Hermitian Clifford–Hermite wavelets: an alternative approach

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Abstract

Clifford analysis is a higher dimensional function theory offering a refinement of classical harmonic analysis, which has proven to be an appropriate framework for developing a higher dimensional continuous wavelet transform theory. In this setting a very specific construction of the wavelets has been established, encompassing all dimensions at once as opposed to the usual tensorial approaches, and being based on generalizations to higher dimension of classical orthogonal polynomials on the real line, such as the radial Clifford–Hermite polynomials, leading to Clifford–Hermite wavelets. More recently, Hermitian Clifford analysis has emerged as a new and successful branch of Clifford analysis, offering yet a refinement of the orthogonal case. In this new setting a Clifford–Hermite continuous wavelet transform has already been introduced in earlier work, its norm preserving character however being expressed in terms of suitably adapted scalar valued inner products on the respective L_2–spaces of signals and of transforms involved. In this contribution we present an alternative Hermitian Clifford–Hermite wavelet theory with Clifford algebra valued inner products, based on an orthogonal decomposition of the space of square integral functions, which is obtained by introducing a new Hilbert transform in the Hermitian setting.

Received by the editors March 2007.
Communicated by F. Bastin.
2000 *Mathematics Subject Classification :* 42C40; 42B10; 30G35; 44A15.
Key words and phrases : continuous wavelet transform; Hermitian Clifford analysis.

1 Introduction

The one–dimensional continuous wavelet transform (CWT) is a successful tool for signal and image analysis, which has found numerous applications in mathematics, physics and engineering (see e.g. [13, 14]). On the real line wavelets constitute a family of functions $\psi_{a,b}$ derived from one original function ψ, called the mother wavelet, by change of scale a (i.e. by dilation) and by change of position b (i.e. by translation):

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-b}{a}\right) \quad , \quad a > 0 \quad , \quad b \in \mathbb{R}\,.$$

On this mother wavelet ψ some conditions have to be imposed. First, ψ is required to be an L_2–function, i.e. a signal of finite energy, which is well localized both in the time and in the frequency domain. Moreover it has to satisfy the so–called admissibility condition

$$C_\psi \equiv 2\pi \int_{-\infty}^{+\infty} \frac{|\mathcal{F}[\psi](u)|^2}{|u|}\, du < +\infty\,,$$

where $\mathcal{F}[\psi]$ denotes the Fourier spectrum of ψ. If ψ is an L_1–function as well, the admissibility condition implies that ψ should have zero momentum, i.e.

$$\int_{-\infty}^{+\infty} \psi(x)\; dx = 0$$

which can only be fulfilled if ψ is an oscillating function, explaining the terminology wavelet. In applications, additional requirements are imposed, among which a given number of vanishing moments, viz.

$$\int_{-\infty}^{+\infty} x^n\; \psi(x)\; dx = 0 \quad , \quad n = 0, 1, \ldots, N\,.$$

This means that the corresponding CWT defined as

$$F(a,b) = < \psi_{a,b}\;, f > = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} \left(\psi\left(\frac{x-b}{a}\right)\right)^c\; f(x)\; dx$$

(where $\cdot^c$ denotes complex conjugation) will filter out polynomial behaviour of the signal f up to degree N, making it adequate at detecting singularities. When considering L_2–functions f and g with respective transforms F and G, the following weighted inner product may be introduced:

$$[F,G] = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_0^{+\infty} \Big(F(a,b)\Big)^c\; G(a,b)\; \frac{da}{a^2}\; db\,.$$

Taking into account the above mentioned admissibility condition for the mother wavelet ψ, the corresponding Parseval formula reads

$$[F,G] = < f, g >\,,$$

showing that, as a consequence of the admissibility condition, the CWT is an isometry, or a norm preserving map, from $L_2(\mathbb{R})$ into $L_2(\mathbb{R}_+ \times \mathbb{R}, C_\psi^{-1} a^{-2}\; da\; db)$.

Higher dimensional CWTs typically originate as tensor products of one–dimensional phenomena. As opposed to these tensorial approaches, Clifford analysis (see e.g. [2, 15, 16]) offers an appropriate framework for developing a higher dimensional CWT theory where all dimensions are encompassed at once as an intrinsic feature (see e.g. [7, 8, 9, 10, 12]). Standard Euclidean Clifford analysis focusses on so–called monogenic functions, i.e. null solutions of the rotation invariant vector valued Dirac operator

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j} \,,$$

where $(e_1, \ldots, e_m)$ forms an orthonormal basis for the quadratic space $\mathbb{R}^{0,m}$ underlying the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. As a function theory, Clifford analysis may be considered both as a generalization of the theory of holomorphic functions in the complex plane and as a refinement of harmonic analysis, since the Dirac operator factorizes the Laplacian. The wavelets developed in this setting are based on Clifford generalizations of classical orthogonal polynomials on the real line. In this respect we explicitly mention the radial Clifford–Hermite polynomials introduced in [20], which were applied to wavelet analysis in the Euclidean Clifford setting in [11].

In earlier work the CWT has also been studied in a new branch of Clifford analysis: Hermitian Clifford analysis, focussing on Hermitian monogenic functions taking values in a complex Clifford algebra or in a complex spinor space (see e.g. [5, 6, 18, 19]). Hermitian monogenicity, h–monogenicity for short, is expressed by means of two mutually h–conjugate complex vector valued Dirac operators, which are invariant under the action of a realization of the unitary group. New Hermitian Clifford–Hermite polynomials were constructed in [5], starting in a natural way from a Rodrigues formula involving both Hermitian Dirac operators mentioned. Due to the specific features of the setting, four different types of polynomials were obtained, two types of even and two types of odd degree. In [3] these polynomials were used as building blocks for Hermitian Clifford–Hermite wavelets. Following the construction of four types of polynomials, also four types of wavelets and corresponding CWTs were introduced, two of even and two of odd order. However, the Hermitian setting necessitated a major adaptation as compared to the Clifford–Hermite wavelets in the orthogonal framework: the Parseval formula, expressing the norm preserving character of the CWT, had to be reformulated in terms of suitably adapted scalar valued inner products on the L_2–spaces of signals and of wavelet transforms.

In this paper we present an alternative Hermitian Clifford–Hermite wavelet theory with Clifford algebra valued inner products on the L_2–spaces of signals and of transforms. This new theory is based on the decomposition of the space of square integrable functions as a direct sum of two orthogonal subspaces $\mathcal{H}^{\pm}$, obtained through the introduction of a new Hilbert transform in the Hermitian setting (see [1]). Moreover, the spaces $\mathcal{H}^{\pm}$ turn out to be the respective kernels of, in each case, two out of the four types of wavelet transforms obtained. Finally, the present approach clearly shows that the Hermitian Clifford–Hermite CWTs offer a refinement of the Clifford–Hermite CWTs in orthogonal Clifford analysis, since the Clifford–Hermite CWT of a given order is seen to split into both Hermitian Clifford–Hermite CWTs of that same order.

2 Hermitian Clifford analysis

Let $\mathbb{R}^{0,m}$ be endowed with a non–degenerate quadratic form of signature $(0,m)$, let $(e_1,\ldots,e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$ and let $\mathbb{R}_{0,m}$ be the real Clifford algebra constructed over $\mathbb{R}^{0,m}$. The non–commutative multiplication in $\mathbb{R}_{0,m}$ is governed by

$$e_j e_k + e_k e_j = -2\delta_{jk} \quad , \quad j,k = 1,\ldots,m\,. \tag{2.1}$$

A basis for $\mathbb{R}_{0,m}$ is obtained by considering for a set $A = \{j_1,\ldots,j_h\} \subset \{1,\ldots,m\}$ the element $e_A = e_{j_1}\ldots e_{j_h}$, with $1 \leq j_1 < j_2 < \ldots < j_h \leq m$. For the empty set $\emptyset$ one puts $e_\emptyset = 1$, the identity element. Any Clifford number a in $\mathbb{R}_{0,m}$ may thus be written as $a = \sum_A e_A a_A$, $a_A \in \mathbb{R}$, or still as $a = \sum_{k=0}^m [a]_k$, where $[a]_k = \sum_{|A|=k} e_A a_A$ is the so–called k–vector part of a $(k = 0,1,\ldots,m)$. The Euclidean space $\mathbb{R}^{0,m}$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(x_1,\ldots,x_m)$ with the Clifford vector $\underline{x}$ given by $\underline{x} = \sum_{j=1}^m e_j x_j$. The product of two vectors is given by

$$\underline{x}\,\underline{y} = \underline{x} \bullet \underline{y} + \underline{x} \wedge \underline{y}\,,$$

where

$$\underline{x} \bullet \underline{y} = -<\underline{x},\underline{y}> = -\sum_{j=1}^m x_j y_j \quad , \quad \underline{x} \wedge \underline{y} = \sum_{i=1}^m \sum_{j=i+1}^m e_i e_j (x_i y_j - x_j y_i)$$

are a scalar and a bivector (or 2–vector) respectively. Note that the square of a vector $\underline{x}$ is scalar valued and equals the norm squared up to a minus sign: $\underline{x}^2 = -<\underline{x},\underline{x}> = -|\underline{x}|^2$. The dual of the vector $\underline{x}$ is the rotation invariant, vector valued first order differential operator

$$\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}\,,$$

called Dirac operator, which factorizes the Laplacian, viz $\Delta_m = -\partial_{\underline{x}}^2$. It is precisely this Dirac operator which underlies the notion of monogenicity of a function, a notion which is the higher dimensional counterpart of holomorphy in the complex plane. A function f defined and differentiable in an open region Ω of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$ is called left–monogenic in Ω if $\partial_{\underline{x}}[f] = 0$.

When allowing for complex constants and moreover taking the dimension to be even, say $m = 2n$, the same set of generators as above, $(e_1,\ldots,e_{2n})$, still satisfying the defining relations (2.1), may in fact also produce the complex Clifford algebra $\mathbb{C}_{2n}$. As $\mathbb{C}_{2n}$ is the complexification of the real Clifford algebra $\mathbb{R}_{0,2n}$, i.e. $\mathbb{C}_{2n} = \mathbb{R}_{0,2n} \oplus i\mathbb{R}_{0,2n}$, any complex Clifford number $\lambda \in \mathbb{C}_{2n}$ may be written as $\lambda = a + ib$, $a,b \in \mathbb{R}_{0,2n}$, leading to the definition of the Hermitian conjugation

$$\lambda^\dagger = (a+ib)^\dagger = \overline{a} - i\overline{b}\,,$$

where the bar denotes the usual conjugation in $\mathbb{R}_{0,2n}$, i.e. the main anti–involution for which $\overline{e}_j = -e_j$, $j = 1,\ldots,2n$. This Hermitian conjugation leads to a Hermitian inner product and its associated norm on $\mathbb{C}_{2n}$ given by

$$(\lambda,\mu) = [\lambda^\dagger \mu]_0 \quad \text{and} \quad |\lambda| = \sqrt{[\lambda^\dagger \lambda]_0} = (\sum_A |\lambda_A|^2)^{1/2}\,.$$

The above framework will be referred to as the Hermitian Clifford setting, as opposed to the traditional Euclidean Clifford setting. Hermitian Clifford analysis then focusses on the null solutions of two Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$, introduced by means of the so–called Witt basis for the complex Clifford algebra $\mathbb{C}_{2n}$:

$$\mathfrak{f}_j = \frac{1}{2}(e_j - i e_{n+j}) \quad , \quad \mathfrak{f}_j^\dagger = -\frac{1}{2}(e_j + i e_{n+j}) \;\; , \;\; j = 1, \ldots, n$$

satisfying the Grassmann identities

$$\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = \mathfrak{f}_j^\dagger \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j^\dagger = 0 \;\; , \;\; j, k = 1, \ldots, n$$

and the duality identities

$$\mathfrak{f}_j \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j = \mathfrak{f}_j^\dagger \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j^\dagger = \delta_{jk} \;\; , \;\; j, k = 1, \ldots, n\,.$$

The Grassmann algebras generated by $(\mathfrak{f}_j)_{j=1}^n$ and $(\mathfrak{f}_j^\dagger)_{j=1}^n$ are denoted by $\mathbb{C}\Lambda_n$ and $\mathbb{C}\Lambda_n^\dagger$ respectively. Using this Witt basis, the vector $(X_1, \ldots, X_{2n}) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{0,2n}$ is identified with the Clifford vector

$$\underline{X} = \sum_{j=1}^n (e_j x_j + e_{n+j} y_j) = \sum_{j=1}^n \mathfrak{f}_j z_j - \sum_{j=1}^n \mathfrak{f}_j^\dagger z_j^c$$

where the complex variables $z_j = x_j + i y_j$ and their complex conjugates $z_j^c = x_j - i y_j$, $j = 1, \ldots, n$ have been introduced. Defining the Hermitian vector variables

$$\underline{z} = \sum_{j=1}^n \mathfrak{f}_j z_j \quad \text{and} \quad \underline{z}^\dagger = (\underline{z})^\dagger = \sum_{j=1}^n \mathfrak{f}_j^\dagger z_j^c\,,$$

the Clifford vector $\underline{X}$ clearly takes the form

$$\underline{X} = \underline{z} - \underline{z}^\dagger\,.$$

To this vector $\underline{X}$ the traditional Dirac operator is associated, rewritten as

$$\partial_{\underline{X}} = \sum_{j=1}^n (e_j \partial_{x_j} + e_{n+j} \partial_{y_j}) = 2\,(\sum_{j=1}^n \mathfrak{f}_j \partial_{z_j^c} - \sum_{j=1}^n \mathfrak{f}_j^\dagger \partial_{z_j}) = 2(\partial_{\underline{z}}^\dagger - \partial_{\underline{z}})\,.$$

Here we have introduced the Hermitian Dirac operators

$$\partial_{\underline{z}} = \sum_{j=1}^n \mathfrak{f}_j^\dagger \partial_{z_j} \quad \text{and} \quad \partial_{\underline{z}}^\dagger = (\partial_{\underline{z}})^\dagger = \sum_{j=1}^n \mathfrak{f}_j \partial_{z_j^c}\,,$$

involving the classical Cauchy–Riemann operators and their complex conjugates in the complex z_j planes, i.e. $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and $\partial_{z_j^c} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$, $j = 1, \ldots, n$. In what follows also a second Clifford vector is considered, viz

$$\underline{X}| = \sum_{j=1}^n (e_j y_j - e_{n+j} x_j) = \frac{1}{i}\sum_{j=1}^n \mathfrak{f}_j z_j + \frac{1}{i}\sum_{j=1}^n \mathfrak{f}_j^\dagger z_j^c = \frac{1}{i}(\underline{z} + \underline{z}^\dagger)$$

with corresponding Dirac operator

$$\partial_{\underline{X}|} = \sum_{j=1}^{n}(e_j\partial_{y_j} - e_{n+j}\partial_{x_j}) = \frac{2}{i}\,(\sum_{j=1}^{n}\mathfrak{f}_j\partial_{z_j^c} + \sum_{j=1}^{n}\mathfrak{f}_j^\dagger\partial_{z_j}) = \frac{2}{i}(\partial_{\underline{z}}^\dagger + \partial_{\underline{z}})\,.$$

Note that the vectors $\underline{X}$ and $\underline{X}|$ are orthogonal, which implies that the Clifford vectors $\underline{X}$ and $\underline{X}|$ anti–commute: $\underline{X}\;\underline{X}| = -<\underline{X},\underline{X}|> + \underline{X}\wedge\underline{X}| = \underline{X}\wedge\underline{X}| = -\underline{X}|\wedge\underline{X} = -\underline{X}|\;\underline{X}$. On account of the isotropy of the Witt basis elements, the Hermitian vector variables and Dirac operators are isotropic as well, i.e.

$$(\underline{z})^2 = (\underline{z}^\dagger)^2 = 0 \quad \text{and} \quad (\partial_{\underline{z}})^2 = (\partial_{\underline{z}}^\dagger)^2 = 0\,, \tag{2.2}$$

from which it directly follows that the Laplacian $\Delta_{2n} = -\partial_{\underline{X}}^2 = -\partial_{\underline{X}|}^2$ allows for the decomposition

$$\Delta_{2n} = 4(\partial_{\underline{z}}\partial_{\underline{z}}^\dagger + \partial_{\underline{z}}^\dagger\partial_{\underline{z}})\,. \tag{2.3}$$

Moreover, one also has that

$$\underline{z}\underline{z}^\dagger + \underline{z}^\dagger\underline{z} = |\underline{z}|^2 = |\underline{z}^\dagger|^2 = |\underline{X}|^2 = |\underline{X}||^2\,.$$

A continuously differentiable function g on $\mathbb{R}^{2n}$ with values in $\mathbb{C}_{2n}$ is called a Hermitian monogenic (or h–monogenic) function if and only if it satisfies the system

$$\partial_{\underline{X}}g = 0 = \partial_{\underline{X}|}g \;\text{ or equivalently }\; \partial_{\underline{z}}g = 0 = \partial_{\underline{z}}^\dagger g\,.$$

The Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$ are invariant under the action of a realisation of the unitary group in terms of the Clifford algebra, see [5, 6]. This group $\widetilde{\mathrm{U}}(n) \subset \mathrm{Spin}(2n)$ is given by

$$\widetilde{\mathrm{U}}(n) = \{s \in \mathrm{Spin}(2n) \mid \exists\theta \geq 0 : \overline{s}I = \exp(-i\theta)I\}\,, \tag{2.4}$$

its definition involving the primitive selfadjoint idempotent I, which is introduced as follows. Put, for each $j = 1, \ldots, n$, $I_j = \mathfrak{f}_j\mathfrak{f}_j^\dagger = \frac{1}{2}(1 - ie_je_{n+j})$, then the I_j are mutually commuting idempotents for which moreover $I_j^\dagger = I_j$. Now, let $I = I_1 \ldots I_n = \mathfrak{f}_1\mathfrak{f}_1^\dagger\mathfrak{f}_2\mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n\mathfrak{f}_n^\dagger$, then obviously $I^2 = I$ and $I^\dagger = I$. The invariance of the operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$ under the action of $\widetilde{\mathrm{U}}(n)$ is then expressed as

$$[\partial_{\underline{z}}, L(s)] = 0 = [\partial_{\underline{z}}^\dagger, L(s)] \quad , \quad s \in \widetilde{\mathrm{U}}(n)\,,$$

where $[.\,,\,.]$ denotes the commutator and $L(s)$ is the so–called ℓ–representation of an arbitrary spin element s (see e.g. [2]).

In the sequel we will use the following definition of the standard Fourier transform in $\mathbb{R}^{2n}$:

$$\mathcal{F}[f](\underline{U}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \exp(-i<\underline{X},\underline{U}>)\; f(\underline{X})\; dV(\underline{X})\,,$$

where $dV(\underline{X})$ denotes the differential form $dV(\underline{X}) = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \ldots \wedge dx_n \wedge dy_n$, and where we have put $\underline{U} = (u_1, \ldots, u_n, v_1, \ldots, v_n)$ and $\underline{X} = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. Let us now rewrite this Fourier transform in terms of the Hermitian vector variables. From the foregoing we know that $\underline{X} = \underline{z} - \underline{z}^\dagger$ and $\underline{U} = \underline{w} - \underline{w}^\dagger$, with $\underline{z} = \sum_{j=1}^n \mathfrak{f}_jz_j$, $z_j = x_j + iy_j$ and $\underline{w} = \sum_{j=1}^n \mathfrak{f}_jw_j$, $w_j = u_j + iv_j$,

$j = 1, \ldots, n$. When passing to these Hermitian vector variables, the Fourier transform takes the form

$$\mathcal{F}[f](\underline{w}, \underline{w}^\dagger) = \frac{i^n}{(4\pi)^n} \int_{\mathbb{C}^n} \exp\left(-2i\mathrm{Re}(\underline{w}, \underline{z})\right) f(\underline{z}, \underline{z}^\dagger)\, d\underline{z} \wedge d\underline{z}^\dagger\,,$$

where we have introduced the differential form

$$d\underline{z} \wedge d\underline{z}^\dagger \equiv dz_1 \wedge dz_1^c \wedge \ldots \wedge dz_n \wedge dz_n^c = (-2i)^n dV(\underline{X})\,.$$

This Fourier transform satisfies the differentiation rules

$$\mathcal{F}[\partial_{\underline{z}} f] = \frac{i}{2}\underline{w}^\dagger\, \mathcal{F}[f] \quad \text{and} \quad \mathcal{F}[\partial_{\underline{z}}^\dagger f] = \frac{i}{2}\underline{w}\, \mathcal{F}[f] \tag{2.5}$$

and the multiplication rules

$$\mathcal{F}[\underline{z} f] = 2i\partial_{\underline{w}}^\dagger\, \mathcal{F}[f] \quad \text{and} \quad \mathcal{F}[\underline{z}^\dagger f] = 2i\partial_{\underline{w}}\, \mathcal{F}[f]\,.$$

Moreover, the Fourier transform is an isometry on $L_2(\mathbb{R}^{2n})$, in other words, for all $f, g \in L_2(\mathbb{R}^{2n})$ the Parseval formula holds:

$$< f, g > = < \mathcal{F}[f], \mathcal{F}[g] > \tag{2.6}$$

with the Clifford algebra valued inner product given by

$$< f, g > = \int_{\mathbb{R}^{2n}} f^\dagger(\underline{X})\, g(\underline{X})\, dV(\underline{X})\,.$$

We now introduce two Hilbert transforms, one in the Clifford vector $\underline{X}$, viz

$$H_{\underline{X}}[f] = \frac{2}{a_{2n+1}}\ \mathrm{Pv}\frac{\overline{\underline{X}}}{r^{2n+1}} * f$$

and a second one in the associated Clifford vector $\underline{X}|$, viz

$$H_{\underline{X}|}[f] = \frac{2}{a_{2n+1}}\ \mathrm{Pv}\frac{\overline{\underline{X}|}}{r^{2n+1}} * f\,,$$

a_{2n+1} denoting the area of the unit sphere S^{2n} in $\mathbb{R}^{2n+1}$. Their Fourier spectra are

$$\mathcal{F}[H_{\underline{X}}[f]](\underline{U}) = i\frac{\underline{U}}{|\underline{U}|}\mathcal{F}[f](\underline{U}) \quad \text{and} \quad \mathcal{F}[H_{\underline{X}|}[f]](\underline{U}) = i\frac{\underline{U}|}{|\underline{U}|}\mathcal{F}[f](\underline{U})\,.$$

From the observation that

$$\mathcal{F}\left[H_{\underline{X}}\left[H_{\underline{X}|}[f]\right]\right](\underline{U}) = -\frac{\underline{U}\,\underline{U}|}{|\underline{U}|^2}\mathcal{F}[f](\underline{U}) = \frac{\underline{U}|\,\underline{U}}{|\underline{U}|^2}\mathcal{F}[f](\underline{U}) = -\mathcal{F}\left[H_{\underline{X}|}\left[H_{\underline{X}}[f]\right]\right](\underline{U})\,,$$

we derive that both Hilbert transforms are anti–commuting, i.e.

$$H_{\underline{X}} \circ H_{\underline{X}|} = -H_{\underline{X}|} \circ H_{\underline{X}}\,.$$

The new transform $\widetilde{H}$, obtained by this composition, up to a factor i, viz

$$\widetilde{H} = -iH_{\underline{X}|} \circ H_{\underline{X}} = iH_{\underline{X}} \circ H_{\underline{X}|}$$

shows the usual properties of a Hilbert transform and plays an important rôle in what follows.

Lemma 2.1. *One has:*
• $\widetilde{H}$ *is a bounded linear operator in* $L_2(\mathbb{R}^{2n})$*;*
• $\widetilde{H}$ *is norm preserving, i.e. for all* $f, g \in L_2(\mathbb{R}^{2n})$ *one has*

$$< f, g > = < \widetilde{H}[f], \widetilde{H}[g] > ;$$

• $(\widetilde{H})^2 = \widetilde{H} \circ \widetilde{H} = \mathbb{I}$*;*
• $\widetilde{H}$ *is selfadjoint, i.e.* $(\widetilde{H})^{\text{adj}} = \widetilde{H}$*;*
• $\widetilde{H}$ *is unitary, i.e.* $(\widetilde{H})^{\text{adj}}\widetilde{H} = \widetilde{H}(\widetilde{H})^{\text{adj}} = \mathbb{I}$*;*
• *the Fourier spectrum of* $\widetilde{H}$ *is given by*

$$\mathcal{F}\Big[\widetilde{H}[f]\Big](\underline{U}) = i \frac{\underline{U}| \, \underline{U}}{|\underline{U}|^2} \mathcal{F}[f](\underline{U}) \, .$$

Next, we introduce the projection operators

$$\mathbb{P}_h^{\pm} = \frac{1}{2}(\mathbb{I} \pm \widetilde{H})$$

for which, apart from $(\mathbb{P}_h^{\pm})^2 = \mathbb{P}_h^{\pm}$, it also holds that $\mathbb{P}_h^{+} + \mathbb{P}_h^{-} = \mathbb{I}$ and $\mathbb{P}_h^{+} \circ \mathbb{P}_h^{-} = \mathbb{P}_h^{-} \circ \mathbb{P}_h^{+} = 0$. This leads to the direct sum decomposition

$$\begin{aligned} L_2(\mathbb{R}^{2n}) &= \mathbb{P}_h^{+}[L_2(\mathbb{R}^{2n})] \oplus \mathbb{P}_h^{-}[L_2(\mathbb{R}^{2n})] \\ g &= \mathbb{P}_h^{+}[g] + \mathbb{P}_h^{-}[g] \, . \end{aligned} \tag{2.7}$$

In frequency space, the decomposition (2.7) of the $L_2(\mathbb{R}^{2n})$ function g reads

$$\begin{aligned} \mathcal{F}[g](\underline{U}) &= \frac{1}{2}\left(1 + i \, \frac{\underline{U}| \, \underline{U}}{|\underline{U}|^2}\right) \mathcal{F}[g](\underline{U}) + \frac{1}{2}\left(1 - i \, \frac{\underline{U}| \, \underline{U}}{|\underline{U}|^2}\right) \mathcal{F}[g](\underline{U}) \\ &= \Psi_h^{+}\mathcal{F}[g](\underline{U}) + \Psi_h^{-}\mathcal{F}[g](\underline{U}) \, , \end{aligned}$$

where we have put

$$\Psi_h^{\pm} = \frac{1}{2}\left(1 \pm i \frac{\underline{U}| \, \underline{U}}{|\underline{U}|^2}\right) \, .$$

The above introduced functions $\Psi_h^{\pm}$ are selfadjoint mutually orthogonal idempotents and can be regarded as a Hermitian analogue of the so–called Clifford–Heaviside functions of orthogonal Clifford analysis (see [17] and [21]).

Lemma 2.2. *The functions* $\Psi_h^{\pm}$ *show the following properties*
• $\Psi_h^{+} + \Psi_h^{-} = 1$*;*
• $(\Psi_h^{\pm})^{\dagger} = \Psi_h^{\pm}$*;*
• $\Psi_h^{+}\Psi_h^{-} = \Psi_h^{-}\Psi_h^{+} = 0$*;*
• $(\Psi_h^{\pm})^2 = \Psi_h^{\pm}$.

3 The Hermitian Clifford–Hermite polynomials

The so–called radial Clifford–Hermite polynomials were introduced by Sommen in [20] as a multidimensional generalization to orthogonal Clifford analysis of the classical Hermite polynomials on the real line. They are defined by means of the Rodrigues formula:

$$H_\ell(\underline{x}) = (-1)^\ell \exp\big(\frac{|\underline{x}|^2}{2}\big) \, \partial_{\underline{x}}^\ell[\exp\big(-\frac{|\underline{x}|^2}{2}\big)] \quad , \quad \ell = 0, 1, 2, \ldots$$

and are orthogonal on $\mathbb{R}^{2n}$ with respect to the exponential weight function $\exp(-|\underline{x}|^2/2)$. For their generalization to the Hermitian setting we restrict ourselves to the basic results; for a detailed account, we refer the reader to [4, 5]. Instead of the single operator $\partial_{\underline{x}}$, we now have two Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$, whence it is natural to consider the following Rodrigues formula for the Hermitian Clifford–Hermite polynomials H_p:

$$H_p(\underline{z}, \underline{z}^\dagger) = \exp\left(\frac{|\underline{z}|^2}{2}\right) D_p(\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger)[\exp\left(-\frac{|\underline{z}|^2}{2}\right)],$$

where $D_p(\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger)$ is a differential operator of order p, consisting of p factors $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$. Taking into account the properties (2.2) and (2.3), it is easily seen that the proposed form of D_p results into four types of differential operators, viz two mutually adjoint types of odd order, given by

$$D^{(1)}_{2p+1} = \partial_{\underline{z}}^\dagger \, \Delta^p_{2n} \quad \text{and} \quad D^{(2)}_{2p+1} = \partial_{\underline{z}} \, \Delta^p_{2n}$$

and two selfadjoint types of even order, given by

$$D^{(3)}_{2p+2} = \partial_{\underline{z}}\partial_{\underline{z}}^\dagger \Delta^p_{2n} \quad \text{and} \quad D^{(4)}_{2p+2} = \partial_{\underline{z}}^\dagger \partial_{\underline{z}} \Delta^p_{2n}\,.$$

Hence we are led to four types of Hermitian Clifford–Hermite polynomials, which may be expressed in terms of the Laguerre polynomials on the real line as

$$\begin{aligned} H^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger) &= (-1)^{p-1} 2^{p-1} p! \; \underline{z} \; L^n_p\left(\frac{|\underline{z}|^2}{2}\right) \\ H^{(2)}_{2p+1}(\underline{z}, \underline{z}^\dagger) &= (-1)^{p-1} 2^{p-1} p! \; \underline{z}^\dagger \; L^n_p\left(\frac{|\underline{z}|^2}{2}\right) \end{aligned}$$

$$\begin{aligned} H^{(3)}_{2p+2}(\underline{z}, \underline{z}^\dagger) &= (-1)^{p-1} 2^{p-1} p! \left(\beta \; L^n_p\left(\frac{|\underline{z}|^2}{2}\right) - \frac{1}{2} \, \underline{z}^\dagger \underline{z} \; L^{n+1}_p\left(\frac{|\underline{z}|^2}{2}\right) \right) \qquad (3.1) \\ H^{(4)}_{2p+2}(\underline{z}, \underline{z}^\dagger) &= (-1)^{p-1} 2^{p-1} p! \left((n-\beta) \; L^n_p\left(\frac{|\underline{z}|^2}{2}\right) - \frac{1}{2} \, \underline{z}\underline{z}^\dagger \; L^{n+1}_p\left(\frac{|\underline{z}|^2}{2}\right) \right), \end{aligned}$$

where β denotes the Clifford number $\beta = \sum_{j=1}^n \mathfrak{f}_j^\dagger \mathfrak{f}_j$. With respect to the Gaussian weight $\exp(-|\underline{z}|^2/2)$ all Hermitian Clifford–Hermite polynomials are found to be mutually orthogonal in $L_2(\mathbb{R}^{2n})$, i.e. for arbitrary degrees k, ℓ and indices $i, j = 1, 2, 3, 4$ they satisfy

$$\int_{\mathbb{R}^{2n}} \exp\left(-\frac{|\underline{z}|^2}{2}\right) \left(H^{(i)}_k(\underline{z}, \underline{z}^\dagger) \right)^\dagger \; H^{(j)}_\ell(\underline{z}, \underline{z}^\dagger) \; dV(\underline{X}) = 0 \quad , \quad \text{with } k \neq \ell \text{ when } i = j\,.$$

4 The Hermitian Clifford–Hermite wavelet kernels of the first type

Following the construction of the four types of Hermitian Clifford–Hermite polynomials in the previous section, also four different families of wavelet kernels with their respective mother wavelets may be introduced. For the first type, this mother wavelet function is

$$\psi^{(1)}_{2p+1}(\underline{z},\underline{z}^\dagger) = \partial^\dagger_{\underline{z}}\Delta^p_m\left[\exp\left(-\frac{|\underline{z}|^2}{2}\right)\right] = H^{(1)}_{2p+1}(\underline{z},\underline{z}^\dagger)\ \exp\left(-\frac{|\underline{z}|^2}{2}\right).$$

In this section and the next one, we will study the family of wavelets and the CWT stemming from $\psi^{(1)}_{2p+1}$. The other three types will be discussed briefly in the last section. In [3] it is verified that the $L_1 \cap L_2$–functions $\psi^{(1)}_{2p+1}$ have zero momentum, i.e.

$$\int_{\mathbb{R}^{2n}} \psi^{(1)}_{2p+1}(\underline{z},\underline{z}^\dagger)\ dV(\underline{X}) = \int_{\mathbb{R}^{2n}} \exp\left(-\frac{|\underline{z}|^2}{2}\right)\ H^{(1)}_{2p+1}(\underline{z},\underline{z}^\dagger)\ dV(\underline{X}) = 0\,. \tag{4.1}$$

So they are good candidates for mother wavelets in $\mathbb{R}^{2n}$, if at least they satisfy an appropriate admissibility condition, an issue which will be treated below. To that end, we already calculate the mother wavelet $\psi^{(1)}_{2p+1}$ in frequency space. Taking into account the decomposition (2.3) of the Laplace operator, it is easily seen that

$$\psi^{(1)}_{2p+1}(\underline{z},\underline{z}^\dagger) = 4^p\ \partial^\dagger_{\underline{z}}(\partial_{\underline{z}}\partial^\dagger_{\underline{z}})^p\left[\exp\left(-\frac{|\underline{z}|^2}{2}\right)\right].$$

Hence, by means of the differentiation rule (2.5) we obtain (see also [3])

$$\mathcal{F}[\psi^{(1)}_{2p+1}(\underline{z},\underline{z}^\dagger)](\underline{w},\underline{w}^\dagger) = (-1)^p\frac{i}{2}\ \underline{w}|\underline{w}|^{2p}\ \exp\left(-\frac{|\underline{w}|^2}{2}\right). \tag{4.2}$$

The mother wavelet should also show a number of vanishing moments, in order to filter out polynomial behaviour. By means of the orthogonality relations of the previous section and the zero momentum condition (4.1), one can prove that (see [3]):

$$\int_{\mathbb{R}^{2n}} P_q(\underline{z}-\underline{z}^\dagger)\ \psi^{(1)}_{2p+1}(\underline{z},\underline{z}^\dagger)\ dV(\underline{X}) = 0 \quad \text{if} \quad q < 2p+1\,.$$

Here P_q is a polynomial of degree q in the variable $\underline{X}$ or equivalently in $\underline{z}-\underline{z}^\dagger$ which may in particular be replaced by either of the functions $(\underline{z}^\dagger\underline{z})^t$, $(\underline{z}\underline{z}^\dagger)^t$, $\underline{z}(\underline{z}^\dagger\underline{z})^s$ or $\underline{z}^\dagger(\underline{z}\underline{z}^\dagger)^s$, $0 \le t \le p$ and $0 \le s < p$, revealing the exact meaning of the term vanishing moments in the Hermitian context.

In [3] a family of wavelet kernels stemming from a mother wavelet $\psi(\underline{z},\underline{z}^\dagger) \in L_1 \cap L_2$ is defined, taking into account not only scaling and translation, but also rotation in space. Starting from the Clifford vector $\underline{X}$ and considering a scaling factor $a > 0$ and a translation vector $\underline{B} \in \mathbb{R}^{2n}$, the corresponding operations are transferred to the Hermitian setting by

$$\frac{\underline{X}-\underline{B}}{a} = \frac{\underline{z}-\underline{b}}{a} - \frac{\underline{z}^\dagger-\underline{b}^\dagger}{a}\,,$$

when $\underline{B} = \underline{b} - \underline{b}^\dagger$. For the rotations, we consider spin elements from the unitary subgroup $\widetilde{\mathrm{U}}(n)$ of Spin$(2n)$, see (2.4), and the h–transformation associated to them, viz

$$h(s) : a \in \mathbb{C}_{2n} \to h(s)[a] = sas^\dagger = sa\overline{s} = sas^{-1}\,,$$

leaving the k–blades of the Grassmann algebras $\mathbb{C}\Lambda_n$ and $\mathbb{C}\Lambda_n^\dagger$ invariant. The corresponding operator action on functions, given by $H(s)[g(\underline{X})] = sg(\overline{s}\underline{X}s)\overline{s}$ thus takes the form

$$H(s)[g(\underline{z}, \underline{z}^\dagger)] = s\; g(\overline{s}\underline{z}s, \overline{s}\underline{z}^\dagger s)\; \overline{s}\,,$$

whence the family of wavelet kernels originating from $\psi(\underline{z}, \underline{z}^\dagger)$ is eventually defined as

$$\psi^{a,\underline{b},s}(\underline{z}, \underline{z}^\dagger) = \frac{1}{a^n}\; s\; \psi\left(\frac{\overline{s}(\underline{z} - \underline{b})s}{a}, \frac{\overline{s}(\underline{z}^\dagger - \underline{b}^\dagger)s}{a}\right)\; \overline{s}\,,$$

a being a positive real number, $\underline{b}$ a vector from the Grassmann algebra $\mathbb{C}\Lambda_n$ and s a spin element belonging to the group $\widetilde{\mathrm{U}}(n)$. Next, invoking the basic calculation rules of the Fourier transform for scaling, translation and rotation, the Fourier transforms of these wavelet kernels are easily found to be

$$\mathcal{F}[\psi^{a,\underline{b},s}](\underline{w}, \underline{w}^\dagger) = a^n \exp\left(-2i\mathrm{Re}(\underline{w}, \underline{b})\right)\; s\; \mathcal{F}[\psi](a\overline{s}\underline{w}s, a\overline{s}\underline{w}^\dagger s)\; \overline{s}\,. \tag{4.3}$$

Returning to the Hermitian Clifford–Hermite mother wavelets of the first type, we observe that $s\;\psi^{(1)}_{2p+1}(\overline{s}\underline{z}s, \overline{s}\underline{z}^\dagger s)\;\overline{s} = \psi^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger)$ for any $s \in$ Spin$(2n)$, showing that, in particular, these mother wavelets are invariant under the action of the unitary group $\widetilde{\mathrm{U}}(n)$. Hence, in this case, we may omit this group action while defining the continuous family of wavelets:

$${\psi^{(1)}_{2p+1}}^{a,\underline{b}}(\underline{z}, \underline{z}^\dagger) = \frac{1}{a^n}\; \psi^{(1)}_{2p+1}\left(\frac{\underline{z} - \underline{b}}{a}, \frac{\underline{z}^\dagger - \underline{b}^\dagger}{a}\right)\,, \tag{4.4}$$

where $a \in \mathbb{R}_+$ and $\underline{b} \in \mathbb{C}\Lambda_n \cap \mathbb{C}^{(1)}_{2n}$.

5 The Hermitian Clifford–Hermite CWT of the first type

In this section, we will use the family of functions (4.4) as kernel functions for a new multidimensional CWT. To this end, take $g \in L_2(\mathbb{R}^{2n})$ and define its Hermitian Clifford–Hermite CWT of the first type by:

$$T_{(1)}[g](a, \underline{b}) = \int_{\mathbb{R}^{2n}} \left({\psi^{(1)}_{2p+1}}^{a,\underline{b}}(\underline{z}, \underline{z}^\dagger)\right)^\dagger\; g(\underline{z}, \underline{z}^\dagger)\; dV(\underline{X})\,. \tag{5.1}$$

In the sequel we will show that all types of Hermitian Clifford–Hermite wavelet transforms take their values in the weighted L_2–space

$$L_2\Big(\mathbb{R}_+ \times \mathbb{C}\Lambda_n \cap \mathbb{C}^{(1)}_{2n}, a^{-(2n+1)}da(-2i)^{-n}(d\underline{b} \wedge d\underline{b}^\dagger)\Big)\,,$$

equipped with the Clifford algebra valued inner product

$$[F, G] = \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n} \int_0^{+\infty} \Big(F(a, \underline{b})\Big)^\dagger\; G(a, \underline{b})\; \frac{da}{a^{2n+1}}(-2i)^{-n}(d\underline{b} \wedge d\underline{b}^\dagger)$$

and corresponding norm

$$\|F\| = \left(\Big[[F,F]\Big]_0\right)^{1/2} = \left(\int_{\mathbb{R}^{2n}\cong\mathbb{C}^n}\int_0^{+\infty} |F(a,\underline{b})|^2 \, \frac{da}{a^{2n+1}}(-2i)^{-n}(d\underline{b}\wedge d\underline{b}^\dagger)\right)^{1/2} .$$

By means of (2.6) and (4.3), (5.1) can be rewritten in frequency space as

$$T_{(1)}[g](a,\underline{b}) = a^n \int_{\mathbb{R}^{2n}} \exp\left(2i\mathrm{Re}(\underline{w},\underline{b})\right) \left(\mathcal{F}[\psi^{(1)}_{2p+1}](a\underline{w},a\underline{w}^\dagger)\right)^\dagger \mathcal{F}[g](\underline{w},\underline{w}^\dagger)\, dV(\underline{U})\,. \tag{5.2}$$

Moreover, (4.2) yields

$$\left(\mathcal{F}[\psi^{(1)}_{2p+1}](a\underline{w},a\underline{w}^\dagger)\right)^\dagger = (-1)^{p+1}\frac{i}{2}a^{2p+1}|\underline{w}|^{2p} \, \exp\left(-\frac{a^2|\underline{w}|^2}{2}\right) \underline{w}^\dagger\,.$$

Hence, (5.2) becomes

$$T_{(1)}[g](a,\underline{b}) = (-1)^{p+1}\frac{i}{2}a^{n+2p+1}\int_{\mathbb{R}^{2n}} \exp\left(2i\mathrm{Re}(\underline{w},\underline{b})\right) |\underline{w}|^{2p} \, \exp\left(-\frac{a^2|\underline{w}|^2}{2}\right)$$
$$\underline{w}^\dagger \, \mathcal{F}[g](\underline{w},\underline{w}^\dagger)\, dV(\underline{U})\,.$$

Next, let us decompose $g \in L_2(\mathbb{R}^{2n})$ as $g = \mathbb{P}^+_h[g] + \mathbb{P}^-_h[g]$ by means of the projection operators introduced in Section 2. In what follows, we shortly denote $g^\pm = \mathbb{P}^\pm_h[g]$. We then know that $\mathcal{F}[g^\pm](\underline{U}) = \Psi^\pm_h \, \mathcal{F}[g](\underline{U})$ or

$$\mathcal{F}[g^+](\underline{w},\underline{w}^\dagger) = \frac{\underline{w}^\dagger\underline{w}}{|\underline{w}|^2}\,\mathcal{F}[g](\underline{w},\underline{w}^\dagger) \quad \text{and} \quad \mathcal{F}[g^-](\underline{w},\underline{w}^\dagger) = \frac{\underline{w}\,\underline{w}^\dagger}{|\underline{w}|^2}\,\mathcal{F}[g](\underline{w},\underline{w}^\dagger)\,,$$

where we have rewritten the idempotents $\Psi^\pm_h$ in the Hermitian variables $\underline{w}$ and $\underline{w}^\dagger$:

$$\Psi^+_h = \frac{\underline{w}^\dagger\underline{w}}{|\underline{w}|^2} \quad \text{and} \quad \Psi^-_h = \frac{\underline{w}\,\underline{w}^\dagger}{|\underline{w}|^2}\,.$$

As $(\underline{w}^\dagger)^2 = 0$, we thus obtain that

$$T_{(1)}[g^+](a,\underline{b}) = 0\,,$$

showing that the Hermitian Clifford–Hermite CWT $T_{(1)}$ has a non–trivial kernel, since any function belonging to the L_2–subspace

$$\mathcal{H}^+ := \{g \in L_2(\mathbb{R}^{2n}) \mid g = \mathbb{P}^+_h[g]\}$$

is mapped to zero. In view of the above, it thus is sufficient to study the Hermitian Clifford–Hermite CWT of the first type acting on g^-, since $T_{(1)}[g](a,\underline{b}) = T_{(1)}[g^-](a,\underline{b})$. We put

$$G^-(a,\underline{b}) = < {\psi^{(1)}_{2p+1}}^{a,\underline{b}}, g^- > \,.$$

In frequency space, this takes the form (see (5.2))

$$G^-(a,\underline{b}) = a^n\,(2\pi)^n\,\mathcal{F}\left[\left(\mathcal{F}[\psi^{(1)}_{2p+1}](a\underline{w},a\underline{w}^\dagger)\right)^\dagger \mathcal{F}[g^-](\underline{w},\underline{w}^\dagger)\right](-\underline{b},-\underline{b}^\dagger)\,. \tag{5.3}$$

Now our aim is to prove that the Hermitian Clifford–Hermite CWT $T_{(1)}$ is a bounded linear operator from the L_2–subspace

$$\mathcal{H}^- := \{g \in L_2(\mathbb{R}^{2n}) \mid g = \mathbb{P}_h^-[g]\}$$

to $L_2\Big(\mathbb{R}_+ \times \mathbb{C}\Lambda_n \cap \mathbb{C}_{2n}^{(1)}, a^{-(2n+1)}\ da\ (-2i)^{-n}\ (d\underline{b} \wedge d\underline{b}^\dagger)\Big)$. To that end we calculate

$$[G_1^-, G_2^-] = \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n} \int_0^{+\infty} \Big(G_1^-(a,\underline{b})\Big)^\dagger\ G_2^-(a,\underline{b})\ \frac{da}{a^{2n+1}}\ (-2i)^{-n}\ (d\underline{b} \wedge d\underline{b}^\dagger)\,,$$

where G_1^- and G_2^- denote the images of two signals $g_1^-, g_2^- \in \mathcal{H}^-$. In view of (5.3) and the Parseval formula (2.6), we obtain

$$[G_1^-, G_2^-] = (2\pi)^{2n}(-2i)^{-n} \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n} \Big(\mathcal{F}[g_1^-](\underline{w},\underline{w}^\dagger)\Big)^\dagger$$
$$\left(\int_0^{+\infty} \mathcal{F}[\psi_{2p+1}^{(1)}](a\underline{w}, a\underline{w}^\dagger)\Big(\mathcal{F}[\psi_{2p+1}^{(1)}](a\underline{w}, a\underline{w}^\dagger)\Big)^\dagger \frac{da}{a}\right) \mathcal{F}[g_2^-](\underline{w},\underline{w}^\dagger)\,(d\underline{w} \wedge d\underline{w}^\dagger)\,. \quad (5.4)$$

By means of the substitution $\underline{w} = \frac{r}{a}\,\underline{\xi}$, $|\underline{\xi}| = 1$, or in other words: $|\underline{w}| = \frac{r}{a}$ and $\underline{\xi} = \frac{\underline{w}}{|\underline{w}|}$, the integral between brackets becomes

$$\int_0^{+\infty} \mathcal{F}[\psi_{2p+1}^{(1)}](a\underline{w}, a\underline{w}^\dagger)\ \Big(\mathcal{F}[\psi_{2p+1}^{(1)}](a\underline{w}, a\underline{w}^\dagger)\Big)^\dagger\ \frac{da}{a}$$
$$= \int_0^{+\infty} \mathcal{F}[\psi_{2p+1}^{(1)}](r\underline{\xi}, r\underline{\xi}^\dagger)\ \Big(\mathcal{F}[\psi_{2p+1}^{(1)}](r\underline{\xi}, r\underline{\xi}^\dagger)\Big)^\dagger\ \frac{dr}{r}\,.$$

Next, using expression (4.2) we find

$$\mathcal{F}[\psi_{2p+1}^{(1)}](r\underline{\xi}, r\underline{\xi}^\dagger)\ \Big(\mathcal{F}[\psi_{2p+1}^{(1)}](r\underline{\xi}, r\underline{\xi}^\dagger)\Big)^\dagger = \frac{r^{4p+2}}{4}\ \exp(-r^2)\ \underline{\xi}\ \underline{\xi}^\dagger \quad ,$$

yielding first

$$\int_0^{+\infty} \mathcal{F}[\psi_{2p+1}^{(1)}](a\underline{w}, a\underline{w}^\dagger)\ \Big(\mathcal{F}[\psi_{2p+1}^{(1)}](a\underline{w}, a\underline{w}^\dagger)\Big)^\dagger\ \frac{da}{a} = \frac{(2p)!}{8}\ \Psi_h^-$$

and next

$$[G_1^-, G_2^-] = (2\pi)^{2n}\,(-2i)^{-n}\,\frac{(2p)!}{8} \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n} \Big(\mathcal{F}[g_1^-](\underline{w},\underline{w}^\dagger)\Big)^\dagger\ \Psi_h^-\ \mathcal{F}[g_2^-](\underline{w},\underline{w}^\dagger)\,(d\underline{w} \wedge d\underline{w}^\dagger)$$

for the integral (5.4). Moreover, having $\Psi_h^- \mathcal{F}[g_2^-](\underline{w},\underline{w}^\dagger) = \mathcal{F}[g_2^-](\underline{w},\underline{w}^\dagger)$, we finally obtain that

$$[G_1^-, G_2^-] = C_{(1)}\ \ < g_1^-, g_2^- >\,, \quad (5.5)$$

with $C_{(1)} = (2\pi)^{2n}\ \frac{(2p)!}{8}$, implying the CWT $T_{(1)}$ not only to be bounded but even to be an isometry from $\mathcal{H}^-$ to $L_2\Big(\mathbb{R}_+ \times \mathbb{C}\Lambda_n \cap \mathbb{C}_{2n}^{(1)}, C_{(1)}^{-1} a^{-(2n+1)} da (-2i)^{-n} (d\underline{b} \wedge d\underline{b}^\dagger)\Big)$. From (5.5) we also obtain the reconstruction formula

$$g_2^-(\underline{z},\underline{z}^\dagger) = C_{(1)}^{-1} \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n} \int_0^{+\infty} {\psi_{2p+1}^{(1)}}^{a,\underline{b}}(\underline{z},\underline{z}^\dagger)\ T_{(1)}[g_2^-](a,\underline{b})\ \frac{da}{a^{2n+1}}\ (-2i)^{-n}\ (d\underline{b} \wedge d\underline{b}^\dagger)\,. \quad (5.6)$$

6 The Hermitian Clifford–Hermite CWTs of types two, three and four

6.1 The Hermitian Clifford–Hermite CWT of type two

The Hermitian Clifford–Hermite mother wavelets of type two are derived from the second type of Hermitian Clifford–Hermite polynomials and take the form

$$\psi^{(2)}_{2p+1}(\underline{z},\underline{z}^\dagger) = \exp(-\frac{|\underline{z}|^2}{2})\ H^{(2)}_{2p+1}(\underline{z},\underline{z}^\dagger) = 4^p\ \partial_{\underline{z}}(\partial^\dagger_{\underline{z}}\partial_{\underline{z}})^p \left[\exp(-\frac{|\underline{z}|^2}{2})\right] .$$

Let us first verify that the $L_1 \cap L_2$–functions $\psi^{(2)}_{2p+1}$ have zero momentum. From the form of the generating differential operator $\partial_{\underline{z}}\Delta^p_{2n}$ one infers that

$$H^{(2)}_{2p+1}(\underline{z},\underline{z}^\dagger) = (-1)^p\ (-\frac{1}{2}\underline{z}^\dagger)\ \widetilde{H}_{2p}(r)$$

where $\widetilde{H}_{2p}(r)$ is a scalar polynomial of degree p in $r^2 = |\underline{z}|^2$. Passing to spherical co–ordinates $\underline{z} = r\ \underline{\Xi}$, with $\underline{\Xi} \in S^{2n-1}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \psi^{(2)}_{2p+1}(\underline{z},\underline{z}^\dagger)\ dV(\underline{X}) &= \frac{(-1)^{p+1}}{2} \int_0^{+\infty} r^{2n} \exp(-\frac{r^2}{2})\, \widetilde{H}_{2p}(r)\, dr \int_{S^{2n-1}} \underline{\Xi}^\dagger\, dS(\underline{\Omega}) \\ &= 0\,,\end{aligned}$$

the integral over the unit sphere S^{2n-1} vanishing since $\underline{\Xi}^\dagger$ is a spherical harmonic. Along with that, the functions $\psi^{(2)}_{2p+1}$ also show a number of vanishing moments:

$$\int_{\mathbb{R}^{2n}} \{(\underline{z}^\dagger\underline{z})^t\ ,\ (\underline{z}\underline{z}^\dagger)^t\ ,\ \underline{z}(\underline{z}^\dagger\underline{z})^s\ ,\ \underline{z}^\dagger(\underline{z}\underline{z}^\dagger)^s\}\ \psi^{(2)}_{2p+1}(\underline{z},\underline{z}^\dagger)\, dV(\underline{X}) = 0$$

for $0 \leq s < p,\ 0 \leq t \leq p$.
Furthermore, the Fourier transforms of our second type mother wavelets read

$$\mathcal{F}[\psi^{(2)}_{2p+1}](\underline{w},\underline{w}^\dagger) = (-1)^p\ \frac{i}{2}\ \underline{w}^\dagger\ |\underline{w}|^{2p}\ \exp(-\frac{|\underline{w}|^2}{2})\,.$$

Again we do not have to take the unitary group $\widetilde{\mathrm{U}}(n)$ into consideration, since for each $s \in \mathrm{Spin}(2n)$ we have $s\,\psi^{(2)}_{2p+1}(\overline{s}\underline{z}s, \overline{s}\underline{z}^\dagger s)\,\overline{s} = \psi^{(2)}_{2p+1}(\underline{z},\underline{z}^\dagger)$.

The corresponding Hermitian Clifford–Hermite CWT of the second type applies to functions $g \in L_2(\mathbb{R}^{2n})$ through

$$T_{(2)}[g](a,\underline{b}) = \int_{\mathbb{R}^{2n}} \left({\psi^{(2)}_{2p+1}}^{a,\underline{b}}(\underline{z},\underline{z}^\dagger)\right)^\dagger\ g(\underline{z},\underline{z}^\dagger)\ dV(\underline{X}) \quad .$$

Similarly as in Section 5, this Hermitian Clifford–Hermite CWT can be rewritten as

$$T_{(2)}[g](a,\underline{b}) = (-1)^{p+1}\ \frac{i}{2}a^{n+2p+1} \int_{\mathbb{R}^{2n}} \exp(2i\mathrm{Re}(\underline{w},\underline{b}))\ |\underline{w}|^{2p}\ \exp\left(-\frac{a^2|\underline{w}|^2}{2}\right) \underline{w}\ \mathcal{F}[g](\underline{w},\underline{w}^\dagger)\ dV(\underline{U})\,.$$

Decomposing $g \in L_2(\mathbb{R}^{2n})$ as $g = g^+ + g^-$ with $g^\pm = \mathbb{P}_h^\pm[g]$, we now have that

$$T_{(2)}[g^-](a, \underline{b}) = 0 \,,$$

since $(\underline{w})^2 = 0$. Hence, we obtain that $T_{(2)}[g](a, \underline{b}) = T_{(2)}[g^+](a, \underline{b})$ and we denote

$$G^+(a, \underline{b}) = < {\psi_{2p+1}^{(2)}}^{a,\underline{b}} \;,\; g^+ > \,.$$

In order to prove the boundedness of the Hermitian Clifford–Hermite CWT $T_{(2)}$ as a linear operator from the L_2–subspace $\mathcal{H}^+$ to $L_2\Big(\mathbb{R}_+ \times \mathbb{C}\Lambda_n \cap \mathbb{C}_{2n}^{(1)}, a^{-(2n+1)}\, da(-2i)^{-n}$ $(d\underline{b} \wedge d\underline{b}^\dagger)\Big)$, we calculate

$$[G_1^+, G_2^+] = \int_{\mathbb{R}^{2n} \cong \mathbb{C}^n} \int_0^{+\infty} \Big(G_1^+(a, \underline{b})\Big)^\dagger \; G_2^+(a, \underline{b}) \; \frac{da}{a^{2n+1}} \; (-2i)^{-n} \; (d\underline{b} \wedge d\underline{b}^\dagger)$$

with G_1^+ and G_2^+ the images of two signals $g_1^+, g_2^+ \in \mathcal{H}^+$. Similarly as in Section 5, we obtain

$$[G_1^+, G_2^+] = (2\pi)^{2n}(-2i)^{-n}\frac{(2p)!}{8} \int_{\mathbb{R}^{2n} \cong \mathbb{C}^n} \Big(\mathcal{F}[g_1^+](\underline{w}, \underline{w}^\dagger)\Big)^\dagger \; \Psi_h^+ \; \mathcal{F}[g_2^+](\underline{w}, \underline{w}^\dagger)\, (d\underline{w} \wedge d\underline{w}^\dagger) \,. \tag{6.1}$$

As $\Psi_h^+ \; \mathcal{F}[g_2^+](\underline{w}, \underline{w}^\dagger) = \mathcal{F}[g_2^+](\underline{w}, \underline{w}^\dagger)$, the integral (6.1) becomes

$$[G_1^+, G_2^+] = C_{(1)} \;\; < g_1^+, g_2^+ >$$

showing also the second type Hermitian Clifford–Hermite CWT to constitute an isometry, now from $\mathcal{H}^+$ to $L_2\Big(\mathbb{R}_+ \times \mathbb{C}\Lambda_n \cap \mathbb{C}_{2n}^{(1)}, C_{(1)}^{-1}\, a^{-(2n+1)}\, da\, (-2i)^{-n}(d\underline{b} \wedge d\underline{b}^\dagger)\Big)$. Moreover, the reconstruction formula takes the form

$$g_2^+(\underline{z}, \underline{z}^\dagger) = \frac{1}{C_{(1)}} \int_{\mathbb{R}^{2n} \cong \mathbb{C}^n} \int_0^{+\infty} {\psi_{2p+1}^{(2)}}^{a,\underline{b}}(\underline{z}, \underline{z}^\dagger)\; T_{(2)}[g_2^+](a, \underline{b}) \; \frac{da}{a^{2n+1}}(-2i)^{-n}(d\underline{b} \wedge d\underline{b}^\dagger) \,.$$

6.2 The Hermitian Clifford–Hermite CWT of type three

In this subsection we study the CWT stemming from the Hermitian Clifford–Hermite mother wavelets of type three given by

$$\psi_{2p+2}^{(3)}(\underline{z}, \underline{z}^\dagger) = \exp\,(-\frac{|\underline{z}|^2}{2})\; H_{2p+2}^{(3)}(\underline{z}, \underline{z}^\dagger) \;\; = 4^p(\partial_{\underline{z}} \partial_{\underline{z}}^\dagger)^{p+1} \left[\exp\,(-\frac{|\underline{z}|^2}{2})\right] \,.$$

First we must verify that this mother wavelet of type three has zero momentum. As for the mother wavelets of types one and two, this condition has to be carefully checked, its proof however now being less straightforward than for the previous types. From (3.1) we obtain the following expression for $H_{2p+2}^{(3)}$:

$$H_{2p+2}^{(3)}(\underline{z}, \underline{z}^\dagger) = \underline{z}^\dagger \underline{z}\; K_{2p}(r) + (-1)^{p+1}\frac{1}{2}\beta\; \widetilde{H}_{2p}(r)$$

with $K_{2p}(r) = (-1)^p 2^{p-2} p!\; L_p^{n+1}(\frac{r^2}{2})$ and $\widetilde{H}_{2p}(r) = 2^p p!\; L_p^n(\frac{r^2}{2})$. Hence we find

$$\int_{\mathbb{R}^{2n}} \psi_{2p+2}^{(3)}(\underline{z}, \underline{z}^\dagger)\; dV(\underline{X}) = \int_0^{+\infty} \exp\,(-\frac{r^2}{2})\, r^{2n+1}\, K_{2p}(r)\, dr \left(\int_{S^{2n-1}} \Xi^\dagger \; \Xi \, dS(\underline{\Omega})\right)$$
$$+ (-1)^{p+1}\frac{1}{2}\,\beta\, a_{2n} \int_0^{+\infty} \exp\,(-\frac{r^2}{2})\, r^{2n-1}\, \widetilde{H}_{2p}(r)\, dr \,,$$

where we have again used spherical co–ordinates $\underline{z} = r\underline{\Xi}$, $\underline{\Xi} \in S^{2n-1}$. From the orthogonality of the radial Clifford–Hermite polynomials of orthogonal Clifford analysis, we know:

$$\int_{\mathbb{R}^{2n}} \exp\left(-\frac{r^2}{2}\right) H_{2p+2}(\underline{X})\, dV(\underline{X}) = 0 \quad , \quad p = 0,1,2,\ldots$$

As one can verify that

$$H_{2p+2}(\underline{X}) = (-1)^{p+1} 4 \left(r^2\, K_{2p}(r) + (-1)^{p+1}\, \frac{n}{2}\, \widetilde{H}_{2p}(r) \right),$$

this implies that for $p = 0,1,2,\ldots$

$$\int_{\mathbb{R}^{2n}} \exp\left(-\frac{r^2}{2}\right) \left(r^2\ K_{2p}(r) + (-1)^{p+1}\ \frac{n}{2}\ \widetilde{H}_{2p}(r) \right) dV(\underline{X}) = 0\,.$$

The whole integrand being a purely radial function, we finally obtain for $p = 0,1,2,\ldots$

$$\int_0^{+\infty} \exp\left(-\frac{r^2}{2}\right) r^{2n+1}\, K_{2p}(r)\, dr = (-1)^p\ \frac{n}{2} \int_0^{+\infty} \exp\left(-\frac{r^2}{2}\right) r^{2n-1}\, \widetilde{H}_{2p}(r)\, dr\,.$$

The above result eventually leads to

$$\int_{\mathbb{R}^{2n}} \psi^{(3)}_{2p+2}(\underline{z}, \underline{z}^\dagger)\, dV(\underline{X}) = (-1)^p \frac{n}{2} \int_0^{+\infty} \exp\left(-\frac{r^2}{2}\right) r^{2n-1}\, \widetilde{H}_{2p}(r)\, dr \left(\int_{S^{2n-1}} \underline{\Xi}^\dagger \underline{\Xi}\, dS(\underline{\Omega}) - \frac{\beta}{n}\, a_{2n} \right) = 0\,,$$

since a subtle calculation yields (see [4])

$$\int_{S^{2n-1}} \underline{\Xi}^\dagger\ \underline{\Xi}\ dS(\underline{\Omega}) = \frac{\beta}{n}\ a_{2n}\,.$$

The zero momentum condition combined with the orthogonality of the Hermitian Clifford–Hermite polynomials again gives rise to a number of vanishing moments, i.e.

$$\int_{\mathbb{R}^{2n}} \{ (\underline{z}^\dagger \underline{z})^t\,,\ (\underline{z}\underline{z}^\dagger)^t\,,\ \underline{z}(\underline{z}^\dagger \underline{z})^s\,,\ \underline{z}^\dagger (\underline{z}\underline{z}^\dagger)^s \}\, \psi^{(3)}_{2p+2}(\underline{z}, \underline{z}^\dagger)\, dV(\underline{X}) = 0$$

for $0 \leq t \leq p$, $0 \leq s \leq p$.
The Fourier transforms of the mother wavelets take the form

$$\mathcal{F}[\psi^{(3)}_{2p+2}](\underline{w}, \underline{w}^\dagger) = (-1)^{p+1} \frac{1}{4} (\underline{w}^\dagger \underline{w}) |\underline{w}|^{2p}\ \exp\left(-\frac{|\underline{w}|^2}{2}\right).$$

Moreover, also the mother wavelets of type three are $\widetilde{\mathrm{U}}(n)$ invariant, yet another result which is not as straightforward as for the mother wavelets of types one and two.

Lemma 6.1. *The Clifford number $\beta = \sum_{j=1}^n \mathfrak{f}_j^\dagger\ \mathfrak{f}_j$ commutes with any $s \in \widetilde{\mathrm{U}}(n)$.*

PROOF. Any $s \in \widetilde{\mathrm{U}}(n)$ may be written as $s = \exp(\epsilon\sigma)$ with $\epsilon \in \mathbb{R}$ small and σ an element of the associated Lie algebra $\tilde{u}(n)$. In [6] it is proved that this Lie algebra $\tilde{u}(n)$ is generated by the following (real) bivectors

$$\begin{array}{rcl} i\mathfrak{f}_j \wedge \mathfrak{f}_j^\dagger & , & j = 1, \ldots, n \\ \mathfrak{f}_j\mathfrak{f}_k^\dagger - \mathfrak{f}_k\mathfrak{f}_j^\dagger & , & j,k = 1, \ldots, n\,,\ j \neq k \\ i(\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k\mathfrak{f}_j^\dagger) & , & j,k = 1, \ldots, n\,,\ j \neq k\,. \end{array}$$

By a lengthy but straightforward computation each of these generators may be shown to commute with β, which directly implies the same property for any element σ of the Lie algebra. In particular we then also have that $\sigma^k\beta = \beta\,\sigma^k$, which finally yields

$$s\,\beta = \exp(\epsilon\sigma)\,\beta = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\,\sigma^k\,\beta = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\,\beta\,\sigma^k = \beta\,\exp(\epsilon\sigma) = \beta\,s\,. \qquad \square$$

The above result is crucial, since, invoking also (3.1), we now easily find

$$s\,\psi_{2p+2}^{(3)}(\overline{s}\underline{z}s, \overline{s}\underline{z}^\dagger s)\,\overline{s} = \psi_{2p+2}^{(3)}(\underline{z}, \underline{z}^\dagger)\,, \qquad \forall s \in \widetilde{\mathrm{U}}(n)\,.$$

The corresponding CWT applies to square integrable functions g through

$$T_{(3)}[g](a,\underline{b}) = \int_{\mathbb{R}^{2n}} \left(\psi_{2p+2}^{(3)}{}^{a,\underline{b}}(\underline{z}, \underline{z}^\dagger)\right)^\dagger\, g(\underline{z}, \underline{z}^\dagger)\; dV(\underline{X})\,.$$

The equivalent expression

$$T_{(3)}[g](a,\underline{b}) = \frac{(-1)^{p+1}}{4} a^{n+2p+2} \int_{\mathbb{R}^{2n}} \exp(2i\mathrm{Re}(\underline{w},\underline{b}))\; |\underline{w}|^{2p}\; \exp\left(-\frac{a^2|\underline{w}|^2}{2}\right) (\underline{w}^\dagger\underline{w})\; \mathcal{F}[g](\underline{w}, \underline{w}^\dagger)\; dV(\underline{U})$$

yields

$$T_{(3)}[g^-](a,\underline{b}) = 0\,.$$

Hence $T_{(3)}[g](a,\underline{b}) = T_{(3)}[g^+](a,\underline{b})$, also denoted by $G^+(a,\underline{b})$. We now find that

$$\begin{aligned} [G_1^+, G_2^+] &= \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n} \int_0^{+\infty} \left(G_1^+(a,\underline{b})\right)^\dagger G_2^+(a,\underline{b})\, \frac{da}{a^{2n+1}}\, (-2i)^{-n}\, (d\underline{b} \wedge d\underline{b}^\dagger) \\ &= (2\pi)^{2n}\, (-2i)^{-n} \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n} \left(\mathcal{F}[g_1^+](\underline{w}, \underline{w}^\dagger)\right)^\dagger \left(\int_0^{+\infty} \mathcal{F}[\psi_{2p+2}^{(3)}](r\underline{\xi}, r\underline{\xi}^\dagger)\right. \\ &\qquad \left.\left(\mathcal{F}[\psi_{2p+2}^{(3)}](r\underline{\xi}, r\underline{\xi}^\dagger)\right)^\dagger \frac{dr}{r}\right) \mathcal{F}[g_2^+](\underline{w}, \underline{w}^\dagger)\, (d\underline{w} \wedge d\underline{w}^\dagger)\,, \end{aligned}$$

again with $\underline{\xi} = \frac{\underline{w}}{|\underline{w}|}$. As

$$\mathcal{F}[\psi_{2p+2}^{(3)}](r\underline{\xi}, r\underline{\xi}^\dagger)\, \left(\mathcal{F}[\psi_{2p+2}^{(3)}](r\underline{\xi}, r\underline{\xi}^\dagger)\right)^\dagger = \frac{1}{16}\, r^{4p+4}\, \exp(-r^2)\, (\underline{\xi}^\dagger\underline{\xi})^2\,,$$

the integral between brackets becomes

$$\int_0^{+\infty} \mathcal{F}[\psi_{2p+2}^{(3)}](r\underline{\xi}, r\underline{\xi}^\dagger)\, \left(\mathcal{F}[\psi_{2p+2}^{(3)}](r\underline{\xi}, r\underline{\xi}^\dagger)\right)^\dagger \frac{dr}{r} = \frac{1}{32}(2p+1)!\; P_h^+\,.$$

Hence we obtain that

$$[G_1^+, G_2^+] = C_{(3)} < g_1^+, g_2^+ >$$

with $C_{(3)} = \frac{(2\pi)^{2n}}{32}\,(2p+1)!$, which implies that the Hermitian Clifford–Hermite CWT $T_{(3)}$ is a bounded linear operator from $\mathcal{H}^+$ to $L_2\Big(\mathbb{R}_+ \times \mathbb{C}\Lambda_n \cap \mathbb{C}_{2n}^{(1)}, a^{-(2n+1)}\, da\, (-2i)^{-n}\, (d\underline{b} \wedge d\underline{b}^\dagger)\Big)$, and an isometry from $\mathcal{H}^+$ to $L_2\Big(\mathbb{R}_+ \times \mathbb{C}\Lambda_n \cap \mathbb{C}_{2n}^{(1)}, C_{(3)}^{-1}\, a^{-(2n+1)}\, da\, (-2i)^{-n}\, (d\underline{b} \wedge d\underline{b}^\dagger)\Big)$. Again, a reconstruction formula is obtained as well:

$$g_2^+(\underline{z}, \underline{z}^\dagger) = \frac{1}{C_{(3)}} \int_{\mathbb{R}^{2n} \cong \mathbb{C}^n} \int_0^{+\infty} {\psi_{2p+2}^{(3)}}^{a,\underline{b}}(\underline{z}, \underline{z}^\dagger)\; T_{(3)}[g_2^+](a, \underline{b})\; \frac{da}{a^{2n+1}}\; (-2i)^{-n}\; (d\underline{b} \wedge d\underline{b}^\dagger)\,.$$

6.3 The Hermitian Clifford–Hermite CWT of type four

Finally we consider the CWT stemming from the Hermitian Clifford–Hermite mother wavelets of type four:

$$\psi_{2p+2}^{(4)}(\underline{z}, \underline{z}^\dagger) = \exp\,(-\frac{|\underline{z}|^2}{2})\; H_{2p+2}^{(4)}(\underline{z}, \underline{z}^\dagger) = 4^p(\partial_{\underline{z}}^\dagger \partial_{\underline{z}})^{p+1} \left[\exp\,(-\frac{|\underline{z}|^2}{2})\right].$$

The proof of the zero momentum condition for the mother wavelets of type four runs along similar lines as the one for the mother wavelets of type three (see subsection 6.2). Again the mother wavelets also have a number of vanishing moments:

$$\int_{\mathbb{R}^{2n}} \{(\underline{z}^\dagger \underline{z})^t\,,\ (\underline{z}\underline{z}^\dagger)^t\,,\ \underline{z}(\underline{z}^\dagger \underline{z})^s\,,\ \underline{z}^\dagger(\underline{z}\underline{z}^\dagger)^s\}\; \psi_{2p+2}^{(4)}(\underline{z}, \underline{z}^\dagger)\, dV(\underline{X}) = 0$$

for $0 \le t \le p$, $0 \le s \le p$.
By means of the differentiation rule (2.5), we obtain the following expressions for their Fourier transforms:

$$\mathcal{F}[\psi_{2p+2}^{(4)}](\underline{w}, \underline{w}^\dagger) = (-1)^{p+1}\frac{1}{4}(\underline{w}\underline{w}^\dagger)|\underline{w}|^{2p}\ \exp\,(-\frac{|\underline{w}|^2}{2})\,.$$

Again, the considered mother wavelets are $\widetilde{\mathrm{U}}(n)$ invariant, whence we do not take the group $\widetilde{\mathrm{U}}(n)$ into consideration when defining the Hermitian Clifford–Hermite CWT $T_{(4)}$:

$$T_{(4)}[g](a, \underline{b}) = \int_{\mathbb{R}^{2n}} \left({\psi_{2p+2}^{(4)}}^{a,\underline{b}}(\underline{z}, \underline{z}^\dagger)\right)^\dagger\ g(\underline{z}, \underline{z}^\dagger)\ dV(\underline{X})\,.$$

This CWT can be rewritten as

$$T_{(4)}[g](a, \underline{b}) = \frac{(-1)^{p+1}}{4} a^{n+2p+2} \int_{\mathbb{R}^{2n}} \exp\,(2i\mathrm{Re}(\underline{w}, \underline{b}))\ |\underline{w}|^{2p}\ \exp\left(-\frac{a^2|\underline{w}|^2}{2}\right)$$
$$(\underline{w}\underline{w}^\dagger)\ \mathcal{F}[g](\underline{w}, \underline{w}^\dagger)\ dV(\underline{U})\,,$$

which implies that

$$T_{(4)}[g^+](a, \underline{b}) = 0\,.$$

We thus obtain $T_{(4)}[g](a,\underline{b}) = T_{(4)}[g^-](a,\underline{b})$, denoted as $G^-(a,\underline{b})$. Similarly as in subsection 6.2, we find

$$\begin{aligned}[G_1^-, G_2^-] &= \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n} \int_0^{+\infty} \Big(G_1^-(a,\underline{b})\Big)^\dagger G_2^-(a,\underline{b}) \frac{da}{a^{2n+1}} (-2i)^{-n} (d\underline{b} \wedge d\underline{b}^\dagger) \\ &= C_{(3)} < g_1^-, g_2^- > .\end{aligned}$$

Hence the Hermitian Clifford–Hermite CWT $T_{(4)}$ is a bounded linear operator from $\mathcal{H}^-$ to $L_2\Big(\mathbb{R}_+ \times \mathbb{C}\Lambda_n \cap \mathbb{C}_{2n}^{(1)}, a^{-(2n+1)}\, da\, (-2i)^{-n} (d\underline{b} \wedge d\underline{b}^\dagger)\Big)$ and moreover an isometry from $\mathcal{H}^-$ to $L_2\Big(\mathbb{R}_+ \times \mathbb{C}\Lambda_n \cap \mathbb{C}_{2n}^{(1)}, C_{(3)}^{-1} a^{-(2n+1)}\, da\, (-2i)^{-n} (d\underline{b} \wedge d\underline{b}^\dagger)\Big)$. The Parseval formula also yields the following reconstruction formula:

$$g_2^-(\underline{z}, \underline{z}^\dagger) = \frac{1}{C_{(3)}} \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n} \int_0^{+\infty} \psi_{2p+2}^{(4)\;a,\underline{b}}(\underline{z}, \underline{z}^\dagger)\, T_{(4)}[g_2^-](a,\underline{b}) \frac{da}{a^{2n+1}} (-2i)^{-n} (d\underline{b} \wedge d\underline{b}^\dagger) .$$

7 Final remarks

In [11] the so–called Clifford–Hermite wavelets in orthogonal Clifford analysis were introduced. In even dimension, we may write them as

$$\psi_\ell(\underline{X}) = (-1)^\ell\, \partial_{\underline{X}}^\ell \left[\exp\left(-\frac{|\underline{X}|^2}{2}\right)\right] = H_\ell(\underline{X})\ \exp\left(-\frac{|\underline{X}|^2}{2}\right) .$$

and we may similarly introduce

$$\psi_\ell(\underline{X}|) = (-1)^\ell\, \partial_{\underline{X}|}^\ell \left[\exp\left(-\frac{|\underline{X}|^2}{2}\right)\right] = H_\ell(\underline{X}|) \exp\left(-\frac{|\underline{X}|^2}{2}\right) .$$

Between these Clifford–Hermite wavelets and their Hermitian analogues, the following relations then hold:

$$\begin{aligned}\psi_{2p+1}(\underline{X}) &= (-1)^{p-1} 2 \left(\psi_{2p+1}^{(1)}(\underline{z}, \underline{z}^\dagger) - \psi_{2p+1}^{(2)}(\underline{z}, \underline{z}^\dagger)\right) \\ \psi_{2p+1}(\underline{X}|) &= (-1)^{p-1} \frac{2}{i} \left(\psi_{2p+1}^{(1)}(\underline{z}, \underline{z}^\dagger) + \psi_{2p+1}^{(2)}(\underline{z}, \underline{z}^\dagger)\right) \\ \psi_{2p+2}(\underline{X}) &= \psi_{2p+2}(\underline{X}|) = (-1)^{p-1} 4 \left(\psi_{2p+2}^{(3)}(\underline{z}, \underline{z}^\dagger) + \psi_{2p+2}^{(4)}(\underline{z}, \underline{z}^\dagger)\right) .\end{aligned}$$

Subsequently, also the corresponding Clifford–Hermite CWT is defined, viz

$$\begin{aligned} T_\ell\ :\ L_2(\mathbb{R}^{2n}) &\longrightarrow L_2(\mathbb{R}_+ \times \mathbb{R}^{2n}, C_\ell^{-1} a^{-(2n+1)} da\ dV(\underline{B})) \\ g &\longrightarrow G_\ell(a, \underline{B}) = < \psi_\ell^{a,\underline{B}}, g > = \int_{\mathbb{R}^{2n}} \left(\psi_\ell^{a,\underline{B}}(\underline{X})\right)^\dagger g(\underline{X})\ dV(\underline{X})\end{aligned}$$

the continuous family of wavelets being given by

$$\psi_\ell^{a,\underline{B}}(\underline{X}) = \frac{1}{a^n}\ \psi_\ell \left(\frac{\underline{X} - \underline{B}}{a}\right) , \quad a \in \mathbb{R}_+ , \quad \underline{B} \in \mathbb{R}^{2n} .$$

with admissibility constants $C_\ell = (2\pi)^{2n}\ \frac{(\ell-1)!}{2}$.

First note that the Hermitian Clifford–Hermite CWTs of types one and two can be considered as a refinement of the above Clifford–Hermite CWT of odd order, since

$$G_{2p+1}(a,\underline{B}) = (-1)^{p-1}2\left(T_{(1)}[g^-](a,\underline{b}) - T_{(2)}[g^+](a,\underline{b})\right)$$

with $\underline{B} = \underline{b} - \underline{b}^\dagger$. Furthermore, the Hermitian Clifford–Hermite CWTs of types three and four can be considered as a refinement of the Clifford–Hermite CWT of even order, since

$$G_{2p+2}(a,\underline{B}) = 4(-1)^{p-1}\left(T_{(3)}[g^+](a,\underline{b}) + T_{(4)}[g^-](a,\underline{b})\right) .$$

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