Analysis of a viscoelastic contact problem with Normal damped response and damage

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Abstract

We consider a mathematical model for the process of contact between a viscoelastic body and a reactive foundation. The material is viscoelastic with internal state variable which may describe the damage of the system. We establish a variational formulation for the model and prove the existence and uniqueness result of the weak solution. Finally we prove a dependence result with respect to the data.

1 Introduction

The damage subject is extremely important in design engineering, since it directly affects the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General novel models for damage were derived in [3, 4] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [5, 6]. The three-dimensional case has been investigated in [9]. The damage function β is restricted to have values between zero and one. When $\beta = 1$ there is no damage in the material, when $\beta = 0$ the material is completely damaged, when $0 < \beta < 1$ there is partial damage and the system has a reduced load carrying capacity. Quasistatic contact problems with damage have been investigated in [5, 7, 8, 11]. In this

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paper, the equation used for the evolution of the damage field is

$$\frac{d\beta}{dt} - k \bigtriangleup \beta + \partial\varphi_K(\beta) \ni \phi(\varepsilon(\mathbf{u}), \beta),$$

where K denotes the set of admissible damage functions defined by

$$K = \{\xi \in H^1(\Omega) \;/\; 0 \leq \xi \leq 1 \quad \text{a.e. in} \;\; \Omega\}.$$

A general viscoelastic constitutive law with damage is given by

$$\boldsymbol{\sigma} \; = \mathcal{A}(\varepsilon(\dot{\mathbf{u}})) + \mathcal{G}(\varepsilon(\mathbf{u}), \beta),$$

where $\mathcal{A}$ is a nonlinear viscosity function and $\mathcal{G}$ is a nonlinear elasticity function which depends on the internal state variable describing the damage of the material caused by elastic deformations.

In the present paper we consider a mathematical model for the process of contact between a viscoelastic body and a reactive foundation. Contact is modelled with the normal damped response condition, see, e.g., [10] . We derive the variational formulation and we prove existence and uniqueness of the weak solution of the model.

The paper is organised as follows. In section 2 we present the notation and some preliminaries. In section 3 we present the mechanical problem, we list the assumptions on the data and give the variational formulation of the problem. In section 4 we state our main existence and uniqueness result. It is based on arguments of time-dependent nonlinear equations with monotone operators, a fixed-point argument and a classical existence and uniqueness result on parabolic equations. In the last section we consider a dependence result of the weak solution with respect to the data.

2 Notation and preliminaries

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [2] .

We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while $(.)$ and $|\ .\ |$ represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. We shall use the notation

$$H = L^2(\Omega)^d = \left\{\mathbf{u} = (u_i) \;/\; u_i \in L^2(\Omega) \;\right\},$$

$$\mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma_{ij}) \;/\; \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \;\right\},$$

$$H_1 = \{\mathbf{u} = (u_i) \;/\; \varepsilon(\mathbf{u}) \in \mathcal{H} \;\},$$

$$\mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \;/\; Div\; \boldsymbol{\sigma} \in H\},$$

where $\varepsilon : H_1 \to \mathcal{H}$ and $Div : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \; Div\, \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Here and below, the indices i and j run between 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i v_i \, dx \qquad \forall\, \mathbf{u}, \mathbf{v} \in H,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},$$

$$(\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \qquad \forall\, \mathbf{u}, \mathbf{v} \in H_1,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}, Div\ \boldsymbol{\tau})_H \;\; \forall\ \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\ .\ |_H$, $|\ .\ |_{\mathcal{H}}$, $|\ .\ |_{H_1}$ and $|\ .\ |_{\mathcal{H}_1}$, respectively.

Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H_1$, we also use the notation $\mathbf{v}$ to denote the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on the boundary Γ given by

$$v_\nu = \mathbf{v}.\boldsymbol{\nu}, \;\; \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \tag{2.1}$$

Similarly, for a regular (say C^1) tensor field $\boldsymbol{\sigma} : \Omega \to S_d$ we define its normal and tangential components by

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu}, \;\; \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{2.2}$$

and we recall that the following Green's formula holds:

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}.\mathbf{v}\, da \qquad \forall\, \mathbf{v} \in H_1. \tag{2.3}$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0, T; X)$ and $W^{k,p}(0, T; X)$, $C(0, T; X)$ denotes the space of continuous functions from $[0, T]$ to X, with the norm

$$|\ \mathbf{f}\ |_{C(0,T;X)} = \max_{t \in [0,T]} |\ \mathbf{f}(t)\ |_X\ .$$

Similarly, $C^1(0, T; X)$ denotes the space of continuously differentiable functions from $[0, T]$ to X, with the norm

$$|\ \mathbf{f}\ |_{C^1(0,T;X)} = \max_{t \in [0,T]} |\ \mathbf{f}(t)\ |_X + \max_{t \in [0,T]} |\ \dot{\mathbf{f}}(t)\ |_X\ .$$

Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1 \times X_2}$. A dot above a variable represents its derivative with respect to time.

3 Mechanical and variational formulations

In this paper, we consider a contact problem which involves the evolution of the mechanical damage in a viscoelastic material. The physical setting is the following. A viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a regular surface Γ that is divided into three disjoint measurable parts Γ_1 , Γ_2 and Γ_3 such that *meas* $(\Gamma_1) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, and, therefore, the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0, T)$ and a body force of density $\mathbf{f}_0$ is applied in $\Omega \times (0, T)$. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approximation of the process.

The body is in contact with a reactive foundation over the contact surface Γ_3. We assume that contact is locked in the tangential direction, or in the stick state, and so the tangential displacement on the contact surface vanishes, i.e.,

$$\mathbf{u}_\tau = 0. \tag{3.1}$$

We assume that contact is modelled with the normal damped response condition ([10]), so,

$$\sigma_\nu = -\alpha \mid \dot{u}_\nu \mid . \tag{3.2}$$

Then, the classical formulation of the mechanical contact problem of a viscoelastic material with damage is as follows.

Problem P. *Find the displacement field* $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to S_d$ *and* $\beta : \Omega \times [0, T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{G}(\varepsilon(\mathbf{u}), \beta), \tag{3.3}$$

$$\dot{\beta} - k \bigtriangleup \beta + \partial\varphi_K(\beta) \ni \phi(\varepsilon(\mathbf{u}), \beta), \tag{3.4}$$

$$Div\ \boldsymbol{\sigma} + \mathbf{f}_0 = 0 \quad \text{in } \Omega \times (0, T)\,, \tag{3.5}$$

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma_1 \times (0, T)\,, \tag{3.6}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on} \quad \Gamma_2 \times (0, T)\,, \tag{3.7}$$

$$\sigma_\nu = -\,\alpha \mid \dot{u}_\nu \mid, \ \mathbf{u}_\tau = 0 \quad \text{on } \Gamma_3 \times (0, T)\,, \tag{3.8}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in} \quad \Omega, \tag{3.9}$$

$$\frac{\partial\beta}{\partial\nu} = 0 \quad \text{on} \ \ \Gamma \times (0, T)\,, \tag{3.10}$$

$$\beta(0) = \beta_0 \quad \text{in} \quad \Omega. \tag{3.11}$$

Here, the relation (3.3) represents the nonlinear viscoelastic constitutive law with internal state variable which involves the damage, the relation (3.4) represents the evolution of the damage field which is governed by the evolution equation where ϕ is the mechanical source of damage growth, assumed to be rather general function of the strains and damage itself, $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K, the relation (3.5) represents the equilibrium equation, since the accelerations are neglected in the equation of motion, leading to the quasistatic approximation of the process. The relations (3.6)-(3.7) are the displacement-traction conditions. Here $\mathbf{u}_0$ is the given initial displacement and β_0 is the initial material damage. (3.10) represents a homogeneous Newmann boundary condition where $\frac{\partial\beta}{\partial\nu}$ represents the normal derivative of β.

We denote by $\mathbf{u}$ the displacement field, by $\boldsymbol{\sigma}$ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linearized strain tensor. To simplify the notation, we do not indicate explicitely the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$.

To obtain a variational formulation of the problem (3.3)-(3.11) we need additional notation. Let V denote the closed subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 \;/\; \mathbf{v} = 0 \text{ on } \Gamma_1,\; \mathbf{v}_\tau = 0 \text{ on } \Gamma_3\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$ which depends only on Ω and Γ_1 such that $\mid \varepsilon(\mathbf{v}) \mid_{\mathcal{H}} \geq C_k \mid \mathbf{v} \mid_{H_1} \forall \mathbf{v} \in V$. On the space V, we consider the inner product and the associated norm given by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \;\; \forall \mathbf{u}, \mathbf{v} \in V, \mid \mathbf{v} \mid_V = \mid \varepsilon(\mathbf{v}) \mid_{\mathcal{H}} \quad \forall \mathbf{v} \in V. \tag{3.12}$$

It follows from Korn's inequality that $\mid . \mid_{H_1}$ and $\mid . \mid_V$ are equivalent norms on V. Therefore $(V, \mid . \mid_V)$ is a real Hilbert space. Moreover by the Sobolev's trace theorem and (3.12), there exists a constant $C_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$\mid \mathbf{v} \mid_{L^2(\Gamma_3)} \leq C_0 \mid \mathbf{v} \mid_V \qquad \forall \mathbf{v} \in V. \tag{3.13}$$

In the study of the mechanical problem (3.3)-(3.11), we make the following assumptions.

The viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies

$$\left\{\begin{array}{l} (a) \text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ such that} \\ \mid \mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2) \mid \leq L_{\mathcal{A}} \mid \varepsilon_1 - \varepsilon_2 \mid \quad \forall \varepsilon_1, \varepsilon_2 \in S_d, \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \mid \varepsilon_1 - \varepsilon_2 \mid^2 \;\; \forall \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (c) \; \mathbf{x} \to \mathcal{A}(\mathbf{x}, \varepsilon) \text{ is Lebesgue measurable on } \Omega. \\ (d) \text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{array}\right. \tag{3.14}$$

The elasticity operator $\mathcal{G} : \Omega \times S_d \times \mathbb{R} \to S_d$ satisfies

$$\left\{\begin{array}{l}(a) \text{ There exists a constant } L_{\mathcal{G}}>0 \text{ Such that}\\ \mid \mathcal{G}(\mathbf{x},\varepsilon_1,\beta_1)-\mathcal{G}(\mathbf{x},\varepsilon_2,\beta_2)\mid \leq L_{\mathcal{G}}(\mid \varepsilon_1-\varepsilon_2\mid + \mid \beta_1-\beta_2\mid)\\ \forall\varepsilon_1,\varepsilon_2\in S_d,\ \forall\beta_1,\beta_2\in\mathbb{R} \text{ a.e. } \mathbf{x}\in\Omega.\\ (b) \text{ For any } \varepsilon\in S_d \text{ and } \beta\in\mathbb{R},\ \mathbf{x}\to\mathcal{G}(\mathbf{x},\varepsilon,\beta) \text{ is Lebesgue measurable on } \Omega.\\ (c) \text{ The mapping } \mathbf{x}\to\mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0})\in\mathcal{H}.\end{array}\right. \tag{3.15}$$

The damage source function $\phi : \Omega \times S_d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\left\{\begin{array}{l}(a) \text{ There exists a constant } L>0 \text{ such that}\\ \mid \phi(\mathbf{x},\varepsilon_1,\beta_1)-\ \phi(\mathbf{x},\varepsilon_2,\beta_2)\mid \leq L(\mid \varepsilon_1-\varepsilon_2\mid + \mid \beta_1-\beta_2\mid)\\ \forall\varepsilon_1,\varepsilon_2\in S_d,\ \forall\beta_1,\beta_2\in\mathbb{R} \text{ a.e. } \mathbf{x}\in\Omega.\\ (b) \text{ For any } \varepsilon\in S_d \text{ and } \beta\in\mathbb{R},\ \mathbf{x}\to\phi(\mathbf{x},\varepsilon,\beta) \text{ is Lebesgue measurable on } \Omega.\\ (c) \text{ The mapping } \mathbf{x}\to\phi(\mathbf{x},\mathbf{0},\mathbf{0})\in\mathcal{H}.\end{array}\right. \tag{3.16}$$

We suppose that the body forces and surface tractions satisfy

$$\mathbf{f}_0 \in C(0,T;H), \quad \mathbf{f}_2 \in C(0,T;L^2(\Gamma_2)^d), \tag{3.17}$$

the function α has the following properties:

$$\alpha \in L^\infty(\Gamma_3), \qquad \alpha(x) \geq \alpha_* > 0 \ \text{ a.e. on } \Gamma_3, \tag{3.18}$$

$$k > 0, \tag{3.19}$$

$$\mathbf{u}_0 \in V, \tag{3.20}$$

$$\beta_0 \in K. \tag{3.21}$$

Next, we define the element $\mathbf{f}(t) \in V$ by

$$(\mathbf{f}(t),\mathbf{v})_V = \int_\Omega \mathbf{f}_0(t).\mathbf{v}\ dx + \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\ ds\,, \tag{3.22}$$

for $\mathbf{v} \in V$, a.e. $t \in (0,T)$, and let $j : V \times V \to \mathbb{R}$, be the functional

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} \alpha \mid u_\nu \mid v_\nu\ ds. \tag{3.23}$$

We note that the conditions (3.17) imply

$$\mathbf{f} \in C(0,T;V). \tag{3.24}$$

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega)^d \to \mathbb{R}$ by

$$a(\xi,\varphi) = k \int_\Omega \nabla\xi.\nabla\varphi\ dx. \tag{3.25}$$

Using standard arguments we obtain the following formulation of the mechanical problem (3.3)-(3.11).

Problem P_V. *Find the displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$ *and a damage field* $\beta : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{G}(\varepsilon(\mathbf{u}(t)), \beta(t)) \quad \text{a.e. } t \in (0,T)\,, \tag{3.26}$$

$$(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V \quad \forall\, \mathbf{v} \in V, \ \text{a.e. } t \in (0,T)\,, \tag{3.27}$$

$$\begin{aligned} \beta(t) \in K \text{ for all } t \in [0,T]\,, \qquad & (\dot{\beta}(t), \xi - \beta(t))_{L^2(\Omega)} + a(\beta(t), \xi - \beta(t)) \\ & \geq (\phi(\varepsilon(\mathbf{u}(t)), \beta(t)), \xi - \beta(t))_{L^2(\Omega)} \quad \forall \xi \in K, \end{aligned} \tag{3.28}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \beta(0) = \beta_0. \tag{3.29}$$

4 An existence and uniqueness result

Our main existence and uniqueness result is the following.

Theorem 4.1. *Let the assumptions (3.14)-(3.21) hold. Then there exists a constant* α_0 *which depends only on* Ω, Γ_1 , Γ_3 *and* $\mathcal{A}$ *such that if* $|\, \alpha \,|_{L^\infty(\Gamma_3)} < \alpha_0$, *then there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \beta\}$ *to the problem* P_V. *Moreover, the solution satisfies*

$$\mathbf{u} \in C^1(0,T;V), \boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1),$$

$$\beta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.1}$$

We conclude that under the assumptions (3.14)-(3.21) the mechanical problem (3.3)-(3.11) has a unique weak solution with the regularity (4.1), provided α is sufficiently small.

The proof of this theorem will be carried out in three steps, It is based on arguments on time-dependent nonlinear equations, a fixed-point theorem and a classical existence and uniqueness result on parabolic equations (see [1 *p*.124]).

Let $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$, then there exists a constant α_0 which depends only on Ω, Γ_1 , Γ_3 and $\mathcal{A}$ such that if $|\, \alpha \,|_{L^\infty(\Gamma_3)} < \alpha_0$, there exists a unique solution $\{\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta\}$ of the following intermediate problem such that, for $t \in [0,T]$,

Problem P_V^η : *Find a displacement field* $\mathbf{u}_\eta : [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma}_\eta : [0,T] \to \mathcal{H}_1$ *such that*

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}_\eta(t)) + \boldsymbol{\eta}(t) \qquad \text{in } \Omega, \tag{4.2}$$

$$(\boldsymbol{\sigma}_\eta(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\dot{\mathbf{u}}_\eta(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V \quad \forall \mathbf{v} \in V, \tag{4.3}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{4.4}$$

Proposition 4.2. *P_V^η has a unique weak solution such that*

$$\mathbf{u}_\eta \in C^1(0,T;V), \boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H}_1). \tag{4.5}$$

Proof. We define the operator $A : V \to V$ by

$$(A\mathbf{u}, \mathbf{v})_V = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\mathbf{u}, \mathbf{v}) \ \ \forall \mathbf{u}, \mathbf{v} \in V. \tag{4.6}$$

Using (4.6), (3.14), (3.23), (3.12) and (3.13), it follows that

$$\mid A\mathbf{u} - A\mathbf{v} \mid_V \leq (L_{\mathcal{A}} + C_0^2 \mid \alpha \mid_{L^\infty(\Gamma_3)}) \mid \mathbf{u} - \mathbf{v} \mid_V \ \forall \mathbf{u}, \mathbf{v} \in V, \tag{4.7}$$

and,

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_V \geq (m_{\mathcal{A}} - C_0^2 \mid \alpha \mid_{L^\infty(\Gamma_3)}) \mid \mathbf{u} - \mathbf{v} \mid_V^2 \ \forall \mathbf{u}, \mathbf{v} \in V. \tag{4.8}$$

Let $\alpha_0 = \frac{m_{\mathcal{A}}}{C_0^2}$, clearly it is a positive constant which depends on Ω, Γ_1, Γ_3 and $\mathcal{A}$. Then A is Lipschitz continuous on V and strongly monotone on V if

$$\mid \alpha \mid_{L^\infty(\Gamma_3)} < \alpha_0. \tag{4.9}$$

Therefore, A is invertible and its inverse A^{-1} is also strongly monotone Lipschitz continuous operator on V. Moreover using Riesz Representation Theorem we may define an element $\mathbf{f}_\eta \in C(0,T;V)$ by

$$(\mathbf{f}_\eta(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V - (\boldsymbol{\eta}\,(t), \varepsilon(\mathbf{v}))_{\mathcal{H}}.$$

It follows now from classical result (see for example [2]) that there exists a unique function $\mathbf{v}_\eta \in C(0,T;V)$ which satisfies

$$A\mathbf{v}_\eta(t) = \mathbf{f}_\eta(t) \text{ a.e. } t \in (0,T)\,. \tag{4.10}$$

From the relation (4.2), we conclude that $\boldsymbol{\sigma}_\eta\ (t) \in C(0,T;\mathcal{H})$. The couple $\{\mathbf{v}_\eta, \boldsymbol{\sigma}_\eta\}$ represents a unique solution of the intermediate problem P_V^η with the following regularity

$$\mathbf{v}_\eta \in C(0,T;V), \boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H}_1). \tag{4.11}$$

Let $\mathbf{u}_\eta : [0,T] \to V$ be the function defined by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{v}_\eta(s)\ ds + \mathbf{u}_0. \tag{4.12}$$

Using (4.11) and (4.12) we find that $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$ satisfies (4.5). ∎

Let $\theta \in C(0,T;L^2(\Omega))$. We suppose that the assumptions of Theorem 4.1 hold and we consider the following intermediate problem.

Problem P_V^θ : *Find a damage field $\beta_\theta : [0,T] \to H^1(\Omega)$ such that $\beta_\theta(t) \in K$, for all $t \in [0,T]$ and*

$$(\dot{\beta}_\theta(t), \xi - \beta_\theta(t))_{L^2(\Omega\,)} + a(\beta_\theta(t), \xi - \beta_\theta(t))$$
$$\geq (\theta, \xi - \beta_\theta(t))_{L^2(\Omega\,)} \quad \forall \xi \in K \ \text{ a.e. } t \in (0,T)\,, \tag{4.13}$$
$$\beta_\theta(0) = \beta_0. \tag{4.14}$$

Proposition 4.3. *Problem P_V^θ has a unique solution β_θ such that*

$$\beta_\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.15}$$

Proof. We use (3.19), (3.21), (3.25) and a classical existence and uniqueness result on parabolic equations (see for instance [1 *p*.124]). ■

As a consequence of Proposition 4.2 and 4.3, we may define the operator $\mathcal{L}$: $C(0,T;\mathcal{H}\times L^2(\Omega)) \to C(0,T;\mathcal{H}\times L^2(\Omega))$ by

$$\mathcal{L}(\boldsymbol{\eta},\theta) = (\mathcal{G}(\varepsilon(\mathbf{u}_\eta),\beta_\theta),\phi(\varepsilon(\mathbf{u}_\eta),\beta_\theta)), \tag{4.16}$$

for all $(\boldsymbol{\eta},\theta) \in C(0,T;\mathcal{H}\times L^2(\Omega))$, we have

Proposition 4.4. *The operator $\mathcal{L}$ has a unique fixed-point*

$$(\boldsymbol{\eta}^*,\theta^*) \in C(0,T;\mathcal{H}\times L^2(\Omega)).$$

Proof. Let $(\boldsymbol{\eta}_1,\theta_1)$, $(\boldsymbol{\eta}_2,\theta_2) \in C(0,T;\mathcal{H}\times L^2(\Omega))$ and let $t\in[0,T]$. Using (3.15), (3.16) and (4.16), we deduce that

$$\begin{aligned} &\mid \mathcal{L}(\boldsymbol{\eta}_1,\theta_1) - \mathcal{L}(\boldsymbol{\eta}_2,\theta_2) \mid_{\mathcal{H}\times L^2(\Omega\ )} \\ &\le C(\mid \mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\mid_V + \mid \beta_{\theta_1}(t) - \beta_{\theta_2}(t) \mid_{L^2(\Omega)}). \end{aligned} \tag{4.17}$$

Here and below C is a positive constant whose value may change from place to place. Let $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2 \in C(0,T;\mathcal{H})$ and use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$ for $i=1,2$. Moreover, using (4.2) and (4.3) we obtain

$$(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_1) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}_2), \varepsilon(\dot{\mathbf{u}}_1) - \varepsilon(\dot{\mathbf{u}}_2))_{\mathcal{H}} + (\eta_1 - \boldsymbol{\eta}_2, \varepsilon(\dot{\mathbf{u}}_1) - \varepsilon(\dot{\mathbf{u}}_2))_V$$

$$+ j(\dot{\mathbf{u}}_1, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2) - j(\dot{\mathbf{u}}_2, \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2) = 0 \text{ a.e.} t \in (0,T)\,.$$

Using (3.12), (3.13), (3.14) and (3.23) we obtain

$$\mid \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t) \mid_V \le C \mid \boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t) \mid_V .$$

Since $\mathbf{u}_1(0) = \mathbf{u}_2(0)$ we have

$$\mid \mathbf{u}_1(t) - \mathbf{u}_2(t) \mid_V \le \int_0^t \mid \dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s) \mid_V ds.$$

From the two previous inequalities we find

$$\mid \mathbf{u}_1(t) - \mathbf{u}_2(t) \mid_V \le C\int_0^t \mid \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s) \mid_{\mathcal{H}} ds. \tag{4.18}$$

We let $\theta_1,\theta_2 \in C(0,T;L^2(\Omega))$ and use the notation $\beta_{\theta i} = \beta_i$ for $i=1,2$. From (4.13) we find

$$\begin{aligned} &(\dot{\beta}_1, \beta_2 - \beta_1)_{L^2(\Omega)} + a(\beta_1,\beta_2-\beta_1) \\ &\ge (\theta_1, \beta_2 - \beta_1)_{L^2(\Omega)} \quad \text{a.e. } t \in (0,T)\,, \end{aligned}$$

and,

$$(\dot{\beta}_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a(\beta_2, \beta_1 - \beta_2)$$

$$\geq (\theta_2, \beta_1 - \beta_2)_{L^2(\Omega)} \quad \text{a.e. } t \in (0,T)\,.$$

Adding the previous inequalities we obtain

$$(\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a(\beta_1 - \beta_2, \beta_1 - \beta_2)$$

$$\leq (\theta_1 - \theta_2, \beta_1 - \beta_2)_{L^2(\Omega)} \quad \text{a.e. } t \in (0,T)\,,$$

which implies that

$$(\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a(\beta_1 - \beta_2, \beta_1 - \beta_2)$$

$$\leq \mid \theta_1 - \theta_2 \mid_{L^2(\Omega)} \mid \beta_1 - \beta_2 \mid_{L^2(\Omega)} \quad \text{a.e. } t \in (0,T)\,.$$

Integrating the previous inequality on $[0,t]$, after some manipulations we obtain

$$\frac{1}{2} \mid \beta_1(t) - \beta_2(t) \mid^2_{L^2(\Gamma_3)} \leq C \int_0^t \mid \theta_1(s) - \theta_2(s) \mid_{L^2(\Omega)} \mid \beta_1(s) - \beta_2(s) \mid_{L^2(\Omega)} ds$$

$$+ C \int_0^t \mid \beta_1(s) - \beta_2(s) \mid^2_{L^2(\Omega)} ds.$$

Applying Gronwall's inequality to the previous inequality yields

$$\mid \beta_1(t) - \beta_2(t) \mid_{L^2(\Omega)} \leq C \int_0^t \mid \theta_1(s) - \theta_2(s) \mid_{L^2(\Omega)} ds. \tag{4.19}$$

Substituting (4.18) and (4.19) in (4.17), we obtain

$$\mid \mathcal{L}(\boldsymbol{\eta}_1, \theta_1) - \mathcal{L}(\boldsymbol{\eta}_2, \theta_2) \mid_{\mathcal{H} \times L^2(\Omega)} \leq$$

$$C \int_0^t \mid (\boldsymbol{\eta}_1, \theta_1)(s) - (\boldsymbol{\eta}_2, \theta_2)(s) \mid_{\mathcal{H} \times L^2(\Omega)} ds. \tag{4.20}$$

Proposition 4.4 is a consequence of the result (4.20) and Banach's fixed-point theorem. ■

Proof. Theorem 4.1**.** Let $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*})$ be the solution to P_V^{η} for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and let β_{θ^*} be the solution of P_V^{θ} for $\theta = \theta^*$. It is easy to see that $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*}, \beta_{\theta^*})$ is the solution to problem P_V and

$$\mathbf{u}_{\eta^*} \in C^1(0,T;V), \boldsymbol{\sigma}_{\eta^*} \in C(0,T;\mathcal{H}_1),$$

$$\beta_{\theta^*} \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

The uniqueness of this solution follows from the uniqueness of the fixed-point of the operator $\mathcal{L}$ defined by (4.16). ■

5 Dependence on α

Let $\alpha_0 = \frac{m_{\mathcal{A}}}{C_0^2}$ and consider a sequence of functions $(\alpha_n)_{n\in\mathbb{N}}$ and a function $\alpha \in L^\infty(\Gamma_3)$ satisfying (3.18) and the following assumptions:

$$\alpha \text{ and } \alpha_n \text{ satisfy (4.9)}, \tag{5.1}$$

$$|\ \alpha_n - \alpha\ |_{L^\infty(\Gamma_3)} \to 0 \text{ when } n \to +\infty. \tag{5.2}$$

We define the functional $j_n(\mathbf{u}, \mathbf{v})$ obtained by substituting α_n in place of α in $j(\mathbf{u}, \mathbf{v})$ given in (3.23). We obtain a sequence of variational problems P_V^n defined for all $n \in \mathbb{N}$ as:

Problem P_V^n. *Find the displacement field* $\mathbf{u}_n : [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma}_n : [0,T] \to \mathcal{H}$ *and a damage field* $\beta_n : [0,T] \to H^1(\Omega)$ *such that*

$$\boldsymbol{\sigma}_n(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}_n(t)) + \mathcal{G}(\varepsilon(\mathbf{u}_n(t)), \beta_n(t)) \quad \text{a.e. } t \in (0,T)\,, \tag{5.3}$$

$$(\boldsymbol{\sigma}_n(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j_n(\dot{\mathbf{u}}_n(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V \quad \forall\, \mathbf{v} \in V, \text{a.e. } t \in (0,T)\,, \tag{5.4}$$

$$\begin{aligned} \beta_n(t) \in K\ ,\ \forall t \in [0,T] \quad & (\dot{\beta}_n(t), \xi - \beta_n(t))_{L^2(\Omega\ )} + a(\beta_n(t), \xi - \beta_n(t)) \\ & \geq (\phi(\varepsilon(\mathbf{u}_n(t)), \beta_n(t)), \xi - \beta_n(t))_{L^2(\Omega\ )} \quad \forall \xi \in K, \end{aligned} \tag{5.5}$$

$$\mathbf{u}_n(0) = \mathbf{u}_0, \qquad \beta_n(0) = \beta_0. \tag{5.6}$$

Theorem 4.1 give us the existence and the uniqueness of the sequence of the solutions $(\mathbf{u}_n, \boldsymbol{\sigma}_n, \beta_n)_{n\in\mathbb{N}}$ satisfying (4.1). We denote by $(\mathbf{u}, \boldsymbol{\sigma}, \beta)$ the solution of the problem P_V having the regularity (4.1). The main result of this section is the following.

Theorem 5.1. *Under the assumptions (3.14)-(3.21) and (5.1)-(5.2), we have that for* $n \to \infty$ *the following hold:*

$$|\ \mathbf{u}_n - \mathbf{u}\ |_{C^1(0,T;V)} \to 0, \tag{5.7}$$

$$|\ \boldsymbol{\sigma}_n - \boldsymbol{\sigma}\ |_{C(0,T;\mathcal{H}_1)} \to 0, \tag{5.8}$$

$$|\ \beta_n - \beta\ |_{L^2(\Omega)} \to 0. \tag{5.9}$$

Proof. Let $n \in \mathbb{N}$ and $t \in [0,T]$. From the relations (5.3), (5.4), (3.26) and (3.27) of the problems P_V^n and P_V, we find

$$\begin{aligned} (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_n) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}), \varepsilon(\dot{\mathbf{u}}_n) - \varepsilon(\dot{\mathbf{u}}))_{\mathcal{H}} &= (\mathcal{G}(\varepsilon(\mathbf{u}), \beta) - \mathcal{G}(\varepsilon(\mathbf{u}_n), \beta_n), \varepsilon(\dot{\mathbf{u}}_n) - \varepsilon(\dot{\mathbf{u}}))_{\mathcal{H}} \\ &+ j(\dot{\mathbf{u}}, \dot{\mathbf{u}} - \dot{\mathbf{u}}_n) - j_n(\dot{\mathbf{u}}_n, \dot{\mathbf{u}} - \dot{\mathbf{u}}_n) \quad \text{a.e. } t \in (0,T)\,. \end{aligned} \tag{5.10}$$

Using the definitions of the functional j and j_n and the relation (3.13), we obtain

$$\begin{aligned} & j(\dot{\mathbf{u}}, \dot{\mathbf{u}} - \dot{\mathbf{u}}_n) - j_n(\dot{\mathbf{u}}_n, \dot{\mathbf{u}} - \dot{\mathbf{u}}_n) \\ & \leq C_0^2\ |\ \alpha_n - \alpha\ |_{L^\infty(\Gamma_3)} |\ \dot{\mathbf{u}}\ |_V |\ \dot{\mathbf{u}}_n - \dot{\mathbf{u}}\ |_V + C_0^2\ |\ \alpha_n\ |_{L^\infty(\Gamma_3)} |\ \dot{\mathbf{u}}_n - \dot{\mathbf{u}}\ |_V^2\ . \end{aligned} \tag{5.11}$$

The relation (3.14) leads

$$(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_n) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}), \varepsilon(\dot{\mathbf{u}}_n) - \varepsilon(\dot{\mathbf{u}}))_{\mathcal{H}} \geq m_{\mathcal{A}}\ |\ \dot{\mathbf{u}}_n - \dot{\mathbf{u}}\ |_V^2\ . \tag{5.12}$$

The relation (3.15) give

$$(\mathcal{G}(\varepsilon(\mathbf{u}),\beta)-\mathcal{G}(\varepsilon(\mathbf{u}_n),\beta_n),\varepsilon(\dot{\mathbf{u}}_n)-\varepsilon(\dot{\mathbf{u}}))_{\mathcal{H}}\leq$$
$$L_{\mathcal{G}}(|\,\mathbf{u}_n-\mathbf{u}\,|_V+|\,\beta_n-\beta\,|_{L^2(\Omega)}). \tag{5.13}$$

Substituting (5.11), (5.12) and (5.13) in (5.10), we obtain

$$(m_{\mathcal{A}}-C_0^2\,|\,\alpha_n\,|_{L^\infty(\Gamma_3)})\,|\,\dot{\mathbf{u}}_n-\dot{\mathbf{u}}\,|_V\leq L_{\mathcal{G}}(|\,\mathbf{u}_n-\mathbf{u}\,|_V+|\,\beta_n-\beta\,|_{L^2(\Omega)})$$
$$+C_0^2\,|\,\alpha_n-\alpha\,|_{L^\infty(\Gamma_3)}|\,\dot{\mathbf{u}}\,|_V\,. \tag{5.14}$$

This majoration can be used if the quantity $|\,\dot{\mathbf{u}}\,|_V$ is bounded independently of n, then keeping in mind (3.26), (3.27) and (4.8) we find

$$(m_{\mathcal{A}}-C_0^2\,|\,\alpha_n\,|_{L^\infty(\Gamma_3)})\,|\,\dot{\mathbf{u}}\,|_V\leq|\,A0\,|+|\,\mathbf{f}\,|_{C(0,T;V)}+L_{\mathcal{G}}\,|\,\mathbf{u}\,|_V\,. \tag{5.15}$$

It follows from Gronwall's inequality that the function $\dot{\mathbf{u}}$ is bounded independently of n. We conclude that the function $\mathbf{u}$ is bounded independently of n. The assumptions (5.1) and (5.2) imply that $\exists\zeta\in\mathbb{R}^*$ and $\exists N_\zeta$ an integer different from zero, such that

$$(m_{\mathcal{A}}-C_0^2\,|\,\alpha_n\,|_{L^\infty(\Gamma_3)})>\zeta\qquad\forall\zeta\geq N_\zeta. \tag{5.16}$$

Keeping in mind the relation (5.15) and (5.16), the inequality (5.14) becomes

$$|\,\dot{\mathbf{u}}_n-\dot{\mathbf{u}}\,|_V\leq C(|\,\mathbf{u}_n-\mathbf{u}\,|_V+|\,\beta_n-\beta\,|_{L^2(\Omega)})+C\,|\,\alpha_n-\alpha\,|_{L^\infty(\Gamma_3)}\,. \tag{5.17}$$

Here and below, we denote by C a positive constant which may depend on the data but is independent of n and whose value may change from place to place. Combining the inequalities (5.5) and (3.28), we deduce that

$$(\dot\beta_n(t)-\dot\beta(t),\beta_n(t)-\beta(t))_{L^2(\Omega\,)}+a(\beta_n(t)-\beta(t),\beta_n(t)-\beta(t))$$
$$\leq(\phi(\varepsilon(\mathbf{u}_n(t)),\beta_n(t))-\phi(\varepsilon(\mathbf{u}(t)),\beta(t)),\beta_n(t)-\beta(t))_{L^2(\Omega\,)}\quad\text{a.e. } t\in[0,T]. \tag{5.18}$$

We obtain from (3.16)

$$\frac{1}{2}\frac{d}{dt}\,|\,\beta_n-\beta\,|^2_{L^2(\Omega)}\leq L(|\,\mathbf{u}_n-\mathbf{u}\,|_V+|\,\beta_n-\beta\,|_{L^2(\Omega)})\,|\,\beta_n-\beta\,|_{L^2(\Omega)}\,. \tag{5.19}$$

Integrating the previous relation on $[0,t]$, keeping in mind (3.29), (5.6) and applying Gronwall's inequality, yields

$$|\,\beta_n(t)-\beta(t)\,|_{L^2(\Omega)}\leq C\int_0^t|\,\mathbf{u}_n(s)-\mathbf{u}(s)\,|_V\,ds, \tag{5.20}$$

and therefore we obtain

$$|\,\beta_n(t)-\beta(t)\,|_{L^2(\Omega)}\leq C\,|\,\mathbf{u}_n-\mathbf{u}\,|_{C^1(0,T;V)}\,. \tag{5.21}$$

Substituting (5.20) in (5.17), by applying Gronwall's inequality, after some algebra, we obtain

$$|\,\mathbf{u}_n-\mathbf{u}\,|_{C^1(0,T;V)}\leq C\,|\,\alpha_n-\alpha\,|_{L^\infty(\Gamma_3)}\,. \tag{5.22}$$

The convergence result (5.7) is now a consequence of (5.22) and (5.2). From (5.3) and (3.26) we find

$$| \boldsymbol{\sigma}_n(t) - \boldsymbol{\sigma}(t) |_{\mathcal{H}} \leq L_{\mathcal{A}} | \dot{\mathbf{u}}_n(t) - \dot{\mathbf{u}}(t) |_V$$

$$+L_{\mathcal{G}}(| \mathbf{u}_n(t) - \mathbf{u}(t) |_V + | \beta_n(t) - \beta(t) |_{L^2(\Omega)}). \tag{5.23}$$

Moreover from (5.20) and after some algebraic manipulations, we obtain

$$| \boldsymbol{\sigma}_n(t) - \boldsymbol{\sigma}(t) |_{\mathcal{H}} \leq C | \mathbf{u}_n - \mathbf{u} |_{C^1(0,T;V)}, \tag{5.24}$$

since $Div\ \boldsymbol{\sigma}_n = Div\ \boldsymbol{\sigma}$, we have

$$| \boldsymbol{\sigma}_n(t) - \boldsymbol{\sigma}(t) |_{\mathcal{H}_1} \leq C | \mathbf{u}_n - \mathbf{u} |_{C^1(0,T;V)}, \tag{5.25}$$

which implies that

$$| \boldsymbol{\sigma}_n - \boldsymbol{\sigma} |_{C(0,T;\mathcal{H}_1)} \leq C | \mathbf{u}_n - \mathbf{u} |_{C^1(0,T;V)} . \tag{5.26}$$

The convergence result (5.8) follows from (5.26) and (5.7), and (5.9) is a consequence of (5.21) and (5.7). ■

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