Generalized Keller Spaces

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Abstract

By means of fields of formal power series, we construct definite Banach spaces over (Krull) valued fields for any cofinal type $\geq \omega_0$ of the value group. Among these spaces are all known non-classical form-Hilbert spaces of infinite dimension.

1 Introduction

In 1979 H.Keller [3] constructed the first example of an infinite-dimensional non-classical orthomodular space. By studying his example and generalizations of it for normed vector spaces over (Krull) valued fields (see f.e. [2], [5]), it seemed quite natural to me to construct these spaces as linear subspaces of formal power series fields. The method of construction is very general and includes also normed vector spaces with higher cofinal type of their value set. But for cofinal types $\geq \omega_1$, these spaces are only definite Banach spaces and, as it is known, no form-Hilbert spaces.

2 Fields of formal power series

We recall the definition and some properties of fields of formal power series (for more details see f.e. [1] or [7]). Let $(\Gamma^{\cdot}, \cdot, \leq)$ be a totally ordered abelian group and F a field. Let f be a mapping from $\Gamma^{\cdot}$ to F and $supp(f) = \{\gamma \in \Gamma^{\cdot} \mid f(\gamma) \neq 0\}$. Let $H = H(\Gamma^{\cdot}, F)$ be the set of all $f \in F^{\Gamma^{\cdot}}$, for which $supp(f)$ is dually well-ordered in the order of $\Gamma^{\cdot}$ ("dually well-ordered" means that each non-empty subset has a largest element). By pointwise addition, i.e. $(f+g)(\gamma) = f(\gamma) + g(\gamma)$ for all $\gamma \in \Gamma^{\cdot}$, and with the multiplication, defined by $(fg)(\gamma) = \sum\limits_{\alpha\beta=\gamma} f(\alpha)g(\beta)$, H is a field, the *field of formal power series*. An element $f \in H$ is said to be a *formal power series*.

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The unit element of H is given by $f(\gamma) = 1$, if $\gamma = 1$ and $f(\gamma) = 0$, if $\gamma \neq 1$; we denote it by 1.

Adjoin to $\Gamma^{\cdot}$ an element 0 such that $0 < \gamma$ and $0\gamma = \gamma 0 = 0 = 00$ for all $\gamma \in \Gamma^{\cdot}$. Put $\Gamma = \Gamma^{\cdot} \cup \{0\}$. The mapping $w : H \to \Gamma$, $0 \neq f \longmapsto \max supp(f)$, $0 \longmapsto 0$, is a (Krull) valuation, i.e.

$w(f) = 0 \Longleftrightarrow f = 0$,
$w(f+g) \leq \max\{w(f), w(g)\}$ and
$w(fg) = w(f)w(g)$.

H has with respect to w the value group $\Gamma^{\cdot}$ and the residue field H_w, isomorphic to F. By identifying $a \in F$ with the formal power series $f_a \in H$, where $f_a(\gamma) = a$ for $\gamma = 1$, and $f_a(\gamma) = 0$ for $\gamma \neq 1$, F is a subfield of H.

$d(f,g) = w(f-g)$ defines an ultrametric distance d on H. The set $B^H_\gamma(f) = \{g \in H \mid w(f-g) \leq \gamma\}$, with $\gamma \in \Gamma^{\cdot}$, $f \in H$, is a ball of H. The ultrametric space (H, d, Γ) is spherically complete, i.e. every chain of balls has a non-empty intersection (see Appendix 6.5). In particular, each chain of the form $\left\{B^H_{\gamma_i}(f_i) \mid i \in I\right\}$, with $0 = \inf \gamma_i$ has a non-empty intersection. Thus H is complete (with respect to the topology defined by the valuation).

Let $\Delta^{\cdot} = \Delta \setminus \{0\}$ be a cofinal subgroup of $\Gamma^{\cdot}$. The set $F[\Delta^{\cdot}]$of formal power series of H, for which the support is a finite subset of $\Delta^{\cdot}$, is a subring of H. Hence the quotient field $F(\Delta^{\cdot})$of $F[\Delta^{\cdot}]$ is a subfield of H. Since H is complete, $F(\Delta^{\cdot})$ has its completion K in H. The restriction of w to K is a valuation of K (again denoted by w) with value group $\Delta^{\cdot}$. As K is a subfield of H, the field H is a vector space over K.

3 Construction of the Banach space E

Let Γ, Δ be as above, i.e. $\Delta^{\cdot} = \Delta \setminus \{0\}$ is a cofinal subgroup of the totally ordered abelian group $\Gamma^{\cdot} = \Gamma \setminus \{0\}$. Let (K, w, Δ) be a valued field and E a vector space over K. The mapping $\| \; \| : E \to \Gamma$ is said to be a *norm* of E, when the following conditions are satisfied for all $x \in E$, $k \in K$:

$\| x \| = 0 \Leftrightarrow x = 0$,
$\| x + y \| \leq \max\{\| x \|, \| y \|\}$,
$\| kx \| = w(k) \| x \|$.

The norm $\| \; \| : E \to \Gamma$ does not need to be surjective. We denote by $\| E \|$ the image of E under $\| \; \|$.

With $\| \; \| : H \to \Gamma$, defined by $\| f \| = w(f)$, $(H, \| \; \|, \Gamma)$ is a normed vector space over the complete field (K, w, Δ) from above.

We now specify the groups $\Delta^{\cdot}$,$\Gamma^{\cdot}$ to obtain the generalized Keller space as an appropriate linear subspace of H. If Λ is a totally ordered abelian group and $\overline{\Lambda}$ its totally ordered root-closed hull (in additive notation, the "divisible hull"), then $\sqrt{\Lambda} = \left\{\overline{\lambda} \in \overline{\Lambda} \mid \overline{\lambda}^2 \in \Lambda\right\}$ is a subgroup of $\overline{\Lambda}$ which contains Λ. Let ρ be finite or a limit ordinal. Let $(\Lambda_\xi)_{\xi<\rho}$ be a strictly increasing family (i.e. $\xi < \xi' < \rho$ implies

$\Lambda_\xi < \Lambda_{\xi'}$) of totally ordered abelian groups with the following properties for all $\xi < \rho$:

(g1) Λ_ξ is a convex subgroup of $\Lambda_{\xi'}$ for each ξ', $\xi < \xi' < \rho$.

(g2) $\Lambda_\xi \neq \sqrt{\Lambda_\xi}$.

(g3) There exists $1 < \gamma_\xi \in \sqrt{\Lambda_\xi}$ such that if $\xi = 1$, $\gamma_\xi \notin \Lambda_\xi$, and if $\xi > 1$, $\gamma_\xi\sqrt{\Lambda_\xi^-} \neq \delta\sqrt{\Lambda_\xi^-}$ for all $\delta \in \Lambda_\xi$, with $\Lambda_\xi^- = \bigcup\limits_{\eta<\xi} \Lambda_\eta$.

Such groups Λ_ξ, $\xi < \rho$, do exist. We give an example (with the operation, written additively):

Let $\Sigma = \prod\limits_{\xi<\rho}{}_f \mathbb{Z}$ be the set of mappings from $\{\xi \mid \xi < \rho\}$ to $\mathbb{Z}$ which have finite support (for $\sigma \in \Sigma$, $supp(\sigma) = \{\xi < \rho \mid \sigma(\xi) \neq 0\}$). By pointwise addition, Σ is an abelian group, which is totally ordered with respect to the anti-lexicographic order, i.e. $\sigma < \sigma'$ if and only if $\sigma(\eta) < \sigma'(\eta)$, where $\eta = \max\{\eta' < \rho \mid \sigma(\eta') \neq \sigma'(\eta')\}$. Replacing $\mathbb{Z}$ by $\frac{1}{2}\mathbb{Z}$, we obtain in the same way the totally ordered group $\frac{1}{2}\Sigma = \prod\limits_{\xi<\rho}{}_f \frac{1}{2}\mathbb{Z}$. For each $\xi < \rho$, let $\Lambda_\xi = \{\sigma \in \Sigma \mid \sigma(\eta) = 0 \text{ for all } \eta > \xi\}$ and $\frac{1}{2}\Lambda_\xi = \left\{\sigma \in \frac{1}{2}\Sigma \mid \sigma(\eta) = 0 \text{ for all } \eta > \xi\right\}$. Then the family $(\Lambda_\xi)_{\xi<\rho}$ is strictly increasing and for each $\xi < \rho$, $\Lambda_\xi \neq \frac{1}{2}\Lambda_\xi$ and Λ_ξ is convex in $\Lambda_{\xi'}$ for all ξ', $\xi < \xi' < \rho$. We have $\Lambda_\xi^- = \bigcup\limits_{\eta<\xi} \Lambda_\eta = \{\sigma \in \Sigma \mid \sigma(\eta') = 0 \text{ for all } \eta' \geq \xi\}$ and $\frac{1}{2}\Lambda_\xi^- = \left\{\sigma \in \frac{1}{2}\Sigma \mid \sigma(\eta') = 0 \text{ for all } \eta' \geq \xi\right\}$. Let $\gamma_\xi \in \frac{1}{2}\Lambda_\xi$ be such that $\gamma_\xi(\xi) = \frac{1}{2}$ and $\gamma_\xi(\eta) = 0$ for $\eta \neq \xi$. Then for all $\sigma \in \Lambda_\xi$, $\gamma_\xi + \frac{1}{2}\Lambda_\xi^- \neq \sigma + \frac{1}{2}\Lambda_\xi^-$, since for all $\sigma' \in \frac{1}{2}\Lambda_\xi^-$, $\frac{1}{2} = \gamma_\xi(\xi) \neq \sigma(\xi) + \sigma'(\xi) = \sigma(\xi) \in \mathbb{Z}$. Thus $(\Lambda_\xi)_{\xi<\rho}$ has the properties (g1),(g2),(g3).

Obviously, $\Sigma = \bigcup\limits_{\xi<\rho} \Lambda_\xi$. If $\rho = \omega_0$, it is this group Σ which is used by H.Keller in [3] for the construction of his example of an orthomodular space of infinite dimension.

We continue our construction of the generalized Keller space as a linear subspace of H. Let $\Delta^{\cdot} = \bigcup\limits_{\xi<\rho} \Lambda_\xi$ and $\Gamma^{\cdot} = \bigcup\limits_{\xi<\rho} \sqrt{\Lambda_\xi} = \sqrt{\Delta^{\cdot}}$. For each $\xi < \rho$, define $e_\xi \in H$ by $e_\xi(\gamma_\xi) = 1$ and $e_\xi(\gamma) = 0$ for $\gamma \neq \gamma_\xi$. Put $e_0 = 1$ and $\rho^* = \{\xi \mid \xi < \rho\} \cup \{0\}$. Let L be the linear span of $\{e_\xi \mid \xi \in \rho^*\}$ over K and E its completion in H. (Since H is a complete vector space, H contains a complete vector space E in which L is dense; see Appendix 6.3). E is a Banach space. We denote the norm $\|\ \|$ of H, restricted to E, again by $\|\ \|$.

4 Construction of an inner product for E

We now assume throughout $w(2) = 1$, so char$F \neq 2$. (This assumption is needed for the proof of the "Ultrametric Pythagoras" and the "Cauchy-Schwarz inequality"; see Appendix 6.1). We shall construct a symmetric bilinear form $< -, - >: E \times E \to K$ which is non-degenerate (this means, if for all $y \in E$, $< x, y >= 0$, then $x = 0$). Such a bilinear form is said to be an *inner product.* A normed vector space with ($w(2) = 1$ and) an inner product $< -, - >$ such that the norm of each element x of the vector space is given by $\| x \| = \sqrt{w(< x, x >)}$ is called a *definite space.* We shall show that E is a definite space and prove this first for the linear span L of $\{e_\xi \mid \xi \in \rho^*\}$.

By choice of the elements e_ξ, $\xi \in \rho^*$, $supp(e_\xi) = \{\gamma_\xi\}$, where $\gamma_0 = 1$. Hence we obtain for the product (in the field H) of e_ξ with itself: $e_\xi^2 = g_\xi$, with $g_\xi(\gamma) = 1$ for $\gamma = \gamma_\xi^2 \in \Delta^{\cdot}$ and $g_\xi(\gamma) = 0$ for $\gamma \neq \gamma_\xi^2$. Thus for all $\xi \in \rho^*$, $g_\xi \in K$. This allows to define $< e_\xi, e_\xi >= g_\xi$, and $< e_\xi, e_\eta >= 0$, if $\eta \neq \xi$. By bilinear extension, we obtain an inner product $< -, - >$ for the linear span L. We show that for $\xi, \eta \in \rho^*$, $\eta < \xi$, and all $0 \neq k_\xi, k_\eta \in K$, $\|k_\xi e_\xi\| \neq \|k_\eta e_\eta\|$. Indeed, if $\|k_\xi e_\xi\| = \|k_\eta e_\eta\|$, then $\|e_\xi\| = \gamma_\xi = w(k_\xi^{-1} k_\eta)\|e_\eta\|$ and hence $\gamma_\xi = \delta_0 \gamma_\eta$, with $\delta_0 = w(k_\xi^{-1} k_\eta) \in \Delta^{\cdot} \cap \sqrt{\Lambda_\xi} = \Lambda_\xi$. This is against property (g3) of the elements γ_ζ, $\zeta \in \rho^*$. Thus if $\xi, \eta \in \rho^*$, $\xi \neq \eta$, $\|k_\xi e_\xi\| \neq \|k_\eta e_\eta\|$ for all $0 \neq k_\xi, k_\eta \in K$. Therefore the norm of an element $x = k_1 e_{\xi_1} + \ldots + k_n e_{\xi_n} \in L$, with $k_1, \ldots, k_n \in K \setminus \{0\}$ and pairwise different $\xi_1, \ldots, \xi_n \in \rho^*$ is given by $\| x \| = \max\{\| k_i e_{\xi_i} \| \mid i = 1, \ldots, n\} = \max\left\{\sqrt{w(< k_i e_{\xi_i}, k_i e_{\xi_i} >)} \mid i = 1, \ldots, n\right\} = \sqrt{w(< x, x >)}$. Hence L is a definite space.

Lemma 1. *$< -, - >: L \times L \to K$ has a continuous extension to an inner product of E such that E is a definite space.*

Proof. Let $a \in L$ and $\varphi_a : L \to K$, $x \mapsto < a, x >$. By the Cauchy-Schwarz inequality (see Appendix 6.1), $w(< a, x >) \leq \| a \| \| x \|$. Hence the linear mapping φ_a is continuous and thus also uniformly continuous.

Let $x \in E \setminus L$. Since E is the completion of L, there exists a Cauchy family $(x_\iota)_{\iota < \lambda}$ of L (see Appendix 6.3) such that $x = \lim\limits_{\iota < \lambda} x_\iota$. Because $\varphi_a : L \to K$ is uniformly continuous, also $(\varphi_a(x_\iota))_{\iota < \lambda}$ is a Cauchy family, and since furthermore K is complete, there exists $k \in K$ such that $k = \lim\limits_{\iota < \lambda} \varphi_a(x_\iota)$. We put $< x, a >= < a, x >= k$. So $< a, x >= \lim\limits_{\iota < \lambda} < a, x_\iota >$. Because the mappings $\| \, \|: E \to \Gamma$ and $w : K \to \Delta$ are continuous (with respect to the order topologies of Γ, resp. Δ), we conclude from $< a, x >= \lim\limits_{\iota < \lambda} < a, x_\iota >$ that $w(< a, x >) = \lim\limits_{\iota < \lambda} w(< a, x_\iota >) \leq \lim\limits_{\iota < \lambda} \| a \| \| x_\iota \| = \| a \| \lim\limits_{\iota < \lambda} \| x_\iota \| = \| a \| \| x \|$. Hence the Cauchy-Schwarz inequality holds for all $a \in L$, $x \in E$. Since addition and multiplication (for E by scalars) are continuous, $< a, - >: E \to K$, $x \mapsto < a, x >$ is linear.

So the mapping $\psi_a : L \to K$, $x \mapsto < x, a >$ is defined for $a \in E$ and has formally the same properties as φ_a from above. Hence ψ_a has a continuous extension to a linear mapping of E. Thus we obtain, if $x = \lim\limits_{\iota < \lambda} x_\iota \in E$, $x_\iota \in L$, $< x, a >= \lim\limits_{\iota < \lambda} < x_\iota, a >$ and $w(< x, a >) \leq \| x \| \| a \|$.

So far we have shown that $< -, - >$ is a bilinear form on E which satisfies the Cauchy-Schwarz inequality. We still have to prove that $< -, - >$ is non-degenerate and that for all $x \in E$, $\| x \| = \sqrt{w(< x, x >)}$.

We show this latter first. The mapping $x \mapsto < x, x >$ from E to K is continuous. Indeed, if $x = 0$, then this follows directly from the Cauchy-Schwarz inequality. So we assume now $x \neq 0$ and $\| x - y \| < \gamma < \| x \|$. Then $w(< y, y > - < x, x >) = w(< y, y > - < x, y > + < x, y > - < x, x >) = w(< y - x, y > + < x, y - x >) \leq \max\{w(< y - x, y >), w(< x, y - x >)\} \leq \max\{\| y - x \| \| y \|, \| x \| \| y - x \|\} < \gamma \| x \|$. Hence the mapping is continuous.

Therefore we obtain for $x = \lim\limits_{\iota<\lambda} x_\iota \in E$, $x_\iota \in L$, $< x, x >= \lim\limits_{\iota<\lambda} < x_\iota, x_\iota >$. Thus $w(< x, x >) = w(\lim\limits_{\iota<\lambda} < x_\iota, x_\iota >) = \lim\limits_{\iota<\lambda} w(< x_\iota, x_\iota >) = \lim\limits_{\iota<\lambda} \| x_\iota \|^2 = \| \lim\limits_{\iota<\lambda} x_\iota \|^2 = \| x \|^2$.

This implies that $< -, - >$ on E is non-degenerate, since from $< x, y >= 0$ for $x \in E$ and all $y \in E$ follows in particular that $< x, x >= 0$, hence $w(< x, x >) = \| x \|^2 = 0$ and thus $x = 0$. ∎

Two elements x, y of a vector space with an inner product $< -, - >$ are called *orthogonal* if $< x, y >= 0$. A set $\{a_i \mid i \in I\}$ of pairwise orthogonal elements $\neq 0$ of a definite Banach space X is said to be an *orthogonal base* of X if the linear span of $\{a_i \mid i \in I\}$ is dense in X. (Since the elements a_i, $i \in I$, are pairwise orthogonal, the set $\{a_i \mid i \in I\}$ is linear independent).

We state now our main result:

Theorem 1. *Let ρ be finite, $\rho = n + 1$, or a limit ordinal. Let $(\Lambda_\xi)_{\xi<\rho}$ be a strictly increasing family of totally ordered abelian groups with the properties (g1)(g2)(g3). Let $\Delta^{\cdot} = \bigcup\limits_{\xi<\rho} \Lambda_\xi$ and $\Gamma^{\cdot} = \sqrt{\Delta^{\cdot}}$. Let F be any field with char$F \neq 2$ and $H = H(\Gamma^{\cdot}, F)$. Let $w : H \to \Gamma$ be the valuation $f \longmapsto \max supp(f)$. Let $K \subset H$ be the completion of the subfield $F(\Delta^{\cdot})$ of H. The field H is with the norm $\| \, \|: H \to \Gamma$, $\| f \| = w(f)$, a normed vector space over (K, w, Δ). Let the elements $e_\xi \in H$, $\xi \in \rho^*$, be chosen as pointed out above, and let $E \subset H$ be the completion of the linear span of $\{e_\xi \mid \xi \in \rho^*\}$ over K. Then E is with the inner product $< -, - >$, defined as a bilinear and continuous extension of $< e_\xi, e_\xi >= e_\xi^2$ and $< e_\xi, e_\eta >= 0$ for $\eta \neq \xi$, a definite Banach space. $\{e_\xi \mid \xi \in \rho^*\}$ is an orthogonal base for E. If $\rho = n + 1$, the cofinal type of Δ is equal to the cofinal type of Λ_n, hence countable or not countable; if ρ is infinite, then Δ has the cofinal type of ρ.*

Proof. In view of the preceding presentations and Lemma 1, we only have to prove the statements about the cofinal type (see Appendix 6.2) of Δ. If $\rho = n + 1$, $\Delta^{\cdot} = \Lambda_n$. Thus Δ has the same cofinal type as Λ_n. If ρ is a limit ordinal, then the set $\left\{\gamma_\xi^2 \mid \xi \in \rho^*\right\}$, with $\gamma_\xi \in \sqrt{\Delta^{\cdot}}$, $\xi \in \rho^*$, chosen according to (g3), is cofinal in $\Delta^{\cdot}$. Indeed, let $1 < \delta \in \Delta^{\cdot} = \bigcup\limits_{\xi<\rho} \Lambda_\xi$. Then there exists $\xi < \rho$ such that $\delta \in \Lambda_\xi$. If $\gamma_{\xi+1}^2 < \delta$, then since Λ_ξ is convex in $\Lambda_{\xi+1}$ and $\gamma_{\xi+1}^2 \in \Lambda_{\xi+1}$, we get $\gamma_{\xi+1}^2 \in \Lambda_\xi$; thus $\gamma_{\xi+1} \in \sqrt{\Lambda_\xi}$ contrary to property (g3) of the elements γ_ζ, $\zeta \in \rho^*$. Therefore $\delta \leq \gamma_{\xi+1}^2$. Hence $\left\{\gamma_\xi^2 \mid \xi \in \rho^*\right\}$ is cofinal in $\Delta^{\cdot}$. This proves $cf\Delta = cf\rho$ (for the notation, see Appendix 6.2). ∎

We call the definite Banach space E of Theorem 1 a *generalized Keller space.* Its construction depends on ρ, $(\Lambda_\xi)_{\xi<\rho}$, $(\gamma_\xi)_{\xi<\rho}$ and F. If we shall refer to such a space, we say the generalized Keller space relative to ρ, $(\Lambda_\xi)_{\xi<\rho}$, $(\gamma_\xi)_{\xi<\rho}$ and F, and we denote it by $E(\rho, (\Lambda_\xi)_{\xi<\rho}, (\gamma_\xi)_{\xi<\rho}, F)$.

5 Some properties of generalized Keller spaces

Let X be a normed vector space with an inner product $< -,- >$. Two *linear subspaces* D, S of X are called *orthogonal* if $< s, d >= 0$ for all $s \in S$, $d \in D$. When S, D are orthogonal, then $S \cap D = \{0\}$, hence the sum $S + D$ is direct. A linear subspace S of X, which is orthogonal to D such that $D + S = X$, is said to be an *orthogonal complement* of D. A *form-Hilbert space* is a definite Banach space for which each closed linear subspace has an orthogonal complement.

The following lemma is well-known from Linear Algebra.

Lemma 2. *Let X be a definite space of finite dimension. Then X has an orthogonal (algebraic) base.*

Proof. Let $\{a_i \mid i = 1, ..., n\}$ be an algebraic base of X. By the Gram-Schmidt orthogonalization process results that $\{e_j \mid j = 1, ..., n\}$, defined inductively by

$e_1 = a_1,$

$e_2 = a_2 - < a_2, e_1 >< e_1, e_1 >^{-1} e_1,$

$\cdots,$

$e_n = a_n - < a_n, e_1 >< e_1, e_1 >^{-1} e_1 - < a_n, e_2 >< e_2, e_2 >^{-1} e_2 - \cdots - < a_n, e_{n-1} >< e_{n-1}, e_{n-1} >^{-1} e_{n-1},$

is an orthogonal algebraic base of E. ■

From this lemma follows directly

Proposition 1. *A finite-dimensional definite Banach space X is a form-Hilbert space.*

Proof. Let $\dim X = n$ and let $D \neq \{0\}$ be a linear subspace of X with $D \neq X$. So $\dim D = m < n$. By the preceding lemma, D has an orthogonal base $\{e_1, ..., e_m\}$. Let $S = \{x \in X \mid < x, e_i >= 0$ for all $i = 1, ..., m\}$. Let $0 \neq x \in X$ and $y = x - \sum_{i=1}^{m} < x, e_i >< e_i, e_i >^{-1} e_i$. So $< y, e_j >= 0$ for all $j = 1, ..., m$. Hence $y \in S$ and thus $x \in D + S$, which proves that X is a form-Hilbert space. ■

From Theorem 1 and Proposition 1 results:

Proposition 2. *The generalized Keller spaces $E(\rho, (\Lambda_\xi)_{\xi<\rho}, (\gamma_\xi)_{\xi<\rho}, F)$ for finite ρ are finite-dimensional form-Hilbert spaces. Their value group $\Delta^{\cdot} = \bigcup_{\xi<\rho} \Lambda_\xi$ has finite or infinite rank, and $\Delta^{\cdot}$ may have any infinite cofinal type.*

We have now a look to the generalized Keller spaces $E(\rho, (\Lambda_\xi)_{\xi<\rho}, (\gamma_\xi)_{\xi<\rho}, F)$ with infinite ρ. As proved in Theorem 1, they have an orthogonal base of cardinality $card\rho$ and their value group $\Delta^{\cdot} = \bigcup_{\xi<\rho} \Lambda_\xi$ has cofinal type $cf\Delta^{\cdot} = cf\rho$. By results of Gross and Künzi (see [2], Theorem 17, Corollary 19 and Corollary 20), each form-Hilbert space with an infinite orthogonal base has necessarily a value group of cofinal type ω_0 and its orthogonal base needs to be countable. Thus all the generalized Keller spaces which don´t satisfy this condition are only definite Banach spaces. However if $cf\Delta = cf\rho \geq \omega_1$, the orthogonal base $\{e_\xi \mid \xi \in \rho^*\}$ of $E = E(\rho, (\Lambda_\xi)_{\xi<\rho}, (\gamma_\xi)_{\xi<\rho}, F)$ is an algebraic base for E as will be shown in Appendix 6.4.

We study now generalized Keller spaces $E(\rho, (\Lambda_\xi)_{\xi<\rho}, (\gamma_\xi)_{\xi<\rho}, F)$ with countable ρ. By using a theorem of [6], we give a criterion, when these spaces are form-Hilbert spaces. To apply this theorem, a further property of the generalized Keller spaces will be needed. We prove this property in the next proposition.

Two *linear subspaces* D_1, D_2 of a normed vector space X over K are called *norm-orthogonal*, if for all $x_1 \in D_1, x_2 \in D_2$, $\| x_1 + x_2 \| = \max\{\| x_1 \|, \| x_2 \|\}$. A *set* $\{a_i \mid i \in I\}$ of elements $a_i \in X$ is said to be *norm-orthogonal* if for each $j \in I$, Ka_j is norm-orthogonal to the linear span of $\{a_i \mid i \in I, i \neq j\}$. A *sequence* $(a_n)_{n<\omega_0}$ of elements $a_n \in X$ is said to be *norm-orthogonal*, if the set $\{a_n \mid n < \omega_0\}$ is norm-orthogonal.

Proposition 3. *Let $E(\rho, (\Lambda_\xi)_{\xi<\rho}, (\gamma_\xi)_{\xi<\rho}, F)$ be a generalized Keller space with countable ρ. Then every strictly decreasing, norm-orthogonal sequence $(f_j)_{j<\omega_0}$ of elements of E converges to* 0 *if and only if $\rho = \omega_0$.*

Proof. We use the notations which were introduced during the construction of the generalized Keller spaces.

1) Assume $\rho \neq \omega_0$. Since $cf\,\rho = \omega_0$, then $\omega_0 + \omega_0 \leq \rho$. The set $\{e_\xi \mid \xi \in \rho^*\}$ is an orthogonal base of E. Hence $\{e_j \mid j < \omega_0\}$ is a form- (so also norm-) orthogonal set and for all j, j', $j < j' < \omega_0$, $\gamma_j = \|e_j\| < \|e_{j'}\| < \|e_{\omega_0}\| = \gamma_{\omega_0}$.

Put $f_j =< e_j, e_j >^{-1} e_j$, $j < \omega_0$. Then the sequence $(f_j)_{j<\omega_0}$ is norm-orthogonal, strictly decreasing and for all $j < \omega_0$, $\|f_j\| > \|e_{\omega_0}\|^{-1} = \gamma_{\omega_0}^{-1}$. Thus if $\rho \neq \omega_0$ there exists a strictly decreasing, norm-orthogonal sequence which does not converge to 0.

2) Let now $\rho = \omega_0$. Let $(f_j)_{j<\omega_0}$ be a strictly decreasing, norm-orthogonal sequence. We wish to show that $(f_j)_{j<\omega_0}$ converges to 0. Since $(f_j)_{j<\omega_0}$ is strictly decreasing, it suffices to prove that $(f_j)_{j<\omega_0}$ has a subsequence which converges to 0.

Because $\|E\| \setminus \{0\} = \bigcup\limits_{0\leq n<\omega_0} \Delta^{\cdot}\gamma_n$, for each $j < \omega_0$, $\| f_j \| = \delta_j \gamma_{n_j}$, $\delta_j \in \Delta^{\cdot}$. If $j < i$, then $\delta_j \gamma_{n_j} > \delta_i \gamma_{n_i}$, hence a) $\delta_j > \delta_i$ or b) $\gamma_{n_j} > \gamma_{n_i}$ and in the latter case $n_j > n_i$. So for $j < \omega_0$, there are only finitely many $i_1, ..., i_m < \omega_0$ such that for $l = 1, ..., m$, $i_l > j$ and $\gamma_{n_j} > \gamma_{n_{i_l}}$. Hence for each $j < \omega_0$, there exists an $i = \varphi(j) < \omega_0$ such that $j < i$ and $\gamma_{n_j} \leq \gamma_{n_i}$. We show that $\gamma_{n_j} = \gamma_{n_i}$ is not possible. Since $\{e_n \mid 0 \leq n < \omega_0\}$ is an orthogonal base for E, each $x \in E$ has a representation as $x = \lim\limits_{m\to\infty} \left(\sum\limits_{n=0}^{m} < x, e_n >< e_n, e_n >^{-1} e_n \right) = \sum\limits_{n=0}^{\infty} < x, e_n >< e_n, e_n >^{-1} e_n$ (see [2], Lemma 16), thus $\lim\limits_{n\to\infty} \|k_n e_n\| = 0$, with $k_n =< x, e_n >< e_n, e_n >^{-1}$. The elements e_n, $0 \leq n < \omega_0$, are pairwise orthogonal, hence $\{e_n \mid n < \omega_0\}$ is norm-orthogonal (see Appendix 6.1), thus $\|x\| = \max\{\|k_n e_n\| \mid 0 \leq n < \omega_0\}$. So if $\gamma_{n_j} = \gamma_{n_i}$, choose m, $0 \leq m < \omega_0$, such that $\gamma_{n_j} = \|e_m\|$, then $\| f_j \| = \|k_m e_m\|$, $k_m =< f_j, e_m >< e_m, e_m >^{-1}$, and $\| f_i \| = \|k'_m e_m\|$, $k'_m =< f_i, e_m >< e_m, e_m >^{-1}$. Therefore $\| k_m^{-1} f_j - (k'_m)^{-1} f_i \| \notin \Delta^{\cdot} \|e_m\|$, which implies that $\| k_m^{-1} f_j - (k'_m)^{-1} f_i \| \neq \max\{\| k_m^{-1} f_j \|, \| (k'_m)^{-1} f_i \|\} \in \Delta^{\cdot} \|e_m\|$. This contradicts the norm-orthogonality of Kf_j, Kf_i. Thus $\gamma_{n_j} < \gamma_{n_i}$ for $i = \varphi(j) > j$.

Let $j_1 = 1$, $j_2 = \varphi(j_1), ..., j_{l+1} = \varphi(j_l), ...$ Then $\{j_l \mid l < \omega_0\}$ is cofinal in ω_0 and for each $l < m$, $\gamma_{j_l} < \gamma_{j_m}$. We prove now that the subsequence $(f_{j_l})_{l<\omega_0}$ of $(f_j)_{j<\omega_0}$ converges to 0. To simplify the notation, we show this directly for the sequence $(f_j)_{j<\omega_0}$ assuming for it the previous property. So we have for all $j < \omega_0$, $\| f_j \| = \delta_j \gamma_{n_j}$ and if $j < i$, $\delta_j \gamma_{n_j} > \delta_i \gamma_{n_i}$, $\gamma_{n_j} < \gamma_{n_i}$ and $\delta_j > \delta_i$. If for some $j < \omega_0$, $\delta_j \gamma_{n_j} \in \sqrt{\Lambda_{n_j}^-}$ (we may assume $n_j \geq 2$), then $\gamma_{n_j}\sqrt{\Lambda_{n_j}^-} = \delta_j^{-1}\sqrt{\Lambda_{n_j}^-}$, with $\delta_j^{-1} \in \Lambda_{n_j}$, which is against property (g3) of the elements γ_n, $n < \omega_0$. Thus if $m_j = \min\left\{m < \omega_0 \mid \delta_j \gamma_{n_j} \in \sqrt{\Lambda_m}\right\}$, then $m_j \geq n_j$. Let $j_0 < \omega_0$. Since $(n_j)_{j<\omega_0}$ is cofinal in ω_0, there exists $i_0 < \omega_0$, $i_0 > j_0$ such that $n_{i_0} > m_{j_0}$. Then $m_{i_0} \geq n_{i_0} > m_{j_0} > n_{j_0}$, furthermore $\delta_{i_0}\gamma_{n_{i_0}} < \delta_{j_0}\gamma_{n_{j_0}}$, hence $\delta_{i_0}\gamma_{n_{i_0}}\sqrt{\Lambda_{m_{j_0}}} < \delta_{j_0}\gamma_{n_{j_0}}\sqrt{\Lambda_{m_{j_0}}} = \sqrt{\Lambda_{m_{j_0}}}$ and therefore $\delta_{i_0}\gamma_{n_{i_0}} < 1$. This implies that $\{\delta_i \gamma_{n_i} \mid i < \omega_0\}$ is coinitial in $\sqrt{\Delta^{\cdot}}$. Thus $\lim\limits_{j<\omega_0} f_j = 0$, and it is proved that, if $\rho = \omega_0$, each strictly decreasing, norm-orthogonal sequence converges to 0. ■

By means of this proposition, it follows from [6], Theorem 3.2.1 (see also Proposition 2.3.2 of [6]) and from [4], Lemma 5.2 (the proof is valid also in this more general situation) that for a countable ρ, with $cf\rho = \omega_0$, $E(\rho, (\Lambda_\xi)_{\xi<\rho}, (\gamma_\xi)_{\xi<\rho}, F)$ is a form-Hilbert space if and only if $\rho = \omega_0$. Hence summarizing, we have:

Theorem 2. *The generalized Keller spaces $E(\rho, (\Lambda_\xi)_{\xi<\rho}, (\gamma_\xi)_{\xi<\rho}, F)$ for infinite ρ are form-Hilbert spaces if and only if $\rho = \omega_0$.*

We give now an example of a generalized Keller space $E(\omega_0, (\Lambda_\xi)_{\xi<\omega_0}, (\gamma_\xi)_{\xi<\omega_0}, F)$ for which the value group $\Delta^{\cdot} = \bigcup\limits_{\xi<\omega_0} \Lambda_\xi$ does not satisfy the condition of Proposition 1.4.4 of [5]. For $\rho = \omega_0$, this condition is equivalent to that each subset of $\Delta^{\cdot}$, which is bounded above, is of cofinal type $\leq \omega_0$. We use additive notation for the operation of $\Delta^{\cdot}$. The group $\Delta^{\cdot}$ will be constructed from totally ordered abelian groups of the form $\prod_{\xi<\lambda}^f \mathbb{Z}$ (such groups and their order are explained immediately below the properties (g1),(g2),(g3)).Let $\Sigma = \prod_{\xi<\omega_0}^f \mathbb{Z}$, $\Sigma' = \prod_{\xi<\omega_1}^f \mathbb{Z}$ and let Λ be their direct product $\Lambda = \Sigma \otimes \Sigma'$, ordered lexicographically, that is $(\sigma, \sigma') < (\tau, \tau')$ if $\sigma < \tau$ or ($\sigma = \tau$ and $\sigma' < \tau'$). For $n < \omega_0$, let $\Sigma_n = \{\sigma \in \Sigma \mid \sigma(j) = 0 \text{ for all } j,\ n < j < \omega_0\}$ and $\Lambda_n = \Sigma_n \otimes \Sigma' = \{(\sigma, \sigma') \in \Lambda \mid \sigma \in \Sigma_n, \sigma' \in \Sigma'\}$. Then $(\Lambda_n)_{n<\omega_0}$ is a strictly increasing sequence of convex subgroups of Λ. Put $\Delta^{\cdot} = \bigcup\limits_{n<\omega_0} \Lambda_n$. Since $\Sigma = \bigcup\limits_{n<\omega_0} \Sigma_n$, we get $\Delta^{\cdot} = \Lambda$. Replacing $\mathbb{Z}$ by $\frac{1}{2}\mathbb{Z}$, we obtain in the same way, $\frac{1}{2}\Delta^{\cdot} = \bigcup\limits_{n<\omega_0} \frac{1}{2}\Lambda_n$, with $\frac{1}{2}\Lambda_n = \frac{1}{2}\Sigma_n \otimes \frac{1}{2}\Sigma'$. Obviously, for all $n < \omega_0$, $\frac{1}{2}\Lambda_n \neq \Lambda_n$. For $n < \omega_0$, let $\sigma_n \in \frac{1}{2}\Sigma_n$ be defined by $\sigma_n(n) = \frac{1}{2}$, $\sigma_n(j) = 0$, if $j < \omega_0$, $j \neq n$. Put $\gamma_n = (\sigma_n, 0) \in \frac{1}{2}\Lambda_n$. It is, if $n \geq 2$, $\Lambda_n^- = \Lambda_{n-1}$ and $\gamma_n + \frac{1}{2}\Lambda_{n-1} \neq (\sigma, \sigma') + \frac{1}{2}\Lambda_{n-1}$ for all $(\sigma, \sigma') \in \Lambda_n$, since otherwise $\sigma_n = \sigma + \tau$, with $\sigma \in \Sigma_n$, $\tau \in \frac{1}{2}\Sigma_{n-1}$, and thus $\frac{1}{2} = \sigma_n(n) = \sigma(n) + \tau(n) \in \mathbb{Z}$, which is absurd. Hence $E(\omega_0, (\Lambda_\xi)_{\xi<\omega_0}, (\gamma_\xi)_{\xi<\omega_0}, F)$ is a generalized Keller space.

We show that $\Delta^{\cdot}$ contains a subset which is bounded above and of cofinal type ω_1. The set $\left\{\sigma'_\xi \mid \xi < \omega_1\right\}$, with $\sigma'_\xi \in \Sigma'$ defined by $\sigma'_\xi(\xi) = 1$ and $\sigma'_\xi(\eta) = 0$ for $\eta \neq \xi$, is cofinal in Σ'. Hence $\Lambda_0 = \left\{(0, \sigma'_\xi) \mid \xi < \omega_1\right\}$ is bounded above and without largest element in $\Delta^{\cdot}$. But there does not exist any countable subset of $\Delta^{\cdot}$ which is cofinal in Λ_0, since if this were the case, it would imply that $cf\Sigma' < \omega_1$, a contradiction.

6 Appendix

6.1 Ultrametric Pythagoras and Cauchy-Schwarz inequality.

The following lemma is well-known (see f.e. [2], Lemma 14).

Lemma 3. *Let* $(X, \|\ \|, \sqrt{\Delta})$ *be a definite space over the valued field* (K, w, Δ)*. Then the following holds for all* $x, y \in X$*:*

1. *("Ultrametric Pythagoras") If* $< x, y >= 0$*, then* $\|x + y\| = \max\{\|x\|, \|y\|\}$.
2. *("Cauchy-Schwarz inequality")* $w(< x, y >) \leq \|x\| \|y\|$.

Proof. (1): Let $< x, y >= 0$ and assume $\|y\| \leq \|x\|$. Since $< x, y >= 0$, we obtain $< x - y, x - y >=< x, x > + < y, y >=< x + y, x + y >$, hence $\|x - y\| = \|x + y\|$. Thus $\|x\| = \|2x\| = \|x + y + x - y\| \leq \max\{\|x + y\|, \|x - y\|\} = \|x + y\| \leq \|x\|$. Therefore $\|x + y\| = \max\{\|x\|, \|y\|\}$.

(2): We may assume $x \neq 0$, $y \neq 0$. Let $k =< x, y >< y, y >^{-1}$ and $b = x - ky$. Then $< b, y >= 0$, hence $< b, ky >= 0$ and therefore by (1), $\|x\| = \|b + ky\| = \max\{\|b\|, \|ky\|\}$. So $\|ky\| \leq \|x\|$. This yields with $k =< x, y >< y, y >^{-1}$ that $w(< x, y >) = w(k)\|y\|^2 = \|ky\| \|y\| \leq \|x\| \|y\|$. ∎

6.2 Cofinal and coinitial type of a totally ordered set.

Let Σ be a totally ordered set. A subset Υ of Σ is said to be *cofinal* in Σ (or a *cofinal subset* of Σ), if for every $\sigma \in \Sigma$ there exists $\tau \in \Upsilon$ such that $\sigma \leq \tau$. We show that there exists a cofinal subset Υ of Σ which is well-ordered (with respect to the order induced by that of Σ). Let $\mathfrak{D}$ be the set of all well-ordered subsets of Σ. We define a partial order on $\mathfrak{D}$ by $\Upsilon \preceq \Upsilon'$ if $\Upsilon \subseteq \Upsilon'$ and $\tau \leq \tau'$ for all $\tau \in \Upsilon$, $\tau' \in \Upsilon' \setminus \Upsilon$. By Zorn's Lemma, there exists a maximal $\Upsilon \in \mathfrak{D}$. Then Υ is cofinal in $\mathfrak{D}$. The *cofinal type* $cf\Sigma$ of Σ is the smallest ordinal λ such that Σ has a cofinal subset which is well-ordered of ordinal type λ. The notions *coinitial subset* of Σ and *coinitial type* $ci\Sigma$ are defined in dual way.

Let λ be a limit ordinal and $(\sigma_\iota)_{\iota<\lambda}$ a family of elements of Σ. We say that $(\sigma_\iota)_{\iota<\lambda}$ is cofinal (resp. coinitial) in Σ, if $\{\sigma_\iota \mid \iota < \lambda\}$ is cofinal (resp. coinitial) in Σ.

6.3 Cauchy families and completions of normed vector spaces

Let $(X, \|\ \|, \Gamma)$ be a normed vector space over the field (K, w, Δ). Let Y be a linear subspace of X. Then Y is with the restriction of $\|\ \|$ to Y a normed vector space. We denote this by $(Y, \|\ \|, \Gamma) \prec (X, \|\ \|, \Gamma)$ and say that X is an *extension* of Y. The extension is called *dense*, if for every $x \in X$, each neighbourhood of x contains an element of Y. Hence $Y \prec X$ is dense, if for every $x \in X$ and for every $0 < \gamma \in \Gamma$ there exists $y \in Y$ such that $\|x - y\| < \gamma$. A dense extension $Y \prec X$ will be denoted by $Y de \prec X$.

Let λ be a limit ordinal and let $(x_\iota)_{\iota<\lambda}$ be a family of elements of X. $(x_\iota)_{\iota<\lambda}$ is said to be a *Cauchy family* if the filter $\mathfrak{F}$ generated by the sets $F_\mu = \{x_\iota \mid \mu \leq \iota < \lambda\}$

is a Cauchy filter with respect to the uniformity on X defined by the norm. This is the case if for every $0 < \gamma \in \Gamma$, there exists ι_0 such that for all ι, μ, $\iota_0 \leq \iota < \mu < \lambda$, $\|x_\mu - x_\iota\| < \gamma$. The family $(x_\iota)_{\iota<\lambda}$ converges to $x \in X$ (denoted by $\lim\limits_{\iota<\lambda} x_\iota = x$), if the filter $\mathfrak{F}$ converges to x, that is, if for every $0 < \gamma \in \Gamma$, there exists ι_0 such that for all ι, $\iota_0 \leq \iota < \lambda$, $\|x - x_\iota\| < \gamma$. X is complete, if each Cauchy family of X converges to an element of X. X is a completion of Y, if X is complete and $Y de \prec X$. When X is a completion of Y, then obviously $\|X\| = \|Y\|$. Indeed, let $0 \neq x \in X$. There exists $y \in Y$ such that $\|x - y\| < \|x\|$. Thus $\|y\| = \max\{\|y - x\|, \|x\|\} = \|x\|$.

Proposition 4. *Let* $(Y, \|\ \|, \Gamma) \prec (X, \|\ \|, \Gamma)$ *be an extension of normed vector spaces and assume that* X *is complete. Then* X *contains a completion of* Y.

Proof. Let $\mathfrak{D}$ be the set of all linear subspaces D of X such that $Y de \prec D$. $\mathfrak{D}$ is (partially) ordered by the relation of inclusion. Let $\{D_i \mid i \in I\}$ be a chain of $\mathfrak{D}$. Then $G = \bigcup\limits_{i\in I} D_i$ is a linear subspace of X and $Y \prec G$. Since $Y de \prec D_i$ for all $i \in I$, Y is dense in G. Hence $G \in \mathfrak{D}$. Thus $\mathfrak{D}$ is inductively ordered, and trivially, $\mathfrak{D} \neq \emptyset$. So there exists, by Zorn's Lemma, a maximal element $G \in \mathfrak{D}$.

We show that G is complete. If not, then there exists a Cauchy family $(x_\iota)_{\iota<\lambda}$ of elements of G which has its limit $z \in X \setminus G$. So G is properly contained in the linear subspace $G' = G + Kz$ of X. We prove that $Y de \prec G'$. Thus we have to show that for every $0 < \gamma \in \Gamma$ and every $g' \in G'$ there exists $y \in Y$ such that $\|g' - y\| < \gamma$. It suffices to prove this for an element of the form $g' = g + z$, $g \in G$. Since $z = \lim\limits_{\iota<\lambda} x_\iota$, $x_\iota \in G$, there exists $\iota_0 < \lambda$ such that $\|z - x_{\iota_0}\| < \gamma$. Because $Y de \prec G$, we find $y_1, y_2 \in Y$ with $\|g - y_1\| < \gamma$ and $\|x_{\iota_0} - y_2\| < \gamma$. Then $y = y_1 + y_2 \in Y$ and $\|g' - y\| = \|g - y_1 + z - x_{\iota_0} + x_{\iota_0} - y_2\| \leq \max\{\|g - y_1\|, \|z - x_{\iota_0}\|, \|x_{\iota_0} - y_2\|\} < \gamma$. Thus $Y de \prec G'$ and therefore $G' \in \mathfrak{D}$. But this contradicts the maximality of G in $\mathfrak{D}$. Hence G is complete.

Since furthermore $Y de \prec G$, G is a completion of Y. ■

6.4 Direct sums of normed vector spaces.

Let $I \neq \emptyset$ be a set and $(X_i)_{i\in I}$ a family of normed vector spaces $(X_i, \|\ \|, \Gamma)$ over the valued field (K, w, Δ). The cartesian product $\prod\limits_{i\in I} X_i$ is (by pointwise addition and multiplication by scalars) a vector space over K. As mentioned in [6], the subset $\bigoplus\limits_{i\in I} X_i$ of elements $f \in \prod\limits_{i\in I} X_i$, for which for every $0 < \varepsilon \in \Gamma$ the set $\{i \in I \mid \|f(i)\| \geq \varepsilon\}$ is finite, is a normed linear subspace of $\prod\limits_{i\in I} X_i$ with respect to the norm $\|f\| = \max\{\|f(i)\| \mid i \in I\} \in \Gamma$. The normed vector space $(\bigoplus\limits_{i\in I} X_i, \|\ \|, \Gamma)$ is called the *direct sum* of the family $(X_i)_{i\in I}$. If all the spaces X_i, $i \in I$, are complete, also $\bigoplus\limits_{i\in I} X_i$ is complete. Let $\prod\limits_{i\in I}{}_f X_i$ be the linear subspace of $\bigoplus\limits_{i\in I} X_i$, which consists of all f, for which *supp*$(f) = \{i \in I \mid f(i) \neq 0\}$ is finite. Obviously, $\prod\limits_{i\in I}{}_f X_i$ is dense in $\bigoplus\limits_{i\in I} X_i$. Hence if all X_i, $i \in I$, are complete, $\bigoplus\limits_{i\in I} X_i$ is the completion of $\prod\limits_{i\in I}{}_f X_i$.

Proposition 5. *Let* $(\bigoplus_{i\in I} X_i, \|\ \|, \Gamma)$ *be the direct sum of the family* $(X_i)_{i\in I}$.

If $cf\Gamma \geq \omega_1$, *then* $\bigoplus_{i\in I} X_i = \prod_{f\, i\in I} X_i$. *If* $cf\Gamma = \omega_0$, *then for each* $f \in \bigoplus_{i\in I} X_i \setminus \prod_{f\, i\in I} X_i$, $supp(f)$ *is countable and* $\lim_{n<\omega_0} f(j_n) = 0$ *for any enumeration* $j_1, j_2, ...$ *of* $supp(f)$.

Proof. 1) We first show the following: Let $f \in \bigoplus_{i\in I} X_i$ and assume, $supp(f)$ is infinite. Let $(\gamma_n)_{n<\omega_0}$ be any sequence of $\Gamma^{\cdot}$ and i_1 any element of $supp(f)$. Then there exist elements $i_n \in supp(f)$, $2 \leq n < \omega_0$, such that for all $n < \omega_0$, $\|f(i_{n+1})\| < \min\{\|f(i_n)\|, \gamma_n\}$.

This follows immediately by induction. Indeed, let $i_1, i_2, ..., i_n$ be elements of $supp(f)$ with the property from above. Then there exists $i_{n+1} \in supp(f)$ such that $\|f(i_{n+1})\| < \min\{\|f(i_n)\|, \gamma_n\}$, since $\{i \in supp(f) \mid \|f(i)\| \geq \min\{\|f(i_n)\|, \gamma_n\}\}$ is finite and $supp(f)$ is infinite.

2) Let $cf\Gamma \geq \omega_1$ and assume that there exists $f \in \bigoplus_{i\in I} X_i$ such that $supp(f)$ is infinite. We choose an arbitrary $i_1 \in supp(f)$ and put $\gamma_n = \|f(i_1)\|$ for all $n < \omega_0$. By part 1), there exist $i_n \in supp(f)$, $2 \leq n < \omega_0$, such that the sequence $(\|f(i_n)\|)_{n<\omega_0}$ is strictly decreasing. Since $ci\Gamma^{\cdot} = cf\Gamma \geq \omega_1$, $(\|f(i_n)\|)_{n<\omega_0}$ is not coinitial in $\Gamma^{\cdot}$. So there exists $0 < \gamma \in \Gamma$ such that for all $n < \omega_0$, $\|f(i_n)\| > \gamma$. But then $f \notin \bigoplus_{i\in I} X_i$, a contradiction. This proves that $\bigoplus_{i\in I} X_i = \prod_{f\, i\in I} X_i$.

We treat now the case $cf\Gamma = \omega_0$. Let $f \in \bigoplus_{i\in I} X_i \setminus \prod_{f\, i\in I} X_i$. Since $ci\Gamma^{\cdot} = \omega_0$, there exists a strictly decreasing sequence $(\gamma_n)_{n<\omega_0}$ which is coinitial in $\Gamma^{\cdot}$. Let $i_1 \in supp(f)$ and choose $i_n \in supp(f)$, $2 \leq n < \omega_0$, according to part 1). For each $n < \omega_0$, the set $J_n = \{i \in supp(f) \mid \|f(i)\| \geq \|f(i_n)\|\}$ is finite. Hence $\bigcup_{n<\omega_0} J_n$ is countable. This implies, since $(\|f(i_n)\|)_{n<\omega_0}$ is coinitial in $\Gamma^{\cdot}$, that $supp(f)$ is countable.

Let now $j_1, j_2, ...$ be any enumeration of $supp(f)$. Let $0 < \varepsilon \in \Gamma$. Since $\|f(i)\| \geq \varepsilon$ for only finitely many $i \in supp(f)$, there exists $n_0 < \omega_0$ such that for all n, $n_0 \leq n < \omega_0$, $\|f(j_n)\| < \varepsilon$. Hence $\lim_{n<\omega_0} f(j_n) = 0$. ■

A norm-orthogonal set $\mathfrak{B} = \{e_i \mid i \in I\}$ of elements $\neq 0$ of a Banach space X is said to be a *norm-orthogonal base* of X, if the linear span of $\mathfrak{B}$ is dense in X. A subset $\mathfrak{B}$ of X is an algebraic base if $\mathfrak{B}$ is linear independent and if the linear span of $\mathfrak{B}$ is equal to X.

We obtain as a corollary from the preceding proposition:

Theorem 3. *Let* $(X, \|\ \|, \Gamma)$ *be a Banach space with* $cf\Gamma \geq \omega_1$. *Then every norm-orthogonal base of* X *is an algebraic base.*

Proof. Let $\mathfrak{B} = \{e_i \mid i \in I\}$ be a norm-orthogonal base of X. The mapping $\varphi : \prod_{f\, i\in I} Ke_i \to X$, $f \longmapsto \sum_{i\in supp(f)} f(i)$ is linear and norm preserving. $\prod_{f\, i\in I} Ke_i$ is mapped under φ onto the linear span F of $\mathfrak{B}$ over K. Hence since by the previous proposition $\prod_{f\, i\in I} Ke_i$ is complete, also F is complete. Thus $F = X$. ■

6.5 Spherical completeness of fields of formal power series.

Let $\Gamma^{\cdot} = \Gamma \setminus \{0\}$ be a totally ordered abelian group, F a field and $H = H(\Gamma^{\cdot}, F)$ the field of formal power series (from $\Gamma^{\cdot}$ to F).

Proposition 6. $H = H(\Gamma^{\cdot}, F)$ *is spherically complete with respect to the valuation* $w : H \to \Gamma$, $0 \neq f \mapsto \max supp(f)$, $0 \mapsto 0$.

Proof. Let $\mathfrak{B} = \{B_i \mid i \in I\}$ be a chain of balls of H. We have to show that $\bigcap \mathfrak{B} \neq \emptyset$. This is clear, if $\mathfrak{B}$ has a smallest element. Therefore we assume now that $\mathfrak{B}$ is without smallest element. Then the coinitial type $ci\mathfrak{B}$ is a limit ordinal λ. Thus there exists a strictly decreasing family $(B_\iota)_{\iota<\lambda}$ of balls $B_\iota = B_{\gamma_\iota}(f_\iota) \in \mathfrak{B}$, labelled by ordinals, such that $\{B_\iota \mid \iota < \lambda\}$ is coinitial in $\mathfrak{B}$. We define a mapping $f : \Gamma^{\cdot} \to F$ by $f(\gamma) = f_\iota(\gamma)$, if there exists $\iota < \lambda$ such that $\gamma > \gamma_\iota$, otherwise $f(\gamma) = 0$.

We have to show that f is well-defined. Assume $\gamma > \gamma_\iota$ and $\gamma > \gamma_{\iota'}$ with $\iota < \iota' < \lambda$. Then $\gamma_\iota > \gamma_{\iota'}$ and $f_{\iota'} \in B_{\gamma_\iota}(f_\iota)$, hence $w(f_{\iota'} - f_\iota) \leq \gamma_\iota$ and therefore $f_{\iota'}(\gamma) = f_\iota(\gamma)$. Thus f is well-defined.

We prove that $supp(f)$ is dually well-ordered. Let Υ be a non-empty subset of $supp(f)$. Let $\gamma \in \Upsilon$. Since $f(\gamma) \neq 0$, there exists $\iota < \lambda$ such that $\gamma > \gamma_\iota$ and $f(\gamma) = f_\iota(\gamma)$. Hence $\gamma \in \Upsilon \cap supp(f_\iota)$. Let $\gamma_0 = \max(\Upsilon \cap supp(f_\iota))$. Then $\gamma_0 \geq \gamma > \gamma_\iota$. If $\gamma' > \gamma_0$, then $\gamma' > \gamma_\iota$ and therefore $f(\gamma') = f_\iota(\gamma')$. Hence if $f_\iota(\gamma') \neq 0$, then $\gamma' \notin \Upsilon$. Thus γ_0 is the largest element of Υ. This shows that $supp(f)$ is dually well-ordered.

So $f \in H$. Since for all $\iota < \lambda$, $w(f - f_\iota) \leq \gamma_\iota$, $f \in \bigcap\limits_{\iota<\lambda} B_\iota = \bigcap \mathfrak{B}$. Hence H is spherically complete. ∎

References

[1] Alling N., Foundations of Analysis over Surreal Number Fields, (North-Holland Mathematics Studies **141**, Notas de Matematica **117**), North-Holland, Amsterdam etc. (1987).

[2] Gross H., Künzi U.M., On a class of orthomodular quadratic spaces, L'Enseignement Math. **31, (**1985), 187-212.

[3] Keller H., Ein nicht-klassischer Hilbertscher Raum, Math. Z. **172**, (1980), 41-49.

[4] Ochsenius H., Schikhof W.,Norm Hilbert spaces, Bull. Pol. Acad. Sci **46** (1998), 233-247.

[5] Ochsenius H., Schikhof W., Banach spaces over fields with an infinite rank valuation, in: p-adic Functional Analysis, Lecture Notes in Pure and Appl. Math. **207**, edited by J. Kakol, N. De Grande-De Kimpe and C. Perez-Garcia. Marcel Dekker (1999), pp. 233-293.

[6] Ochsenius H., Schikhof W.,Norm Hilbert spaces over Krull valued fields, Indag. Mathem., N.S., **17** (2006), 65-84.

[7] Priess-Crampe S., Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen, (Erg. d. Math. u. ihrer Grenzgeb. **98**), Springer, Heidelberg etc. (1983).

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