Enhanced delay to bifurcation*

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Abstract

We present an example of slow-fast system which displays a full open set of initial data so that the corresponding orbit has the property that given any ϵ and T, it remains to a distance less than ϵ from a repulsive part of the fast dynamics and for a time larger than T. This example shows that the common representation of generic fast-slow systems where general orbits are pieces of slow motions near the attractive parts of the critical manifold intertwined by fast motions is false. Such a description is indeed based on the condition that the singularities of the critical set are folds. In our example, these singularities are transcritical.

1 Introduction

A first approximation for the time evolution of fast-slow systems dynamics is often seen as follows. A generic orbit jumps close to an attractive part of the equilibrium set of the fast dynamics. It evolves then slowly close to this attractive part until that, under the influence of the slow dynamics, this attractive part bifurcates into a repulsive one. Then the generic orbit jumps to another attractive part of the fast equilibrium set until it also looses its stability and then jumps again in search of another attractor and so on. This approach is indeed only a first approximation. It is a quite meaningful one because in this setting one can explain many phenomena like for instance hysteresis cycles, relaxation oscillations, bursting oscillations and more complicated alternance of pulsatile and surge patterns of coupled neurons populations. It can also serve as a good representation model of the so-called ultrastability introduced in Cybernetics by R. Ashby ([1]) (as the property that some systems display an adaptation under some external influences to switch in a reversible or

*This research was supported by the ACI "Instantbio"
1991 *Mathematics Subject Classification :* Primary 34C29, 34C25,58F22.
Key words and phrases : Slow-fast systems, Dynamical Bifurcations.

non-reversible manner from a stable state to another one) and used in life sciences by H. Atlan ([2], [10]).

Also, this approach does not take into account canards. Discovered by E. Benoit, Callot, M. and F. Diener ([5]) using non-standard analysis, canards are limit periodic sets (limit when $\epsilon \to 0$) of the van der Pol system:

$$\epsilon \dot{x} = y - f(x) = y - (\frac{x^3}{3} + x^2) \tag{1.1}$$

$$\dot{y} = x - c(\epsilon) \tag{1.2}$$

where $c(\epsilon)$ ranges between some bounds:

$$c_0 + \exp(-\alpha/\epsilon) < c(\epsilon) < c_0 + \exp(-\beta/\epsilon), \quad (\alpha > \beta > 0). \tag{1.3}$$

The surprising aspect is that a part of the canard coincides with a repulsive piece of the critical manifold. More recent contributions of Dumortier and Roussarie ([8]) yield new (standard) proofs for the existence of such limit-periodic sets. There are now several evidences showing the relevance of this notion to explain experimental facts observed in physiology. Note that canards occur in two-parameters families in some narrow set of parameters.

The system we present here, although quite simple, displays a kind of recurrence of the canard effect which enhances the delay to bifurcation.

2 Transcritical Dynamical Bifurcation

The classical transcritical bifurcation occurs when the parameter λ in the equation:

$$\dot{x} = -\lambda x + x^2, \tag{2.4}$$

crosses $\lambda = 0$. Equation 2.4 displays two equilibria, $x = 0$ and $x = \lambda$. For $\lambda > 0$, $x = 0$ is stable and $x = \lambda$ is unstable. After the bifurcation, $\lambda < 0$, $x = 0$ is stable and $x = \lambda$ is unstable. The two axis have "exchanged" their stability.

The terminology "Dynamical Bifurcation" refers to the situation where the bifurcation parameter is replaced by a slowly varying variable. In the case of the transcritical bifurcation, this yields:

$$\dot{x} = -yx + x^2 \tag{2.5}$$

$$\dot{y} = -\epsilon, \tag{2.6}$$

where ϵ is assumed to be small.

This yields

$$\dot{x} = -(-\epsilon t + y_0)x + x^2, \quad (y_0 = y(0))$$

which is an integrable equation of Bernoulli type. Its solution displays:

$$x(t) = \frac{x_0 \exp[-Y(t)]}{1 - x_0 \int_0^t \exp[-Y(u)]du} \quad (x_0 = x(0)$$

$$Y(t) = \int_0^t y(s)ds = \int_0^t (-\epsilon s + y_0)ds = -\epsilon \frac{t^2}{2} + y_0 t.$$

If we fix an initial data (x_0, y_0), $y_0 > 0$, $0 < x_0 < y_0/2$, and we consider the solution with this initial data we find easily that it takes time $t = y_0/\epsilon$ to reach the axis $y = 0$. If x_0 is quite small, that means the orbit stays closer and closer of the attractive part of the critical manifold untill it reaches the axis $x = y$ and then coordinate x start increasing. But now consider time cy_0/ϵ, $1 \geq c \geq 2$. Then a straightforward computation shows that $Y(t) = c(1 - \frac{c}{2})\frac{y_0^2}{\epsilon} = \frac{k}{\epsilon}$, and that

$$\lim_{\epsilon \to 0} \int_\epsilon^{2y_0/\epsilon} \exp(\epsilon \frac{t^2}{2} - y_0 t) dt = \frac{2}{y_0}.$$

This shows that if ϵ is small enough and if the initial data satifies $x_0 < y_0/2$, then:

$$x(t) = O(\frac{x_0 \exp(-k/\epsilon)}{1 - 2x_0/y_0}) < x_0.$$

This yields that, despite the repulsiveness of the axis $x = 0, y < 0$, the orbit remains for a very long time close to $x = 0$, indeed $x(t) < x_0$. Note that after a larger time the orbit blows away from this repulsive axis. This phenomenon, although quite simply explained, is of the same nature as the delay to bifurcation discovered for the dynamical Hopf bifurcation. See for instance ([3], [6], [9], [11]). This well-known effect is instrumental in the example we construct in this article. Some related work has been done in computing exit points through the passage of single turning points, for example [4], [7].

3 An example of system with enhanced delay

Consider the equation:

$$\dot{x} = (1 - x^2)(x - y)$$
$$\dot{y} = \epsilon x.$$

The fast dynamics displays the invariant lines $x = -1$, $x = 1$, and $y = x$. A quick analysis shows that, as the slow variable y varies, the fast system undergoes two transcritical bifurcations near the points $(-1, -1)$ and $(1, 1)$. As we recalled in the first paragraph, a typical orbit near $(-1, -1)$ first displays a "delay" along the repulsive part $(x = -1, y < -1)$ of the slow manifold. Then, by hysteresis, it jumps to the attractive part $(x = 1, y < 1)$ till it reaches the other transcritical bifurcation where it again displays another delay along the repulsive part $(x = 1, y > 1)$. Then it jumps again to $(x = -1, y > 1)$ and starts again. There is such a mechanism of successive enhancements of the delay after several turns generated by the hysteresis. After this intuitive explanation, we give now a formal proof of the:

Theorem
For all initial data inside the strip $-1 < x < 1$, for all δ and for all T, the corresponding orbit spends a time larger than T within a distance less than δ to the repulsive part of the slow manifold.

Proof
Inside the strip $\mid x \mid < 1$, it is convenient to use the variable u: $x = \tanh u$. The system yields the equations:

$$\dot{u} = \tanh u - y$$

$$\dot{y} = \epsilon \tanh u.$$

Consider the function

$$\Phi(u, y) = \frac{1}{2}(\tanh u - y)^2 + \epsilon \ln(\cosh u),$$

and its time derivation along the flow. This displays:

$$\frac{d}{dt}\Phi(u, y) = (\frac{\dot{u}}{\cosh u})^2$$

Hence the function Φ is strictly increasing along the flow (Lyapunov function for the flow).
Note now that if $(u(t), y(t))$ is a solution then $(-u(t), -y(t))$ is also a solution. To study the orbits of the system, we can restrict to initial data $u = u_0 \geq 0$ and $y = y_0$. The first step of the proof is to show that all orbits intersect both axes $u = 0$ and $y = u$ in infinitely many points.
Assume first $y_0 \geq \tanh u_0$. Then $\dot{u}(0) \leq 0$. But

$$\frac{d}{dt}(y - \tanh u) = \epsilon \tanh u + \frac{y - \tanh u}{\cosh^2 u}$$

shows that $y - \tanh u$ grows hence remains positive. Assume that u would remain always positive. Then, as $y - \tanh u > 0$, u is monotone decreasing. Hence there exists l such that $u \to l$ as $t \to +\infty$.
If $l > 0$, $\dot{y} = \epsilon \tanh u$ implies (via the mean value theorem) $y \to +\infty$ but then $\dot{u} \to -\infty$ and (mean value theorem) contradiction with $u > 0$.
If $l = 0$, $\dot{u} + y \to 0$. But y is monotone increasing. If $y \to +\infty$, then $\dot{u} \to -\infty$ and again contradiction. If y tends to a finite limit m, then $\dot{u} \to -m$ and again contradiction. Hence all orbits with initial data (u_0, y_0) with $y_0 \geq \tanh u_0 \geq 0$ intersect the axe $u = 0$.
Consider now the case $y_0 < \tanh u_0$. The variable u is first strictly increasing (as soon as $y < \tanh u$), hence positive and so y is increasing. Assume that $\tanh u - y$ would remain positive along the orbit. Then as $t \to +\infty$, u would tend to a limit m (eventually $m = +\infty$).As $\dot{y} \to m$, mean value theorem would imply $y \to +\infty$ and again a contradiction with $\dot{u} \to -\infty$. So the orbit necessarily intersects the axe $y = \tanh u$ and ultimately the axe $u = 0$ by the preceding argument. By symmetry, we can also show the existence of two sequences of times (t_n) and (θ_n) such that:

$$t_n < \theta_n < t_{n+1}$$

$$x(t_n) = x(t_{n+1}) = 0$$

$$x(\theta_n) = y(\theta_n)$$

$$y(t_n) = (-1)^n a_n, \quad a_n > 0.$$

In the second part of the proof we show that the sequence (a_n) is unbounded. Consider now the function

$$2\Phi(u,y) = w(x,y) = (x-y)^2 - \epsilon \ln \mid 1-x^2 \mid,$$

which satisfies:

$$\dot{w} = 2(1-x^2)(x-y)^2/\epsilon.$$

As w is strictly increasing, this yields:

$$w(t_n) = a_n^2 < w(\theta_n) = -\epsilon \ln(1-\xi_n^2) < w(t_{n+1}) = a_{n+1}^2,$$

with $\xi_n = x(\theta_n)$. Integration along the flow of $(\dot{w}) = 2\dot{x}(x-y)$ yields:

$$a_{n+1}^2 - a_n^2 = \int_{t_n}^{t_{n+1}} 2\dot{x}(x-y)dt = 2\int_{t_n}^{t_{n+1}} x\dot{y}dt$$

$$= 2\epsilon \int_{t_n}^{t_{n+1}} x^2 dt \leq 2\epsilon(t_n - t_{n+1}).$$

This shows that if (t_n) converges to a finite limit then so does (a_n) and (ξ_n). Now integration along the flow of $\dot{y} = \epsilon u$ yields:

$$a_n + a_{n+1} \leq \epsilon \int_{t_n}^{t_{n+1}} \mid \tanh u \mid dt \leq \epsilon(t_{n+1} - t_n),$$

and this shows that the sequence of times (t_n) is necessarily unbounded. Now assume that the sequence (a_n) would be bounded. Then, as the sequence (t_n) tends to $+\infty$, the function w would be bounded on the orbit. But then so would be both the two functions $(x-y)^2$ and $-\ln(1-x^2)$. But then there would exist a constant α such that $(1-x^2) \geq \alpha$ along the orbit and

$$\dot{w} \geq 2\alpha(x-y)^2 \geq 2\alpha w,$$

hence $\mathrm{e}^{-2\alpha t} w(t)$ increasing and contradiction with the fact that w would be bounded. Last step is classical in slow-fast dynamics. Consider the unbounded sequence of points $(0, a_n)$ on the orbit. By Tikhonov's theorem there is an invariant curve which passes near that point. By Takens's theorem the flow is conjugated to the fast flow along this invariant curve. Hence necessarily there is for all orbits inside the strip, all δ and all T a part of the orbit which remains at a distance less than δ of the repulsive parts of the boundary of the strip for a time larger than T.

4 Asymptotics of the system outside the strip

Theorem

Given any initial point (x_0, y_0) outside the strip $\mid x \mid \leq 1$, the corresponding orbit is asymptotic to $y = x$.

Proof

We can always assume that $x_0 > 1$ because the system is symmetric relatively to the origin. The equations yield:

$$\epsilon \frac{dx}{dy} = (x - \frac{1}{x})(y-x).$$

So if $y \geq x$ and (x_0, y_0), (x_1, y_1) are two points on the same orbit with $x_0 < x_1$, we get:

$$(x_0 - \frac{1}{x_0})(y - x) \leq \epsilon \frac{dx}{dy} \leq (x_1 - \frac{1}{x_1})(y - x).$$

Set:

$$\alpha_i = \frac{1}{\epsilon}(x_i - \frac{1}{x_i}), \quad i = 0, 1$$

this yields

$$\frac{dx}{dy} - \alpha_0(y - x) \geq 0,$$

$$\frac{dx}{dy} - \alpha_1(y - x) \leq 0,$$

hence

$$\frac{d}{dy}(\mathrm{e}^{\alpha_0 y} x) \geq \alpha_0 y \mathrm{e}^{\alpha_0 y}$$

$$\frac{d}{dy}(\mathrm{e}^{\alpha_1 y} x) \leq \alpha_1 y \mathrm{e}^{\alpha_1 y}.$$

Integration between y_0 and y_1 yields

$$\mathrm{e}^{\alpha_0 y_0}(y_0 - x_0 - \frac{1}{\alpha_0}) \geq \mathrm{e}^{\alpha_0 y_1}(y_1 - x_1 - \frac{1}{\alpha_0}),$$

and

$$\mathrm{e}^{\alpha_1 y_0}(y_0 - x_0 - \frac{1}{\alpha_1}) \leq \mathrm{e}^{\alpha_1 y_1}(y_1 - x_1 - \frac{1}{\alpha_1}).$$

The second inequality shows that if $y_0 = x_0$, then

$$\mathrm{e}^{\alpha_1 y_0} \geq \mathrm{e}^{\alpha_1 y_1}(1 - \alpha_1(y_1 - x_1)),$$

and thus that the orbit stays above the line $y = x$. If we now choose x_0 (and α_0) large enough, the first inequality displays:

$$(y_1 - x_1) \leq \frac{1}{2\alpha_0}.$$

This shows that the orbit is asymptotic to $y = x$.

References

[1] R. Ashby, *Cybernetics*, New York, Academic Press, 1959.

[2] H. Atlan, *L'organisation biologique et la théorie de l'information*, Paris, Hermann, 1972.

[3] S.M. Baer, T. Erneux and J. Rinzel, *The slow passage through a Hopf bifurcation: Delay memory effects and resonance* SIAM Journal on Applied mathematics, vol. 49, n1 (Feb 1989), 55–71.

[4] E. Benoit, *Equations différentielles: relation entrée-sortie* Comptes rendus de l'Académie des Sciences de Paris, **293**, série I, 293-296 (1981).

[5] E. Benoit, J.-L. Callot, F. Diener and M. Diener, *Chasse au canard.* Collectanea Mathematica, Barcelone, 31(1-3); 37-119 (1981).

[6] B. Candelpergher, F. Diener and M. Diener, *Retard à la bifurcation : du local au global* in *Bifurcations of Planar Vector Fields* proceedings of the Luminy conference, JP Françoise and R. Roussarie (Edts), Lecture Notes in Mathematics **1455**, Springer, Berlin (1990), 1–19.

[7] P. De Maesschalk, F. Dumortier, *Time analysis and entry-exit relation near planar turning points.* Journal of Diff. Equations, vol. 215, Issue 2, 225-267 (2005).

[8] F. Dumortier and R. Roussarie, *Canard cycles and center manifolds* with an appendix by Cheng Zhi Li. *Mem. Amer. Math. Soc.* 121 (1996), n577.

[9] T. Erneux, E.L. Reiss, L.J. Holden, M. Georgiou, *Slow passage through bifurcation and limit points. Asymptotic theory and applications.* Dynamic Bifurcations (Luminy, 1990), Ed. E. Benoit, Lecture Notes in Mathematics **1493**, Springer, Berlin (1991), 14–28.

[10] J.-P. Françoise, Oscillations en biologie : Analyse qualitative et modèles, Collection: Mathématiques et Applications **46**, Springer, (2005).

[11] H. Zoladek, *Remarks on the delay of the loss of stability of systems with changing parameters* in *Bifurcations of Planar Vector Fields* proceedings of the Luminy conference, JP Françoise and R. Roussarie (Edts), Lecture Notes in Mathematics **1455**, (1990), 393–396.

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