Contractive Mappings in Spaces with Vector Norm

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Abstract

In this paper we study the contractive mappings in vector space with vector norm. We suppose that values of the norm belong to a σ-complete vector lattice. The results extend the Banach fixed point theorem for differential equations, under the condition to retain the convergence of successive approximations to the unique solution.

1 Introduction

It is well-known that the successive approximations method is an iterative technique to prove the existence and approximation of solution of a (deterministic or stochastic) differential equation. If the coefficients of the equation satisfy the standard Lipschitz condition, than this method can be formulated as the Banach fixed point theorem relative to an integral operator on the space of functions associated in a natural way to the differential equation.

But this standard conditions is not satisfied for important classes of differential equations which arise in applications, therefore is necessary to weaken the Banach fixed point theorem, under condition to retain the convergence of successive approximations to the unique solution of the differential equation. Attempts to weaken the Banach fixed point theorem abound in literature. In our context we remark that the operator associated to the differential equation is compatible with the natural ordering of the underlying function space. Thus, one attempt is led to study the Banach fixed point theorem for operators in ordered vector spaces.

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This is possible in various ways. For example, is possible to consider that this operator is defined on an ordered Banach space. L.V. Kantorovicz ([10], Ch. XVIII, section 4) has generalized the Banach fixed point theorem in a different direction, considering that the norm of a function is an element belonging to a vector lattice. We say that this norm is a vector norm.

A survey on this subject can be found in R. Cristescu [6],[7].

In this paper, we study the contractive mappings in vector space with vector norm, but we suppose that values of the norm belong to a σ-complete vector lattice. In order to show the importance of these abstract results, in [3], [4] are given some nontrivial applications to stochastic differential equations. We thus obtain a fixed point result of successive approximation method used by Yamada [14] and Barbu and Bocşan [1] for stochastic differential equations in finite-dimensional respectively infinite-dimensional Hilbert spaces.

We note that the metric spaces, in which the metric take values in an ordered space, where first introduced by D. Kurepa [11]. Then E. Popa [12] and L.B. Ciric [5] studied the contraction type mappings in thus spaces. On the other hand, O. Hadzic [9] studied the fixed points in topological vector spaces considering thus spaces as a paranormed space, in which the norm take values in a topological semifield.

In sections 2 and 4 we recall the results in lattice theory, respectively in linear spaces with vector norm. In section 3 we introduce the notions of c-maps and b-maps and establish some properties. These results are used in section 5 for the notion of contraction map and to prove a fixed point theorem for these mappings.

The results obtained in this paper are used in [3], [4] for the study of the stochastic differential equations under non-Lipschitz conditions, of the type as in [1].

2 Convergence in σ-complete vector lattices

Let R be a σ-complete vector lattice with the order relation " $\leq$ ". Thus, each countable subset A which is upper bounded has a least upper bound denoted by $\sup_{a\in A} a$ or $\bigvee_{a\in A} a$. Also, each countable subset B which is lower bounded has a greatest lower bound denoted by $\inf_{a\in A} a$ or $\bigwedge_{a\in A} a$. Thus, for a bounded sequence $(r_n) \subset R$ the lower and upper limits

$$\underline{\lim}\, r_n = \bigvee_{n=1}^{\infty} \bigwedge_{m=n}^{\infty} r_m, \quad \overline{\lim}\, r_n = \bigwedge_{n=1}^{\infty} \bigvee_{m=n}^{\infty} r_m \tag{2.1}$$

is well-defined.

It is well-known [6] that in a σ-complete vector lattice a sequence (r_n) converges in order (or is o-convergent) if and only if it is bounded and $\underline{\lim}\, r_n = \overline{\lim}\, r_n$. In this case the common value r of these limits is called o-limit of (r_n) and we write $r_n \xrightarrow{o} r$.

In what follows we use the following properties of σ-convergence.

Proposition 2.1 **[6]** *Let (r_n) and (p_n) be two sequences in R.*
(i) If (r_n) is nondecreasing and upper bounded, then $r_n \xrightarrow{o} r$ where $r = \bigvee_{n=1}^{\infty}$.
ii) If (r_n) is nonincreasing and lower bounded, then $r_n \xrightarrow{o} r$ where $r = \bigwedge_{n=1}^{\infty}$.
(iii) If $r_n \xrightarrow{o} r$ and $p_n \xrightarrow{o} p$ and if $r_n \leq p_n$ for all n, then $r \leq p$.

(iv) If $0 \le r_n \le p_n$ for all n and if $p_n \xrightarrow{o} 0$ then $r_n \xrightarrow{o} 0$.
(v) If $r_n \xrightarrow{o} r$ and if (r_{n_k}) is a subsequence of (r_n), then $r_{n_k} \xrightarrow{o} r$.
(vi) If $r_n \xrightarrow{o} r$ and $r_n \xrightarrow{o} p$ then $r = p$.

For a sequence satisfying (i) (respectively (ii)) we write $r_n \uparrow r$ (respectively $r_n \downarrow r$).

Let R_+ be the cone of all nonnegative elements of R. We consider a subset P of R_+ which is also a σ-complete sub-lattice of R. Thus, P satisfies the following three properties:

(I) $0 \in P$ and $r \ge 0$ for all $r \in P$.
(II) If $(r_n) \subset P$ is an upper bounded sequence, then $\bigvee_{n=1}^{\infty} \in P$.
(III) If $(r_n) \subset P$ is a nonincreasing sequence, then $\bigwedge_{n=1}^{\infty} \in P$.

Since a sequence in P is lower bounded, the property (ii) of Proposition 2.1 can be reformulated.

Corollary 2.2 *If $(r_n) \subset P$ is a nonincreasing sequence, then $r_n \downarrow r$ where $r = \bigwedge_{n=1}^{\infty}$ and $r \in P$.*

Now, for an upper bounded sequence $(r_n) \subset P$ we define the sequence (ρ_n) by

$$\rho_n = \bigvee_{m=n}^{\infty} r_m. \tag{2.2}$$

Clearly, by (II) and (2.2) it follows that

$$(\rho_n) \subset P \text{ and } 0 \le r_n \le \rho_n \text{ for all } n. \tag{2.3}$$

Moreover,

$$(\rho_n) \text{ is nonincreasing} \tag{2.4}$$

and we have the following result.

Theorem 2.3 *For an upper bounded sequence $(r_n) \subset P$ the following statements are equivalent:*
(i) $r_n \xrightarrow{o} 0$
(ii) $\rho_n \downarrow 0$.

Proof. Suppose (i) holds, therefore $\overline{\lim}\, r_n = 0$. $(\rho_n) \subset P$ and (ρ_n) is nonincreasing in conformity with (2.3) and (2.4). Applying Corollary 2.2 it follows that (ρ_n) is o-convergent to $\rho = \bigwedge_{n=1}^{\infty} \rho_n = \bigwedge_{n=1}^{\infty} \bigvee_{m=n}^{\infty} r_m = \overline{\lim}\, r_n = 0$. Since $\rho_n \ge 0$, we obtain (ii).

Using (2.3) and Proposition 2.1, (iv) we obtain that (ii) implies (i). ■

We now recall that in R can be defined the module $|r|$ of $r \in R$. It is well known that $|r|$ can be calculated by the formula

$$|r| = r \bigvee (-r). \tag{2.5}$$

The following result is well-known [6].

Theorem 2.4 *Let $(r_n) \subset R$. Then $r_n \xrightarrow{o} r \in R$ if and only if there exists a sequence $(\rho_n) \subset R,\ \rho_n \downarrow 0$ such that*

$$|r_n - r| \leq \rho_n \text{ for all } n. \tag{2.6}$$

3 c-maps and b-maps on P

Let R be a σ-complete vector lattice and P a subset of R satisfying the conditions (I), (II) and (III) of Section 2.

Definition 3.1 *A map $\Phi : P \to P$ is called to be a c-map on P if it satisfies the following properties.*
(a) $\Phi(0) = 0$ and $\Phi(r) \geq r$ implies $r = 0$.
(b) Φ is nondecreasing.
(c) Φ is o-continuous, i.e. $r_n \to r$ implies $\Phi(r_n) \to \Phi(r)$.

Theorem 3.2 *Let Φ a c-map on P. If the sequence $(r_n) \subset P$ satisfies*
(i) (r_n) is upper bounded
(ii) $r_{n+1} \leq \Phi(r_n)$ for all $n \geq 1$,
then $r_n \xrightarrow{o} 0$.

Proof. Since (r_n) is upper bounded and $(r_n) \subset P$, from (II) it follows that the sequence (ρ_n) defined by (2.2) is in P.

Moreover (ρ_n) satisfies (2.4) and thus from Corollary 2.2 we obtain $\rho_n \downarrow \rho$ where $\rho = \bigwedge_{n=1}^{\infty} \rho_n$ and $\rho \in P$.

Now, for a fixed n and $m \geq n$ by (2.3) and (2.4) we have

$$r_m \leq \rho_m \leq \rho_n. \tag{3.1}$$

Since Φ is nondecreasing, by (ii) and (3.1) we obtain

$$r_{m+1} \leq \Phi(\rho_m) \leq \Phi(\rho_n). \tag{3.2}$$

But $\rho_{n+1} = \bigvee_{k=n+1}^{\infty} r_k = \bigvee_{m=n}^{\infty} r_{m+1}$ and then (3.2) implies

$$\rho_{n+1} \leq \Phi(\rho_n) \text{ for all } n. \tag{3.3}$$

Since (ρ_n) is o-convergent to $\rho \in P$ and Φ is o-continuous, then passing to limit in (3.3) and using the properties (iii) and (v) from Proposition 2.1 we obtain $\rho \leq \Phi(\rho)$. The property (a) from Definition 3.1 implies $\rho = 0$ and applying Theorem 2.3 results that $r_n \xrightarrow{o} 0$. ■

Remark 3.3 From the above proof it follows that Theorem 3.2 remains true if we suppose that Φ is continuous only for nonincreasing sequences.

Definition 3.4 *A map $\Psi : P \to P$ is called to be a b-map on P if it satisfies the following properties:*
(a) Ψ is nondecreasing;
(b) Ψ has a fixed point $r^* \neq 0$.

Theorem 3.5 *Let Ψ a b-map on P and let $r^* \neq 0$ a fixed point of Ψ. Let $r_0,\ r_1 \in P$. Then*
(i) If $r_0 \leq r^*$ and $r_1 \leq \Psi(r_0)$, then $r_1 \leq r^*$.
(i) If $r_0 \geq r^*$ and $r_1 \geq \Psi(r_0)$, then $r_1 \geq r^*$.

Proof. Since Ψ is nondecreasing and $r_0 \leq r^*$ it follows that $r_1 \leq \Psi(r_0) \leq \Psi(r^*) = r^*$. Thus (i) is proved and similarly can be proved (ii). ∎

For $r \in P$ define
$$[0, r] = \{r' : 0 \leq r' \leq r\}.$$
For $r_1,\ r_2 \in P,\ r_1 \leq r_2$ we set
$$[r_1, r_2] = \{r : r_1 \leq r \leq r_2\}.$$

We have the following result.

Corollary 3.6 *Let r_1^* and r_2^* two fixed points of a b-map Ψ on P. Suppose $r_1^* \leq r_2^*$. Then*
$$\Psi([r_1^*, r_2^*]) \subseteq [r_1^*, r_2^*].$$

Proof. For $r \in \Psi([r_1^*, r_2^*])$ there exists $r_0 \in [r_1^*, r_2^*]$ such that $r = \Psi(r_0)$. Applying Theorem 3.5 with $r^* = r_2^*$ and $r_1 = r$ we obtain $r \leq r_2^*$.
Similarly can be proved that $r \geq r_1^*$. ∎

Remark 3.7 From the proof of Corollary 3.6 and Theorem 3.5 (i) we have
$$\Psi([0, r^*]) \subseteq [0, r^*]$$
if $r^* \neq 0$ is a fixed point of a b-map Ψ on P.

4 Linear space with vector P-norm

Let R be a real σ-complete vector lattice and P a subset of R satisfying the conditions (I), (II) and (III) of Section 2. Furthermore , suppose that P is a cone.

Let $\mathcal{X}$ a real linear space. A vector P- norm on $\mathcal{X}$ is a map $p : \mathcal{X} \to P,\ p(x) = ||x||$ which satisfies the following axioms
(i) $||x|| \neq 0$ if $x \neq 0$;
(ii) $||\lambda x|| = |\lambda| ||x||$;
(iii) $||x + y|| \leq ||x|| + ||y||$
for all x and y in $\mathcal{X}$ and every λ a real number.

¿From this axioms follows that $||0|| = 0$ and $||x|| = 0$ implies $x = 0$.

Clearly, $(\mathcal{X}, p)$ is a normed lattice in the sense of Kantorovich and Akilov [[10], Ch. XVIII, section 4]. As in this book we say that a sequence $(x_n) \subset \mathcal{X}$ is

(a) convergent to $x \in \mathcal{X}$ if the sequence $r_n = ||x_n - x|| \in P$ is o-convergent to 0;

(b) Cauchy if there exists a sequence $(\rho_n) \subset P,\ \rho_n \downarrow 0$ such that for each n we have

$$||x_n - x_m|| \leq \rho_n \text{ for all } m \geq n. \tag{4.1}$$

Proposition 4.1 *A sequence $(x_n) \in \mathcal{X}$ is convergent to $x \in \mathcal{X}$ if and only if there exists a sequence $(\rho_n) \subset P,\ \rho_n \downarrow 0$ such that*

$$||x_n - x|| \leq \rho_n \textit{ for all } n. \tag{4.2}$$

Proof. Let $r_n = ||x_n - x||$. If $x_n \to x$, then by definition $r_n \xrightarrow{o} 0$ and therefore $(r_n) \subset P$ is bounded. The sequence (ρ_n) defined by (2.2) is nonincreasing and thus (4.2) holds. Applying Theorem 2.3 we obtain $\rho_n \downarrow 0$.

For converse implication we apply Proposition 2.1 (iv) and obtain $r_n \downarrow 0$. ■

Proposition 4.2 *Let a sequence $(x_n) \subset \mathcal{X}$ and (x_{n_k}) a subsequence of (x_n).*

(i) If (x_n) converges to $x \in \mathcal{X}$, then (x_{n_k}) converges to x.

(ii) If (x_n) converges to x and y then $x = y$.

Proof. Let (ρ_n) the sequence as in Proposition 4.1. Then by (4.2) we have

$$||x_{n_k} - x|| \leq \rho_{n_k} \text{ for all } k. \tag{4.3}$$

Thus, for the sequence $x'_k = x_{n_k}, \rho'_k = \rho_{n_k}$ the inequality (4.2) holds. Applying Proposition 2.1, (v) and Proposition 4.1 we obtain the convergence of x'_k to x.

For the part (ii), from axiom (iii) we have

$$0 \leq ||x - y|| \leq ||x - x_n|| + ||x_n - y|| \text{ for all } n. \tag{4.4}$$

Passing to limit in (4.4) we obtain $||x - y|| = 0$ and thus $x = y$. ■

We say that a subset A of $\mathcal{X}$ is bounded if the set $\{||a|| : a \in A\} \subset P$ is bounded (in order). Then we have.

Proposition 4.3 *If the sequence $(x_n) \subset \mathcal{X}$ is convergent or Cauchy, then it is bounded.*

Proof. If (x_n) converges to x, let (ρ_n) the sequence as in Proposition 4.1. From axiom (iii) we have

$$0 \leq ||x_n|| \leq ||x_n - x|| + ||x|| \leq \rho_n + ||x|| \leq \rho_1 + ||x|| \text{ for all } n.$$

Hence, there exists $M = \rho_1 + ||x|| \in P$ such that $0 \leq ||x_n|| \leq M$.

Thus, (x_n) is bounded.

Similarly, if (x_n) is Cauchy then for $n \geq 1$ we have

$$0 \leq ||x_m|| \leq ||x_1 - x_m|| + ||x_1|| \leq \rho_1 + ||x_1|| \text{ for all } m \geq 1$$

and again we obtain $0 \leq ||x_m|| \leq M$ where $M = \rho_1 + ||x_1||$, i.e. (x_n) is bounded. ∎

Theorem 4.4 *(i) The following inequality holds*

$$|||x|| - ||y||| \leq ||x - y|| \text{ for all } x, y \in \mathcal{X}. \tag{4.5}$$

(ii) The vector norm is continuous on $\mathcal{X}$.

Proof. (i) From axiom (iii) we have $||x|| \leq ||x - y|| + ||y||$ which is equivalent with $||x|| - ||y|| \leq ||x - y||$. Similarly, we have $||y|| - ||x|| \leq ||x - y||$ and thus, by (2.5) we obtain (4.5).
(ii) Let $x_n \to x$. By (4.5) it follows that

$$|||x_n|| - ||x||| \leq ||x_n - x|| \text{ for all } n. \tag{4.6}$$

Denoting by $r_n = ||x_n - x||$ we have $r_n \xrightarrow{o} 0$. Applying Theorem 2.3 and using (2.3) and (2.4) it results that Theorem 2.4 can be applied and thus $||x_n|| \xrightarrow{o} ||x||$. ∎

Recall [10] that the space $\mathcal{X}$ is called complete if each Cauchy sequence is convergent to an element of $\mathcal{X}$.

5 Contractions maps in spaces with vector norm

Let R and P as in Section 4 and $(\mathcal{X}, P, || \cdot ||)$ a real linear space with a vector norm.

Definition 5.1 *We say that a map $G : \mathcal{X} \to \mathcal{X}$ is a contraction map on $\mathcal{X}$ with respect to the c-map Φ on P (or P-contraction, or Φ-contraction) if*

$$||Gx - Gy|| \leq \Phi(||x - y||) \text{ for all } x, y \in \mathcal{X}. \tag{5.1}$$

If for a map $G : \mathcal{X} \to \mathcal{X}$ there exists a c-map Φ on P such that (5.1) holds, then we say that G is a contraction on $\mathcal{X}$.

Proposition 5.2 *Let G a contraction on $\mathcal{X}$. Then*
(i) G is continuous
(ii) If G has a fixed point, then it is unique.

Proof. (i) Let $x_n \to x$ and $r_n = ||x_n - x||$. Applying (5.1) we have

$$0 \leq ||Gx_n - Gx|| \leq \Phi(r_n) \text{ for all } n. \tag{5.2}$$

But $x_n \to x$ is equivalent with $r_n \xrightarrow{o} 0$. Then, from Definition 3.1, we obtain $\Phi(r_n) \xrightarrow{o} \Phi(0) = 0$. Passing to limit in (5.2) and using Proposition 2.1, (iv) it follows that $||Gx_n - Gx|| \xrightarrow{o} 0$, i.e $Gx_n \to Gx$.

(ii) Let x and y two fixed points of G and $r = ||x - y||$. Since $r = ||Gx - Gy||$, then (5.1) implies $r \leq \Phi(r)$ and by the definition of a c-map we obtain $r = 0$. Thus, $x = y$. ∎

Remark 5.3 Suppose that in Definition 3.1 the continuity of Φ holds only for nonincreasing sequence. Let (ρ_n) defined by (2.2). Then from Theorem 2.3 we have $\rho_n \downarrow 0$. Since Φ is nondecreasing and (ρ_n) satisfies (2.3), from (5.2) we obtain

$$0 \leq ||Gx_n - Gx|| \leq \Phi(\rho_n) \text{ for all } n.$$

Thus, as in the proof of (i), we obtain the continuity of G.

Theorem 5.4 *Let G a contraction on $\mathcal{X}$. If*
(a) $\mathcal{X}$ is complete
(b) the sequence of successive approximations

$$x_{n+1} = Gx_n, \ n \geq 0, \ x_0 \in \mathcal{X} \tag{5.3}$$

is bounded.

Then G has a fixed point $x \in \mathcal{X}$ and $x_n \to x$.

Proof. Let Φ a c-map satisfying (5.1) and let $r_{n,m} = ||x_n - x_m||$.
Since (x_n) is bounded, it follows that there exists $M \in P$ such that

$$0 \leq ||x_n|| \leq M \text{ for all } n \geq 0. \tag{5.4}$$

Then, by the axioms of vector norm we have

$$0 \leq r_{n,m} \leq ||x_n|| + ||x_m|| \leq M + M = 2M \in P, \ n \geq 0, \ m \geq 0.$$

Thus, $\{r_{n,m} : n \geq 0, \ m \geq 0\}$ is an upper bounded subset of P and since P satisfies the property (II), Section 2, it follows that the sequence (r_n) defined by

$$r_n = \bigvee_{m,p \geq n}^{\infty} r_{m,p}, \ n \geq 0 \tag{5.5}$$

is in P.

But the sequence (r_n) defined by (5.5) is nonincreasing. From Proposition 2.1, (ii) results

$$r_n \downarrow r, \text{ where } r = \bigwedge_{n=0}^{\infty} r_n. \tag{5.6}$$

Since r_n is in P and P satisfies (III) from Section 2, it follows that

$$r \in P. \tag{5.7}$$

On the other hand, G is a Φ-contraction and by (5.1), for $m, p \geq n$, we obtain $r_{m+1,p+1} = ||x_{m+1} - x_{p+1}|| = ||Gx_m - Gx_p|| \leq \Phi(||x_m - x_p||) = \Phi(r_{m,p}) \leq \Phi(r_n)$.

Thus we have

$$r_{m+1,p+1} \leq \Phi(r_n) \text{ for all } m, p \geq n. \tag{5.8}$$

But $r_{n+1} = \bigvee_{m,p\geq n+1}^{\infty} r_{m,p} = \bigvee_{m,p\geq n}^{\infty} r_{m+1,p+1}$ and using (5.8) we obtain

$$r_{n+1} \leq \Phi(r_n) \text{ for all } n \geq 0. \tag{5.9}$$

Therefore the sequence $(r_n) \subset P$ satisfies the conditions of Theorem 3.2. Thus

$$r_n \downarrow 0. \tag{5.10}$$

Since by (5.5) we have

$$||x_n - x_m|| = r_{n,m} \leq r_n \text{ for all } m \geq n \tag{5.11}$$

and since (5.10) holds, it follows that (4.1) holds, therefore (x_n) is a Cauchy sequence. But $\mathcal{X}$ is complete, hence there exists $x \in \mathcal{X}$ such that $x_n \to x$. Let (ρ_n) the sequence given by Proposition 4.1. Then, by (5.1) we have

$$0 \leq ||Gx_n - Gx|| \leq \Phi(||x_n - x||) \leq \Phi(\rho_n) \text{ for all } n. \tag{5.12}$$

Using the continuity of Φ we have $\Phi(\rho_n) \downarrow 0$. Thus, applying Proposition 4.1 we obtain $Gx_n \to Gx$. By (5.3) it follows that (Gx_n) is a subsequence of (x_n), hence $Gx_n \to x$. Applying Proposition 4.2 (ii) we obtain $Gx = x$ and the Theorem is proved. ∎

Definition 5.5 *We say that a map $G : \mathcal{X} \to \mathcal{X}$ is bounded if there exists a b-map Ψ on P such that*

$$||Gx|| \leq \Psi(||x||) \textit{ for all } x \in \mathcal{X}. \tag{5.13}$$

If (5.13) holds we say that G is Ψ-bounded.

Theorem 5.6 *Suppose that the map $G : \mathcal{X} \to \mathcal{X}$ is Ψ-bounded and let $r^* \neq 0$ a fixed point of Ψ.*

Let $x_0 \in \mathcal{X}$ and $r_0 = ||x_0||$. If

$$r_0 \leq r^*, \tag{5.14}$$

then the sequence of successive approximations of G defined by (5.3) is bounded and

$$||x_n|| \leq r^* \textit{ for all } n. \tag{5.15}$$

Proof. Let $r_1 = ||x_1|| = ||Gx_0||$. By (5.13) we have $r_1 \leq \Psi(||x_0||) = \Psi(r_0)$. Using (5.14) it follows from Theorem 3.5, (i) that $r_1 \leq r^*$. Thus, (5.15) holds for $n = 0, 1$ and by mathematical induction it can be proved that (5.15) holds for all n. ∎

Combining the above results we obtain the following.

Theorem 5.7 *Let $G : \mathcal{X} \to \mathcal{X}$ a Ψ-bounded contraction on the linear space with vector norm and let r^*, $r^* \neq 0$ a fixed points of Ψ. If*

(a) $\mathcal{X}$ is complete

(b) $r_0 = ||x_0||$ *satisfies*

$$r_0 \leq r^*$$

then the sequence of successive approximations given by

$$x_{n+1} = Gx_n, \ n \geq 0, \ x_0 \in \mathcal{X}$$

is convergent to the unique fixed point of G.

Proof. From Theorem 5.6 it follows that (x_n) satisfies (5.15), hence (x_n) is bounded. Theorem 5.4 implies that (x_n) converges to a fixed point x of G, which is unique, in conformity with Proposition 5.2. ■

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