Almost summability and unconditionally Cauchy series

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Abstract

In this paper, we obtain new characterizations of weakly unconditionally Cauchy series and unconditionally convergent series through the summability obtained by the Banach-Lorentz convergence. We study new spaces associated to a series in a Banach space and obtain a new version of the Orlicz-Pettis theorem by means of the almost summability.

1 Introduction

Let X be a normed space. For any given series $\sum_i x_i$ in X, let us consider the set $S(\sum_i x_i)$ (respectively $S_w(\sum_i x_i)$) of sequences $(a_i)_i \in \ell_\infty$ such that $\sum_i a_i x_i$ converges (respectively converges for the weak topology). The set $S(\sum_i x_i)$ (respectively $S_w(\sum_i x_i)$), endowed with the sup norm, will be called the space of convergence (respectively weak convergence) associated to the series $\sum_i x_i$.

Definition 1.1. A series $\sum_i x_i$ in a normed space X is said to be a weakly unconditionally Cauchy (*wuc*) series if for each $\varepsilon > 0$ and $f \in X^*$, an $n_0 \in \mathbb{N}$ can be found such that for each finite subset $F \subset \mathbb{N}$ with $F \cap \{1, \dots, n_0\} = \varnothing$ is $\sum_{i \in F} |f(x_i)| < \varepsilon$.

As a consequence, $\sum_i x_i$ is a *wuc* series in X if and only if each functional $f \in X^*$ satisfies that $\sum_{i=1}^{\infty} |f(x_i)| < \infty$.

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In [8] it is proved that a normed space X is complete if and only if for every weakly unconditionally Cauchy (*wuc*) series $\sum_i x_i$, the space $S(\sum_i x_i)$ is also complete.

Diestel [5] proves the following characterization, that will be used several times in this paper:

Theorem 1.1. *Let $\sum_i x_i$ be a series in a normed space X. Then, the series $\sum_i x_i$ is wuc if and only if there exists $H > 0$ such that*

$$\begin{aligned} H &= \sup\{\|\sum_{i=1}^{n} a_i x_i\| : n \in \mathbb{N}, |a_i| \leqslant 1, i \in \{1, \dots, n\}\} = \\ &= \sup\{\|\sum_{i=1}^{n} \varepsilon_i x_i\| : n \in \mathbb{N}, \varepsilon_i \in \{-1, 1\}, i \in \{1, \dots, n\}\} = \\ &= \sup\{\sum_{i=1}^{n} |f(x_i)| : f \in B_{X^*}\}, \end{aligned}$$

where B_{X^*} denotes the closed unit ball in X^*.

In [2], Banach introduces the following concept of convergence and proves its existence:

Definition 1.2. A linear map $\varphi : \ell_\infty \to \mathbb{R}$ is said to be a Banach limit if verifies the following properties:

1. $\varphi((a_n)_n) = \lim_n a_n$ if $(a_n)_n \in c$, where c denotes the space of convergent real sequences.
2. $\varphi((a_n)_n) \geqslant 0$ if $a_n \geqslant 0$ for each $n \in \mathbb{N}$.
3. $\varphi((a_n)_n) = \varphi((a_{n+1})_n)$.

The almost convergence of a bounded real sequence is defined as follows:

Definition 1.3. A sequence $(a_n)_n$ in ℓ_∞ is said to be almost convergent to $s \in \mathbb{R}$ if for each Banach limit φ it is satisfied that $\varphi((a_n)_n) = s$. We denote this by $AC \lim_n a_n = s$.

Lorentz, in 1948 [6] proves that $AC \lim_n a_n = s$ if and only if

$$\lim_{i\to\infty} \frac{1}{i+1}(a_j + a_{j+1} + \cdots + a_{j+i}) = s \qquad \text{uniformly in } j \in \mathbb{N}.$$

In this work, we denote by X a general normed space. We will say that a sequence $(x_n)_n$ in X is almost convergent to $x_0 \in X$, and we write $AC \lim_n x_n = x_0$, if $\lim_{i\to\infty} \frac{1}{i+1}(x_j + x_{j+1} + \cdots + x_{j+i}) = x_0$ uniformly in $j \in \mathbb{N}$. In the following proposition, we check that if $(x_n)_n$ is almost convergent, then it is necessary a bounded sequence.

Proposition 1.1. *Let X be a normed space and $(x_n)_n$ an almost convergent sequence in X. Then $(x_n)_n \in \ell_\infty(X)$.*

Proof. Let $(x_n)_n$ be a sequence in X such that $AC \lim_n x_n = x_0$ for some $x_0 \in X$. We can fix $\varepsilon > 0$ and $i_0 \in \mathbb{N}$ satisfying that

$$\left\| \sum_{k=j}^{j+i} \frac{x_k}{i+1} \right\| \leqslant \|x_0\| + \varepsilon$$

for every $i \geqslant i_0$ and $j \in \mathbb{N}$.
Then, we have that for every $j \in \mathbb{N}$ is

$$\|x_j\| = \left\| \frac{i_0+2}{i_0+1} \sum_{k=j}^{j+i_0+1} \frac{x_k}{i_0+2} - \sum_{k=j+1}^{j+i_0+1} \frac{x_k}{i_0+1} \right\| \leqslant \left(\frac{i_0+2}{i_0+1} + 1 \right) (\|x_0\| + \varepsilon),$$

where the last term is a fixed constant, what concludes the proof. ■

Definition 1.4. We will say that a series $\sum_i x_i$ in X is almost convergent to $x_0 \in X$, and we will denote it by $AC \sum_i x_i = x_0$, if $AC \lim_n s_n = x_0$, where $s_n = \sum_{i=1}^{n} x_i$ is the sequence of partial sums.
Therefore, $AC \sum_i x_i = x_0$ if and only if

$$\lim_{i\to\infty} \left(\sum_{k=1}^{j} x_k + \frac{1}{i+1} \sum_{k=1}^{i} (i-k+1)\, x_{j+k} \right) = x_0$$

uniformly in $j \in \mathbb{N}$.

Definition 1.5. We will say that x_0 is the weak almost limit of a sequence $(x_n)_n$, and we will write $wAC \lim x_n = x_0$, if each function $f \in X^*$ verifies that $AC \lim_n f(x_n) = f(x_0)$.

2 Spaces of sequences defined by the almost summability of a series.

Let X be a normed space and $\sum_i x_i$ a series in X. We define the sets

$$S_{AC}(\sum_i x_i) = \{(a_i)_i \in \ell_\infty : AC \sum_i a_i x_i \text{ exists}\}$$
$$S_{wAC}(\sum_i x_i) = \{(a_i)_i \in \ell_\infty : wAC \sum_i a_i x_i \text{ exists}\}.$$

These are vector subspaces of ℓ_∞, and we consider them endowed with the sup norm.

Theorem 2.1. *Let X be a Banach space and $\sum_i x_i$ a series in X. Then $\sum_i x_i$ is wuc (weakly unconditionally Cauchy) if and only if $S_{AC}(\sum_i x_i)$ is complete.*

Proof. Consider $\sum_i x_i$ to be a *wuc* series. It will be enough to prove that $S_{AC}(\sum_i x_i)$ is closed in ℓ_∞. Let $(a^n)_n$ be a sequence in $S_{AC}(\sum_i x_i)$, $a^n = (a_i^n)_i$ for each $n \in \mathbb{N}$, and let also be $a^0 \in \ell_\infty$ such that $\lim_n \|a^n - a^0\| = 0$. We will show that $a^0 \in S_{AC}(\sum_i x_i)$. Let $H > 0$ be such that

$$H \geqslant \sup\{\|\sum_{i=1}^{n} a_i x_i\| : n \in \mathbb{N}, |a_i| \leqslant 1, i \in \{1, \ldots, n\}\}.$$

For each natural n, there exists $y_n \in X$ such that $y_n = AC\sum_i a_i^n x_i$. We will check that $(y_n)_n$ is a Cauchy sequence.

If $\varepsilon > 0$ is given, there exists an n_0 such that if $p, q \geqslant n_0$, then $\|a^p - a^q\| < \varepsilon/3H$. If $p, q \geqslant n_0$ are fixed, there exists $i \in \mathbb{N}$ verifying

$$\begin{aligned} \|y_p - (\sum_{k=1}^{j} a_k^p x_k + \frac{1}{i+1}\sum_{k=1}^{i}(i-k+1)a_{j+k}^p x_{j+k})\| &< \frac{\varepsilon}{3} &\quad (1)\\ \|y_q - (\sum_{k=1}^{j} a_k^q x_k + \frac{1}{i+1}\sum_{k=1}^{i}(i-k+1)a_{j+k}^q x_{j+k})\| &< \frac{\varepsilon}{3} &\quad (2) \end{aligned}$$

for each $j \in N$. Then, if $p, q \geqslant n_0$ we have that

$$\begin{aligned} \|y_p - y_q\| &\leqslant (1) + (2) + \\ &+ \|\sum_{k=1}^{j}(a_k^p - a_k^q)x_k + \sum_{k=1}^{i}\frac{i-k+1}{i+1}(a_{j+k}^p - a_{j+k}^q)x_{j+k}\|, \quad (3) \end{aligned}$$

where $(3) \leqslant \varepsilon/3$. Therefore, since X is a Banach space, there exists $y_0 \in X$ such that $\lim_n \|y_n - y_0\| = 0$. We will check that $AC\sum_i a_i^0 x_i = y_0$, that is,

$$\lim_{i\to\infty}\left(\sum_{k=1}^{j} a_k^0 x_k + \frac{1}{i+1}\sum_{k=1}^{i}(i-k+1)a_{j+k}^0 x_{j+k}\right) = y_0, \text{ uniformly in } j \in \mathbb{N}.$$

If $\varepsilon > 0$ is given, we can fix a natural n such that $\|a^n - a^0\| < \varepsilon/3H$ and $\|y_n - y_0\| < \varepsilon/3$. Now, we can also fix i_0 such that for every $i \geqslant i_0$ is

$$\|y_n - (\sum_{k=1}^{j} a_k^n x_k + \frac{1}{i+1}\sum_{k=1}^{i}(i-k+1)a_{j+k}^n x_{j+k})\| < \frac{\varepsilon}{3}$$

for every $j \in \mathbb{N}$.

Then, if $i \geqslant i_0$ it is satisfied that

$$\begin{aligned}
&\|y_0 - (\sum_{k=1}^{j} a_k^0 x_k + \frac{1}{i+1}\sum_{k=1}^{i}(i-k+1)a_{j+k}^0 x_{j+k})\| \leqslant \|y_0 - y_n\| + \\
+ \quad &\|y_n - (\sum_{k=1}^{j} a_k^n x_k + \frac{1}{i+1}\sum_{k=1}^{i}(i-k+1)a_{j+k}^n x_{j+k})\| + \\
+ \quad &\|\sum_{k=1}^{j}(a_k^n - a_k^0)x_k + \frac{1}{i+1}\sum_{k=1}^{i}(i-k+1)(a_{j+k}^n - a_{j+k}^0)x_{j+k}\| \leqslant \frac{2\varepsilon}{3} + \\
+ \quad &\|a^n - a^0\| \left(\sum_{k=1}^{j} \frac{(a_k^n - a_k^0)}{\|a^n - a^0\|}x_k + \sum_{k=1}^{i} \frac{(i-k+1)(a_{j+k}^n - a_{j+k}^0)}{(i+1)\|a^n - a^0\|}x_{j+k}\right) \leqslant \\
\leqslant \quad &\frac{2\varepsilon}{3} + \frac{\varepsilon}{3H}H \leqslant \varepsilon
\end{aligned}$$

for every $j \in \mathbb{N}$. Thus $(a_n^0)_n \in S_{AC}(\sum_i x_i)$.

Conversely, if $S_{AC}(\sum_i x_i)$ is closed, since $c_{00} \subset S_{AC}(\sum_i x_i)$, we deduce that $c_0 \subset S_{AC}(\sum_i x_i)$. Suppose that $\sum_i x_i$ is not a *wuc* series. Then, there exists $f \in X^*$ verifying $\sum_{i=1}^{\infty} |f(x_i)| = +\infty$.

We can choose a natural n_1 such that $\sum_{i=1}^{n_1} |f(x_i)| > 2 \cdot 2$, and for $i \in \{1, \dots, n_1\}$ we define $a_i = 1/2$ if $f(x_i) \geqslant 0$ or $a_i = -1/2$ if $f(x_i) < 0$.

There exists $n_2 > n_1$ such that $\sum_{i=n_1+1}^{n_2} |f(x_i)| > 3 \cdot 3$, and for $i \in \{n_1+1, \dots, n_2\}$ we define $a_i = 1/3$ if $f(x_i) \geqslant 0$ or $a_i = -1/3$ if $|f(x_i)| < 0$.

In this manner we obtain an increasing sequence $(n_k)_k$ in $\mathbb{N}$ and a sequence $a = (a_i)_i$ in c_0 such that $\sum_{i=1}^{\infty} a_i f(x_i) = +\infty$. Since $(a_i)_i \in S_{AC}(\sum_i x_i)$, it follows that $AC\sum_i a_i x_i$ exists and therefore $\left(\sum_{i=1}^{n} a_i f(x_i)\right)_n$ is a bounded sequence, which is a contradiction, and we are done. ■

Then, we have the following result:

Corollary 2.1. *Let X be a Banach space and $\sum_i x_i$ a series in X. Then, $\sum_i x_i$ is a wuc series if and only if for each sequence $(a_i)_i \in c_0$ it is satisfied that $AC\sum_i a_i x_i$ exists.*

Proof. Let $\sum_i x_i$ be a *wuc* series in X. Then, we have that $S_{AC}(\sum_i x_i)$ is complete. Since $c_{00} \subset S_{AC}(\sum_i x_i)$, we deduce that $c_0 \subset S_{AC}(\sum_i x_i)$, that is, $AC\sum_i a_i x_i$ exists for every sequence $(a_i)_i \in c_0$. The converse was proved at the end of the previous demonstration. ■

Remark. Let X be a normed space and $\sum_i x_i$ a series in X. We consider the linear map $T : S_{AC}(\sum_i x_i) \to X$ defined by $T(a) = AC\sum_i a_i x_i$.

Suppose that $\sum_i x_i$ is a *wuc* series and consider $H = \sup\{\|\sum_{i=1}^n a_i x_i\| : n \in \mathbb{N}, |a_i| \leqslant 1, i \in \{1,\dots,n\}\}$. Then, it is easy to check that if $a \in S_{AC}(\sum_i x_i)$ then $\|T(a)\| = \|AC\sum_i a_i x_i\| \leqslant H\|a\|$, and therefore T is continuous.

Conversely, if T is continuous and $\{a_1,\dots,a_j\} \subset [-1,1]$, it is satisfied that $\|\sum_{i=1}^{j} a_i x_i\| = \|AC\sum_{i=1}^{\infty} a_i x_i\| \leqslant \|T\|$ (considering $a_i = 0$ if $i > j$), which implies that $\sum_i x_i$ is a *wuc* series.

In the next theorem we study the completeness of the space $S_{wAC}(\sum_i x_i)$.

Theorem 2.2. *Let X be a Banach space and $\sum_i x_i$ a series in X. Then, $\sum_i x_i$ is a wuc series if and only if $S_{wAC}(\sum_i x_i)$ is complete.*

Proof. Consider $\sum_i x_i$ to be a *wuc* series. It will be enough to prove that $S_{wAC}(\sum_i x_i)$ is closed in ℓ_∞. Let $(a^n)_n$ be a sequence in $S_{wAC}(\sum_i x_i)$, $a^n = (a_i^n)_i$ for each $n \in \mathbb{N}$, and let also be $a^0 \in \ell_\infty$ such that $\lim_n \|a^n - a^0\| = 0$. We will show that $a^0 \in S_{wAC}(\sum_i x_i)$. Let $H > 0$ be such that

$$H \geqslant \sup\{\|\sum_{i=1}^{n} a_i x_i\| : n \in \mathbb{N}, |a_i| \leqslant 1, i \in \{1,\dots,n\}\}.$$

For each natural n, there exists $y_n \in X$ such that $y_n = wAC\sum_i a_i^n x_i$. We will check that $(y_n)_n$ is a Cauchy sequence.

If $\varepsilon > 0$ is given, there exists n_0 such that if $p,q \geqslant n_0$, then $\|a^p - a^q\| < \varepsilon/3H$. We fix $p,q \geqslant n_0$, and we have that there exists $f \in S_{X^*}$ (unit sphere in X^*) verifying $\|y_p - y_q\| = |f(y_p - y_q)|$. Since $AC\sum_i a_i^p f(x_i) = f(y_p)$ and $AC\sum_i a_i^q f(x_i) = f(y_q)$, there exists $i \in \mathbb{N}$ such that

$$|f(y_p) - (\sum_{k=1}^{j} a_k^p f(x_k) + \frac{1}{i+1}\sum_{k=1}^{i}(i-k+1)a_{j+k}^p f(x_{j+k}))| < \frac{\varepsilon}{3} \tag{4}$$

$$|f(y_q) - (\sum_{k=1}^{j} a_k^q f(x_k) + \frac{1}{i+1}\sum_{k=1}^{i}(i-k+1)a_{j+k}^q f(x_{j+k}))| < \frac{\varepsilon}{3} \tag{5}$$

for each $j \in N$. Then, if $p,q \geqslant n_0$ we have that

$$\begin{aligned}\|y_p - y_q\| &= |f(y_p) - f(y_q)| \leqslant (4) + (5) + \\ &+ \ |\sum_{k=1}^{j}(a_k^p - a_k^q)f(x_k) + \sum_{k=1}^{i}\frac{i-k+1}{i+1}(a_{j+k}^p - a_{j+k}^q)f(x_{j+k})|, \end{aligned} \tag{6}$$

where (6) $\leqslant \varepsilon/3$. Therefore, since X is a Banach space, there exists $y_0 \in X$ such that $\lim_n \|y_n - y_0\| = 0$. We will check that $wAC\sum_i a_i^0 x_i = y_0$.

If $\varepsilon > 0$ is given, we can fix a natural n such that $\|a^n - a^0\| < \varepsilon/3H$ and $\|y_n - y_0\| < \varepsilon/3$. Consider a functional $f \in B_{X^*}$. We have that there exists $i_0 \in \mathbb{N}$

such that if $i \geqslant i_0$ is

$$|f(y_n) - (\sum_{k=1}^{j} a_k^n f(x_k) + \frac{1}{i+1} \sum_{k=1}^{i} (i-k+1) a_{j+k}^n f(x_{j+k}))| < \frac{\varepsilon}{3}$$

for each $j \in \mathbb{N}$. But then, if $i \geqslant i_0$ and $j \in \mathbb{N}$, we have that

$$\begin{aligned} |f(y_0) - (\sum_{k=1}^{j} a_k^0 f(x_k) + \frac{1}{i+1} \sum_{k=1}^{i} (i-k+1) a_{j+k}^0 f(x_{j+k}))| &\leqslant |f(y_0 - y_n)| + \\ + \quad |f(y_n) - (\sum_{k=1}^{j} a_k^n f(x_k) + \frac{1}{i+1} \sum_{k=1}^{i} (i-k+1) a_{j+k}^n f(x_{j+k}))| &+ \\ + \quad |\sum_{k=1}^{j} (a_k^n - a_k^0) f(x_k) + \frac{1}{i+1} \sum_{k=1}^{i} (i-k+1)(a_{j+k}^n - a_{j+k}^0) f(x_{j+k})| &\leqslant \varepsilon, \end{aligned}$$

that is, $wAC \sum_i a_i^0 x_i = y_0$, and $a^0 \in S_{wAC}(\sum_i x_i)$.

Conversely, suppose that $S_{wAC}(\sum_i x_i)$ is complete, which implies that $c_0 \subset S_{wAC}(\sum_i x_i)$. Suppose that there exists $f \in X^*$ verifying $\sum_{i=1}^{\infty} |f(x_i)| = +\infty$.

Then, as we did in *Theorem 2.1*, a sequence $a = (a_i)_i$ in c_0 can be obtained such that $\sum_i a_i f(x_i) = +\infty$, Since $a \in S_{wAC}(\sum_i x_i)$, there will exist $x_0 \in X$ such that $wAC \sum_i a_i x_i = x_0$, and it will be $AC \sum_i a_i f(x_i) = x_0$. But this implies that the sequence $\left(\sum_{i=1}^{n} a_i f(x_i) \right)_n$ is bounded, which is a contradiction. ■

Remark. Let X be a Banach space and $\sum_i x_i$ a series in X. We consider the linear map $T : S_{wAC}(\sum_i x_i) \to X$ defined by $T(a) = wAC \sum_i a_i x_i$. We will show that $\sum_i x_i$ is a *wuc* series if and only if T is continuous.

We define $H = \sup\{\| \sum_{i=1}^{n} a_i x_i \| : n \in \mathbb{N}, |a_i| \leqslant 1, i \in \{1, \ldots, n\}\}$ and take $a \in S_{wAC}(\sum_i x_i)$. Then, $wAC \sum_i a_i x_i = x_0$ exists , and we can take $f \in S_{X^*}$ such that $|T(a)| = |f(T(a))| = |AC \sum_i a_i f(x_i)| \leqslant H \|a\|$.

Conversely, suppose that T is continuous. Then, if $\{a_1, \ldots a_n\} \subset [-1, 1]$, we have that $\| \sum_{i=1}^{n} a_i x_i \| = \| wAC \sum_{i=1}^{\infty} a_i x_i \| \leqslant \|T\|$ (considering $a_i = 0$ if $i > n$), and this implies that $\sum_i x_i$ is a *wuc* series.

From the previous theorem and its proof, the following corollary can be easily proved:

Corollary 2.2. *Let X be a Banach space and $\sum_i x_i$ a series in X. Then, the following are equivalent:*

1. $\sum_i x_i$ *is a wuc series.*
2. $S_{wAC}(\sum_i x_i)$ *is complete.*

3. $c_0 \subset S_{wAC}(\sum_i x_i)$ *($wAC\sum_i a_i x_i$ exists for each $a = (a_i)_i \in c_0$).*

Let us see that the hypothesis of completeness in the two previous theorems is completely necessary. The following argument has been taken from [8]:

Let X be a non-complete normed space. Then, it is easy to prove that there exists a sequence $\sum_i x_i$ in X such that $\|x_i\| < 1/i2^i$ and $\sum_i x_i = x^{**} \in X^{**} \setminus X$. Then, we have that $AC\sum_i x_i = x^{**}$. If we consider the series $\sum_i z_i$ defined by $z_i = ix_i$ for each $n \in \mathbb{N}$, we have that $\sum_i z_i$ is a *wuc* series. Consider the sequence $a = (a_i)_i \in c_0$ given by $a_i = 1/i$. It is satisfied that $AC\sum_i a_i z_i \in X^{**} \setminus X$, and therefore $a \notin S_{AC}(\sum_i z_i)$ and $a \notin S_{wAC}(\sum_i z_i)$.

Let X be a normed space and X^* its dual space. Let also $\sum_i f_i$ be a series in X^*. It is known [5] that $\sum_i f_i$ is *wuc* if and only if $\sum_i |f_i(x)| < \infty$ for each $x \in X$.

Now, we consider the vector space

$$S_{*\text{-}wAC}(\sum_i f_i) = \{(a_i)_i \in \ell_\infty : *\text{-}wAC\sum_i a_i f_i \text{ exists}\},$$

where $*\text{-}wAC\sum_i a_i f_i = f_0$ if and only if $AC\sum_i a_i f_i(x) = f_0(x)$ for each $x \in X$.

Theorem 2.3. *Let X be a normed space. It is satisfied that 1 $\Rightarrow$ 2 $\Rightarrow$ 3, where:*

1. $\sum_i f_i$ *is a wuc series.*

2. $S_{*\text{-}wAC}(\sum_i f_i) = \ell_\infty$.

3. If $x \in X$ and $M \subset \mathbb{N}$, then $AC\sum_{i\in M} f_i(x)$ exists.

Besides, if X is a barrelled normed space, the three items are equivalent.

Proof. From the $*$-weak compacity of B_{X^*} we deduce that *1* $\Rightarrow$ *2*. It is clear that *2* $\Rightarrow$ *3*.

We suppose now that X is barrelled, and we will prove that *3* $\Rightarrow$ *1*. Effectively, our goal is to prove that $E = \{\sum_{i=1}^n a_i f_i : n \in \mathbb{N}, |a_i| \leqslant 1, i \in \{1,\ldots,n\}\}$ is pointwise bounded for each $x \in X$, and therefore E is bounded, which implies that $\sum_i f_i$ is a *wuc* series. Suppose that E is not pointwise bounded, that is, there exists $x_0 \in X$ such that $\sum_i |f_i(x_0)| = +\infty$. Then, we can choose a subset $M \subset \mathbb{N}$ such that $\sum_{i\in M} f_i(x_0) = \pm\infty$. But, by hypothesis, $AC\sum_{i\in M} f_i(x_0)$ exists, which is a contradiction. ■

3 Almost convergence and Orlicz-Pettis theorem.

The Orlicz-Pettis theorem gives us a characterization of *wuc* series in Banach spaces through the weak convergence of all its subseries.

Theorem 3.1. *Let X be a Banach space and $\sum_i x_i$ a series in X such that for each $M \subset \mathbb{N}$ we have that $w\sum_{i\in M} x_i$ exists. Then $\sum_i x_i$ is uco (unconditionally convergent).*

By means of the almost convergence of Banach-Lorentz, we are going to obtain a new version of this theorem.

Theorem 3.2. *Let X be a Banach space and $\sum_i x_i$ a series in X such that for each $M \subset \mathbb{N}$ we have that $wAC\sum_{i\in M} x_i$ exists. Then $\sum_i x_i$ is uco (unconditionally convergent).*

Proof. First, we are going to prove that $\sum_i x_i$ is a *wuc* series. If $\sum_i x_i$ were not *wuc*, then we can find a set $M \subset \mathbb{N}$ and $f \in X^*$ such that $\sum_{i\in M} f(x_i) = +\infty$. But by hypothesis, there exists $x_0 \in X$ such that $AC\sum_{i\in M} f(x_i) = f(x_0)$, which is a contradiction.

Finally, we will show that if $M \subset \mathbb{N}$, then $w\sum_{i\in M} x_i$ exists, and by the classic Orlicz-Pettis theorem, we will have that $\sum_i x_i$ is *uco*.

Effectively, if $M \subset \mathbb{N}$, we have that there exists $x_0 \in X$ such that $wAC\sum_{i\in M} x_i = x_0$, but if $f \in X^*$, then $\sum_{i\in M} f(x_i)$ exists, and it will be

$$\sum_{i\in M} f(x_i) = AC\sum_{i\in M} f(x_i) = f(x_0),$$

and therefore $w\sum_{i\in M} x_i = x_0$. ■

The following corollary is deduced as an evident consequence:

Corollary 3.1. *Let X be a Banach space, and $\sum_i x_i$ a series in X. Then, the following are equivalent:*

1. $\sum_i x_i$ *is uco.*
2. $AC\sum_i a_i x_i$ *exists for each* $a = (a_i)_i \in \ell_\infty$.
3. $wAC\sum_i a_i x_i$ *exists for each* $a = (a_i)_i \in \ell_\infty$.

Remark. Let X be a Banach space, and $\sum_i x_i$ a *wuc* series in X. Then, for each $a = (a_i)_i \in \ell_\infty$, it is satisfied that $\sum_i a_i x_i$ is a *wuc* series, which implies that

$$S_{AC}(\sum_i x_i) \subset S_w(\sum_i x_i)$$

where $S_w(\sum_i x_i) = \{(a_i)_i \in \ell_\infty : w\sum_i a_i x_i \text{ exists}\}$, but we don't know what conditions us to obtain the equality of both spaces.

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