limits_{k=j+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\ a_k z^{k-p}}\right\}>-\beta\,|b| \quad (z\in U). \tag{2.3}$$

Setting $z=r(0\leq r<1)$ in (2.3), we observe that the expression in the denominator of the left-hand side of (2.3) is positive for $r=0$ and also for $(0<r<1)$.

Thus, by letting $r \to 1^-$ through real values, (2.3) leads us the desired assertion (2.1) of Lemma 1.

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we find from (1.6) that

$$\left| \frac{z(\Omega_z^{(\lambda,p)} f(z))'}{\Omega_z^{(\lambda,p)} f(z)} - p \right| = \left| \frac{- \sum\limits_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \ (k-p) a_k z^{k-p}}{1 - \sum\limits_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \ a_k z^{k-p}} \right|$$

$$\leq \frac{\sum\limits_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \ (k-p) a_k}{1 - \sum\limits_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k}$$

$$\leq \frac{\beta\,|b| \left\{ 1 - \sum\limits_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \ a_k \right\}}{1 - \sum\limits_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k} = \beta\,|b|\,.$$

Hence, by the maximum modulus theorem, we have $f(z) \in K_j(p, \lambda, b, \beta)$, which evidently completes the proof of Lemma 1.

Remark 1. *(i) Putting $\lambda = 0, \beta = j = 1$ and $b = p - \alpha, 0 \leq \alpha < p,$ in Lemma 1, we obtain the result obtained by Owa [12 , Theorem 2.3];*

(ii) Putting $\beta = j = \lambda = 1$ and $b = p - \alpha,\ 0 \leq \alpha < p,$ in Lemma 1, we obtain the result obtained by Owa [12, Theorem 2.4].

Similarly, we can prove the following lemma.

Lemma 2. *Let the function $f(z) \in T(j, p)$ be define by (1.1). Then $f(z) \in L_j(p, \lambda, b, \beta, \mu)$ if and only if*

$$\sum_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} [p + \mu(k-p)] a_k \leq p\beta\,|b|\,. \tag{2.4}$$

Remark 2. *(i) Putting $\lambda = 0, \beta = j = 1$ and $b = 1 - \frac{\alpha}{p}, (0 \leq \alpha < p)$, in Lemma 2, we obtain the result obtained by Lee et al. [7, Lemma 2];*

(ii) Putting $\beta = \lambda = j = 1$ and $b = 1 - \frac{\alpha}{p},\ (0 \leq \alpha < p)$, in Lemma 2, we obtain the result obtained by Aouf [4 , Theorem 1].

Our first inclusion relation involving $N_{j,\delta}(h)$ is given in the following theorem.

Theorem 1. *Let*

$$\delta = \frac{(j+p)\beta\,|b|\,\Gamma(p+1)\Gamma(j+p+1-\lambda)}{(j+\beta\,|b|)\Gamma(j+p+1)\Gamma(p+1-\lambda)} \quad (p > |b|), \tag{2.5}$$

then

$$K_j(p, \lambda, b, \beta) \subset N_{j,\delta}(h). \tag{2.6}$$

Proof. Let $f(z) \in K_j(p,\lambda,b,\beta)$.Then, in view of the assertion (2.1) of Lemma 1, we have

$$\frac{\Gamma(j+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(j+p+1-\lambda)} \quad (j+\beta\,|b|)\sum_{k=j+p}^{\infty} a_k \leq$$

$$\sum_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\,(k+\beta\,|b|-p)a_k \leq \beta\,|b|\,, \tag{2.7}$$

which readily yields

$$\sum_{k=j+p}^{\infty} a_k \leq \frac{\beta\,|b|\,\Gamma(p+1)\Gamma(j+p+1-\lambda)}{(j+\beta\,|b|)\Gamma(j+p+1)]\Gamma(p+1-\lambda)}. \tag{2.8}$$

Making use of (2.1) again, in conjunction with (2.8), we get

$$\frac{\Gamma(j+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(j+p+1-\lambda)} \sum_{k=j+p}^{\infty} ka_k \leq \beta\,|b| +$$

$$(p-\beta\,|b|)\frac{\Gamma(j+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(j+p+1-\lambda)} \sum_{k=j+p}^{\infty} a_k$$

$$\leq \beta\,|b| + \frac{\beta\,|b|\,(p-\beta\,|b|)}{(j+\beta\,|b|)} = \frac{(j+p)\beta\,|b|}{(j+\beta\,|b|)}.$$

Hence

$$\sum_{k=j+p}^{\infty} ka_k \leq \frac{(j+p)\beta\,|b|\,\Gamma(p+1)\Gamma(j+p+1-\lambda)}{(j+\beta\,|b|)\Gamma(j+p+1)\Gamma(p+1-\lambda)} = \delta \quad (p > |b|) \tag{2.9}$$

which, by means of the definition (1.11), establishes the inclusion relation (2.6) asserted by Theorem 1.

Remark 3. *(i) Putting (a) $\lambda = 0$, (b) $\lambda = 1$ in Theorem 1, we obtain the corresponding results for the classes $S_j(p,b,\beta)$ and $C_j(p,b,\beta)$, respectively;*

(ii) Putting $\lambda = 0$ and $p = 1$ in Theorem 1, we obtain the result obtained by Altintas et al. [2, Theorem 1 with $\lambda = 0$];

(iii) Putting $\lambda = p = 1$ in Theorem 1, we obtain the result obtained by Altintas et al. [2, Theorem 1 with $\lambda = 1$].

In a similar manner, by applying the assertion (2.4) of Lemma 2 instead of the assertion (2.1) of Lemma 1 to functions in the class $L_j(p,\lambda,b,\beta,\mu)$, we can prove the following inclusion relationship.

Theorem 2. *If*

$$\delta = \frac{(j+p)p\beta\,|b|\,\Gamma(p+1)\Gamma(j+p+1-\lambda)}{(p+\mu j)\Gamma(j+p+1)\Gamma(p+1-\lambda)} \quad (\mu > 1) \tag{2.10}$$

then

$$L_j(p,\lambda,b,\beta,\mu) \subset N_{j,\delta}(h). \tag{2.11}$$

Remark 4. *Putting* $p = \lambda = 1$ *in Theorem 2, we obtain the result obtained by Altintas et al. [2, Theorem 2].*

3 Neighborhoods for the classes $\mathbf{K}_j^{(\alpha)}(\mathbf{p},\boldsymbol{\lambda},\mathbf{b},\boldsymbol{\beta})$ and $\mathbf{L}_j^{(\alpha)}(\mathbf{p},\boldsymbol{\lambda},\mathbf{b},\boldsymbol{\beta},\boldsymbol{\mu})$

In this section, we determine the neighborhood for each of the classes

$$K_j^{(\alpha)}(p,\lambda,b,\beta) \text{ and } L_j^{(\alpha)}(p,\lambda,b,\beta,\mu)\ ,$$

which we define as follows. A function $f(z) \in T(j,p)$ is said to be in the class $K_j^{(\alpha)}(p,\lambda,b,\beta)$ if there exists a function $g(z) \in K_j(p,\lambda,b,\beta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha \quad (z \in U;\ 0 \le \alpha < p). \tag{3.1}$$

Analogously, a function $f(z) \in T(j,p)$ is said to be in the class $L_j^{(\alpha)}(p,\lambda,b,\beta,\mu)$ if there exists a function $g(z) \in L_j(p,\lambda,b,\beta,\mu)$ such that the inequality (3.1) holds true.

Theorem 3. *If* $g(z) \in K_j(p,\lambda,b,\beta)$ *and*

$$\alpha = p - \frac{\delta(j+\beta\,|b|)\Gamma(j+p+1)\Gamma(p+1-\lambda)}{(j+p)\left\{(j+\beta\,|b|)\Gamma(j+p+1)\Gamma(p+1-\lambda) - \beta\,|b|\,\Gamma(p+1)\Gamma(j+p+1-\lambda)\right\}}, \tag{3.2}$$

then

$$N_{j,\delta}(g) \subset K_j^{(\alpha)}(p,\lambda,b,\beta), \tag{3.3}$$

where

$$\delta \le p(j+p)\left\{1 - \beta\,|b|\,\Gamma(p+1)\Gamma(j+p+1-\lambda)\left[(j+\beta\,|b|)\Gamma(j+p+1)\Gamma(p+1-\lambda)\right]^{-1}\right\}. \tag{3.4}$$

Proof. Suppose that $f(z) \in N_{j,\delta}(g)$. We find from (1.9) that

$$\sum_{k=j+p}^{\infty} k\,|a_k - b_k| \le \delta, \tag{3.5}$$

which readily implies that

$$\sum_{k=j+p}^{\infty} |a_k - b_k| \leq \frac{\delta}{j+p} \quad (p, j \in N). \tag{3.6}$$

Next, since $g(z) \in K_j(p, \lambda, b, \beta)$, we have [cf. equation (2.8)]

$$\sum_{k=j+p}^{\infty} b_k \leq \frac{\beta |b| \Gamma(p+1)\Gamma(j+p+1-\lambda)}{(j+\beta |b|)\Gamma(j+p+1)\Gamma(p+1-\lambda)}, \tag{3.7}$$

so that

$$\left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum\limits_{k=j+p}^{\infty} |a_k - b_k|}{1 - \sum\limits_{k=j+p}^{\infty} b_k}$$

$$\leq \frac{\delta}{j+p} \cdot \frac{(j+\beta |b|)\Gamma(j+p+1)\Gamma(p+1-\lambda)}{\{(j+\beta |b|)\Gamma(j+p+1)\Gamma(p+1-\lambda) - \beta |b| \Gamma(p+1)\Gamma(j+p+1-\lambda)\}}$$

$$= p - \alpha, \tag{3.8}$$

provided that α is given by (3.2). Thus, by the above definition, $f(z) \in K_j^{(\alpha)}(p, \lambda, b, \beta)$ for α given by (3.2). This evidently proves Theorem 3.

Remark 5. *(i) Putting $\lambda = 0$, and $p = 1$ in Theorem 3, we obtain the result obtained by Altintas et al. [2, Theorem 3 with $\lambda = 0$];*

(ii) Putting $\lambda = p = 1$ in Theorem 3, we obtain the result obtained by Altintas et al. [2, Theorem 3 with $\lambda = 1$].

The proof of Theorem 4 below is similar to that of Theorem 3 above.

Theorem 4. *If $g(z) \in L_j(p, \lambda, b, \beta, \mu)$ and*

$$\alpha = p - \frac{\delta(p+\mu j)\Gamma(j+p+1)\Gamma(p+1-\lambda)}{(j+p)\{(p+\mu j)\Gamma(j+p+1)\Gamma(p+1-\lambda) - p\beta |b| \Gamma(p+1)\Gamma(j+p+1-\lambda)\}}, \tag{3.9}$$

then

$$N_{j,\delta}(g) \subset L_j^{(\alpha)}(p, \lambda, b, \beta, \mu), \tag{3.10}$$

where

$$\delta \leq p(j+p)\left\{ 1 - p\beta |b| \Gamma(p+1)\Gamma(j+p+1-\lambda) \left[(p+\mu j)\Gamma(j+p+1)\Gamma(p+1-\lambda)\right]^{-1} \right\}. \tag{3.11}$$

(i) Putting $\lambda = 0$, and $\lambda = 1$ in Theorem 4, we obtain the corresponding results for the classes $P_j^{(\alpha)}(p, b, \beta, \mu)$ and $R_j^{(\alpha)}(p, b, \beta, \mu)$, respectively;

(ii) Putting $p = \lambda = 1$ in Theorem 4, we obtain the result obtained by Altintas et al. [2 , Theorem 4].

References

[1] O. Altintas and S. Owa, Neighborhoods of certain analytic functions with negative coefficients, Internat. J. Math. Math. Sci. 19(1996), 797-800.

[2] O. Altintas, O. Ozkan and H. M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, Appl. Math. Letters 13(2000), no.3, 63-67.

[3] O. Altintas, O. Ozkan and H. M. Srivastava, Neighborhoods of a certain family of multivalent functions with negative coefficients, Comput. Math. Appl. 47(2004), 1667-1672.

[4] M. K. Aouf, A subclass of analytic p-valent functions with negative coefficients, Utilitas Math. 46(1994), 219-231.

[5] M. P. Chen, H. Irmak and H. M. Srivastava, Some families of multivalently analytic functions with negative coefficients, J. Math. Anal. Appl. 214 (1997), 67 4-690.

[6] A. W. Goodman, Univalent functions and nonanalytic curves, Proc. Amer. Math. Soc. 8(1957), 598-601.

[7] S. K. Lee, S. Owa and H. M. Srivastava, Basic properties and characterizations of a certain class of analytic functions with negative coefficients, Utilitas Math. 36(1989), 121-128.

[8] G. Murugusundaramoorthy and H. M. Srivastava, Neighborhoods of certain classes of analytic functions of complex order, J. Inequal. Pure Appl. Math. 5(2)(2004), Article 24 , 1-8 (electronic).

[9] H. Orhan and E. Kadioglu, Neighborhoods of a class of analytic functions with negative coefficients, Tamsui Oxford J. Math. Sci. 20 (2004), 135-142.

[10] H. Orhan and M. Kamali, Neighborhoods of a class of analytic functions with negative coefficients, Acta Math. Acad. Paedagog. Nyhazi. (N.S.)21 (2005), no.1, 55-61 (electronic).

[11] S. Owa, On the distortion theorems. I, Kyungpook Math. J. 18(1978), 55-59.

[12] S. Owa, On certain classes of p-valent functions with negative coefficients, Simon Stevin 59(1985), 385-402.

[13] R. K. Raina and H. M. Srivastava, Inclusion and neighborhood properties of some analytic and multivalent functions, J. Inequal. Pure Appl. Math. 7(1) (2006), Article 5, 1-6 (electronic).

[14] St. Ruscheweyh, Neighborhoods of univalent functions, Proc. Amer., Math. Soc. 8 (1981), 521-527.

[15] H. M. Srivastava and M.K.Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I and II, J. Math. Anal. Appl. 171(1992), 1-13; ibid. 192(1995),673-688.

[16] H. M. Srivastava and H. Orhan, Coefficient inequalities and inclusion some families of analytic and multivalent functions, Appl. Math. Letters, 20(2007), 686 - 691.

[17] H. M. Srivastava and S. Owa (Editors), Univalent Functions, Fractional Calculus, and Their Applications, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[18] H. M. Srivastava and S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

Faculty of Science
Mansoura University
Mansoura 35516, Egypt
E-mail: mkaouf127@yahoo.com