Some Basic Questions and Conjectures on Elation Generalized Quadrangles, and their Solutions

Koen Thas*

Abstract

In this paper, I will start with posing three fundamental and old questions on (elation) generalized quadrangles, and survey tersely answers on these questions coming from recent work of S. E. Payne and the author of this paper. I will then introduce a fourth question posed recently by S. E. Payne, and will provide a general answer to this question, a result independently obtained by R. Rostermundt for the Hermitian quadrangles $H(3, q^2)$, q even, in an entirely different fashion. Finally, I will show that this answer yields examples of elation generalized quadrangles for which the automorphism group fixing the elation point is not induced by the automorphisms of the elation group fixing the associated 4-gonal family.

1 Standard Conjectures and Questions on Elation Generalized Quadrangles

This paper can be seen as a sequel to S. E. Payne and K. Thas [10] and K. Thas and S. E. Payne [16].

We refer to the monograph [9] for an introduction to the theory of generalized quadrangles. Also, one might want to use the survey [7] and the recent book [14] for further information and updates.

Let $\mathcal{S}$ be a thick generalized quadrangle (denoted "GQ" throughout), and let p be a point of $\mathcal{S}$. Then $\mathcal{S}$ is an *elation generalized quadrangle (EGQ)* with *elation point*

***Acknowledgment.** The author is a Postdoctoral Fellow of the Fund for Scientific Research — Flanders (Belgium).

p and *elation group* G, if G is an automorphism group of $\mathcal{S}$ which fixes p linewise and which acts sharply transitively on the points of $\mathcal{S}$ which are non-collinear with p. Sometimes we write $(\mathcal{S}^{(p)}, G)$ or $\mathcal{S}^{(p)}$ for $\mathcal{S}$. An *elation* about p is either the identity $\mathbf{1}$, or an automorphism fixing p linewise and fixing no point not collinear with p. Note that this does not imply that an elation ϕ about p acts freely on these points, that is, that $\langle\phi\rangle$ is a group of elations about p. An EGQ is always *thick* in this paper: if (s,t) is its order, then $s,t > 1$.

In Chapter 8 of [9], the following is quoted: "In general it seems to be an open question as to whether or not the set of elations about a point must be a group." In the same chapter of *loc. cit.* the authors study *translation generalized quadrangles (TGQ's)*, which are just EGQ's with an abelian elation group, and show that *all elations* about the elation point are in this group (cf. 8.6.4 of [9]). We will call the aforementioned question "Question (1)".

Most of the known GQ's are, up to duality, EGQ's with at least one elation point. (In fact, each known GQ is as such, or is constructed from an EGQ.) We therefore formulate the following specialization of Question (1):

Question (2). *Given an EGQ $\mathcal{S}^{(p)}$, is the set of elations about p a group?*

The following question makes sense if the answer is "not always":

Question (2′). *Given an EGQ $\mathcal{S}^{(p)}$,* when *is the set of elations about p a group?*

Let $\mathcal{F}$ be a Kantor-Knuth semifield flock of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3,q)$, $q = p^h$, with p an odd prime power (see Chapter 3 of [14] for more details). Let $\mathcal{S}(\mathcal{F})$ be the corresponding *flock GQ* [7, 12] of order (q^2, q). It is well-known that each flock GQ has a 'special' point (∞) for which there is a group of elations K making it into an EGQ with elation point (∞). Recently, S. E. Payne and K. Thas [16] constructed an elation θ about (∞) with the following two properties:

- θ^p is an involution fixing some subGQ of order q pointwise;
- θ^2 is contained in K.

So θ is not an element of K. This suggested the definition of a "standard elation" about a point p: this is an elation ϕ for which $\langle\phi\rangle$ is a group of elations. The following natural question arises:

Question (3). *Given an EGQ $\mathcal{S}^{(p)}$, is the set of standard elations about p a group?*

In the same way as for Question (2), one could now also formulate Question (3′).

2 4-Gonal Families and Elation Generalized Quadrangles

Suppose $(\mathcal{S}^{(p)}, G)$ is an EGQ of order (s,t), $s \neq 1 \neq t$, with elation point p and elation group G, and let q be a point of $P \backslash p^{\perp}$ (P is the point set). Let $L_0, L_1, \ldots, L_t$ be the lines incident with p, and define r_i and M_i by $L_i I r_i I M_i I q$, $0 \leq i \leq t$. Put $H_i = \{\theta \in G \parallel M_i^\theta = M_i\}$ and $H_i^* = \{\theta \in G \parallel r_i^\theta = r_i\}$, and $\mathcal{J} = \{H_i \parallel 0 \leq i \leq t\}$. Then $|G| = s^2 t$ and $\mathcal{J}$ is a set of $t+1$ subgroups of G, each of order s. Also, for each i, H_i^* is a subgroup of G of order st containing H_i as a subgroup. Moreover, the following two conditions are satisfied:

(K1) $H_i H_j \cap H_k = \{\mathbf{1}\}$ for distinct i, j and k;

(K2) $H_i^* \cap H_j = \{\mathbf{1}\}$ for distinct i and j.

Conversely, if G is a group of order $s^2 t$ and $\mathcal{J}$ (respectively $\mathcal{J}^*$) is a set of $t+1$ subgroups H_i (respectively H_i^*) of G of order s (respectively of order st), where $H_i \leq H_i^*$ for each i, and if the Conditions (K1) and (K2) are satisfied, then the H_i^* are uniquely defined by the H_i, and $(\mathcal{J}, \mathcal{J}^*)$ or $\mathcal{J}$ is said to be a *4-gonal family of Type* (s,t) *in* G.

Let $(\mathcal{J}, \mathcal{J}^*)$ be a 4-gonal family of Type (s,t) in the group G of size $s^2 t$, $s \neq 1 \neq t$. Define an incidence structure $\mathcal{S}(G, \mathcal{J})$ as follows.

- Points of $\mathcal{S}(G, \mathcal{J})$ are of three kinds: (i) elements of G; (ii) right cosets $H_i^* g$, $g \in G$, $i \in \{0, 1, \ldots, t\}$; (iii) a symbol (∞).
- Lines are of two kinds: (a) right cosets $H_i g$, $g \in G$, $i \in \{0, 1, \ldots, t\}$; (b) symbols $[H_i]$, $i \in \{0, 1, \ldots, t\}$.
- Incidence. A point g of Type (i) is incident with each line $H_i g$, $0 \leq i \leq t$. A point $H_i^* g$ of Type (ii) is incident with $[H_i]$ and with each line $H_i h$ contained in $H_i^* g$ as a set. The point (∞) is incident with each line $[H_i]$ of Type (b). There are no further incidences.

It is straightforward to check that the incidence structure $\mathcal{S}(G, \mathcal{J})$ is a GQ of order (s,t). Moreover, if we start with an EGQ $(\mathcal{S}^{(p)}, G)$ to obtain the family $\mathcal{J}$ as above, then we have that $(\mathcal{S}^{(p)}, G) \cong \mathcal{S}(G, \mathcal{J})$. Hence, *a group of order* $s^2 t$ *admitting a 4-gonal family is an elation group of a suitable elation generalized quadrangle.* These results were first noted by W. M. Kantor in [6].

3 Some Recent Results by S. E. Payne and K. Thas

3.1 Recent results on Question (3)

For non-classical flock GQ's there is a complete answer to Question (3):

Theorem 3.1 (S. E. Payne and K. Thas [10]). *Let* $\mathcal{S}(\mathcal{F})$ *be a non-classical flock GQ of order* (q^2, q). *Then the set of standard elations about* (∞) *is a group. This group is the usual elation group* K. *When* q *is odd, the same conclusion holds in the classical case.* ■

Remark 3.2. In [10], the condition that $\mathcal{S}(\mathcal{F})$ be non-classical for q even was forgotten in the statement of the theorem (cf. Theorem 2.4 and Theorem 6.1).

The point (∞) of a flock GQ is a *regular point*: for each point $x \not\sim (\infty)$, we have $|\{(\infty), x\}^{\perp\perp}| = q + 1$ (where the order of the GQ is (q^2, q)). We call an EGQ with regular elation point a *skew translation generalized quadrangle* or *STGQ*.[1] For an STGQ of order (s, t) with elation point p, there is an automorphism group of size t fixing $p^\perp$ pointwise. A point with this property is a *center of symmetry*.

Generalizing Theorem 3.1, the following was also obtained in [10].

Theorem 3.3 (S. E. Payne and K. Thas [10]). *Let $\mathcal{S}^{(p)}$ be an STGQ of order (s, t), $s, t > 1$. Then we have two possibilities:*

(a) *the set of standard elations about p is a group;*

(b) *$s = t^2$, s is a power of 2, and there is a $W(t)$-subGQ containing p fixed pointwise by an involution of $\mathcal{S}^{(p)}$.* ■

Remark 3.4. Examples of (b) yield possible counter examples to Question (3). Such STGQ's will play a central role in §4.

Note that by X. Chen [2] and independently D. Hachenberger [5], s and t are powers of the same prime for an STGQ with these parameters.

3.2 Recent results on Question (2)

From the next theorem, it will follow that "most of the time", the answer to Question (2) is that the set of elations about an elation point is *not* a group, and it also explains *precisely* why. Recall first that a *whorl* about a point of a GQ is an automorphism of the GQ fixing the point linewise.

Theorem 3.5 (K. Thas and S. E. Payne [16]). *Let $(\mathcal{S}^{(p)}, G)$ be an EGQ, and let W be the group of all whorls about p. Then the set of elations about p is a group if and only if there is no nontrivial element in W fixing more than one point non-collinear with p if and only if W is a Frobenius group.* ■

Note that "a priori" it is not needed to know that the parameters of the GQ are powers of the same prime. The proof is elementary, and uses Burnside's Lemma.

Remark 3.6. It is important to remark that this theorem also settles the original Question (1) in general; one only has to consider orbits in $P \setminus p^\perp$ — where P is the point set of the GQ — of the group of all whorls about p, instead of considering one orbit $P \setminus p^\perp$.

[1] This is a slight abuse of the original definition (see [14]), but it is an equivalent one and will suffice for our purposes.

The known examples.

It is convenient to mention the known (classes of) examples of EGQ's with elation point p for which the set of elations about p is *not* a group. These classes are treated in detail in [16].

- *The classical and dual classical examples.* $W(q)$ with q odd; $H(3, q^2)$; $H(4, q^2)$; $H(4, q^2)^D$.
- *Flock GQ's.* The flock GQ's with an even number of points on a line.
- *Dual flock GQ's (which are EGQ's).* As the order is (q, q^2) for some q, it can be shown that not more than one point not collinear with the elation point can be fixed by a nonidentity collineation. So there are no examples possible.
- *TGQ's.* There are no examples possible.
- *Dual TGQ's (which are EGQ's).* GQ's $\mathcal{S}^D$, where $\mathcal{S}$ is a TGQ of order (q, q^2), q odd, that is good at some element.

4 Elation Generalized Quadrangles with Non-Isomorphic Elation Groups

The following fundamental question (especially in construction theory for GQ's) was recently posed by S. E. Payne [8]:

QUESTION. *Let $\mathcal{S}^{(p)}$ be an EGQ. Can p be an elation point for non-isomorphic elation groups?*

In this section, we will consider a class of GQ's which do admit non-isomorphic elation groups, thus answering Payne's question affirmatively. The only known examples of this class are $H(3, q^2)$-GQ's with q even.

Lemma 4.1. *Let $\mathcal{S}'$ and $\mathcal{S}''$ be distinct $W(q)$-subGQ's in a GQ $\mathcal{S}$ of order (q, q^2). Suppose $p \in \mathcal{S}' \cap \mathcal{S}''$ is such that the lines of $\mathcal{S}'$ through p are those of $\mathcal{S}''$ through p. Furthermore, suppose $\theta \neq \mathbf{1}$ is an involution that fixes $\mathcal{S}'$ pointwise. Then $\mathcal{S}''$ is stabilized by θ.*

Proof. Let $z \not\sim p$ be a point of $\mathcal{S}''$. Then $\{p, z, z^\theta\}$ is a triad of $\mathcal{S}$, so $|\{p, z, z^\theta\}^{\perp\perp}| \leq q + 1$ by 1.2.4 of [9]. As p is regular in $W(q)$, it follows that $z^\theta \in \mathcal{S}''$. Lemma 4.2.5 of [14] implies that $\mathcal{S}''^\theta = \mathcal{S}''$. ∎

STANDING HYPOTHESIS. 1. *For now, $\mathcal{S}^{(p)} = \mathcal{S}$ is an EGQ of order (q^2, q), q even, with elation group H. Also, $\mathcal{S}'$ is a subGQ of order (s', q), $s' > 1$, which is fixed elementwise by a nontrivial collineation θ of $\mathcal{S}$. By* [16]*, we then have that $\mathcal{S}' \cong W(q)$, and that θ is an involution.*

Suppose W is the group of all whorls about p, and let S_2 be a Sylow 2-subgroup of W which contains H. Then S_2 clearly has size $2q^5$. Put $H' = \theta H$, so that

$S_2 = H \cup H'$. As $\mathcal{S}' \cong W(q)$, and as each point of $W(q)$ is regular, one observes that for each point $z \not\sim p$, the pair $\{p, z\}$ is regular, so that p is a regular point of $\mathcal{S}$. This implies that two distinct subGQ's of order q containing p can only intersect in a very restricted manner (using for instance Lemma 4.2.5 of [14]): either they share the lines through p and the points (of the subGQ's) incident with one of these lines, or they intersect in the points and lines of a dual grid of order $(1, q)$. Let θ' and θ'' be two distinct nontrivial involutions in S_2 that respectively fix the subGQ's $\mathcal{S}_{\theta'}$ and $\mathcal{S}_{\theta''}$ (of order q) pointwise. Suppose that they intersect in a dual grid as above. Then there is a point $z \not\sim p$ for which $\{p, z\}^{\perp\perp} \subseteq \mathcal{S}_{\theta'} \cap \mathcal{S}_{\theta''}$. Since both θ' and θ'' fix z, we immediately have a contradiction since $\theta' \neq \theta''$ and $|[S_2]_z| = 2$. So all subGQ's of order q that are fixed pointwise by a nontrivial involution in S_2 mutually do not share points not collinear with p. This implies that if $\mathcal{S}_1$ and $\mathcal{S}_2 \neq \mathcal{S}_1$ are two such subGQ's, there is some line MIp so that $M^\perp \cap \mathcal{S}_1 = M^\perp \cap \mathcal{S}_2$. Also, it follows easily that the number of such subGQ's is q^2, and that the associated involutions are mutually conjugate in S_2. Note also that all whorls of S_2 which are not elations about p are contained in H'. The group S_2 is non-cyclic; if it were cyclic, then H would be abelian, implying in its turn that there are more lines through a point than points incident with a line (since $\mathcal{S}$ is then a TGQ, cf. Chapter 8 of [9]). As S_2 is non-cyclic, a result of P. Deligne [3] implies that S_2 has at least three subgroups of size q^5 (one of which is H). Suppose $H'' \neq H$ is a subgroup of S_2 of order q^5. If H'' does not contain any of the q^2 involutions of above, then H'' is an elation group ("first case"). If H'' contains at least one such involution ("second case"), it contains all of them since they are mutually conjugate, and since H'' is a normal subgroup of S_2 (as a group of index 2). In that case, put $H_1 = H'' \cap H$, and $H_2 = H' \cap H''$. So $|H_1| = |H_2| = q^5/2$. Then it is straightforward to see that

$$H_1 \cup \theta[H \setminus H_1] = H^-$$

is an elation group of size q^5. As the first and second case are equivalent, we keep using the notation of the second case. We put $H_4 = H \setminus H_1$ and $H_3 = \theta H_4$.

Suppose LIp, and let $xILIp \neq x$. By $H(x, L, p)$, we denote the subgroup of H of collineations that fix x and p linewise, and xp pointwise (we call such collineations "root-elations").

Standing Hypothesis. 2. *For all LIp and $xILIp \neq x$, we have that $|H(x, L, p)| = q^2$. Also, $H^2 \leq Z(H)$, where $H^2 = \{h^2 \parallel h \in H\}$ and $Z(H)$ is the center of H.*

Since $|H(x, L, p)| = q^2$ for all L and x as above, and since these groups generate H, it is straightforward to show that $Z(H)$ is the group of symmetries about p. In fact, one observes now easily that $Z(H) = Z(H^-)$. Let $H(x, L, p)$ be a root-group; then $H(x, L, p)^2 \leq Z(H)$, so that $H(x, L, p)^2 = \{\mathbf{1}\}$. So all such root-groups are elementary abelian. Now consider $\theta\phi \in H^-$, where $\phi \in H_4$ is a non-trivial root-elation in $H(z, M, p)$ with $z \in \mathcal{S}_\theta$ which does not fix $\mathcal{S}_\theta$ (it is an easy exercise that such a ϕ exists for suitable z). Then $(\theta\phi)^2 = [\theta, \phi^{-1}] = [\theta, \phi]$ clearly cannot be the identity, while it fixes z linewise. So $(\theta\phi)^2 \notin Z(H^-)$, so that $H \ncong H^-$.

We have obtained the following theorem.

Theorem 4.2. *Let $\mathcal{S} = (\mathcal{S}^{(p)}, H)$ be an EGQ of order (q^2, q), where q is even, which contains a subGQ $\mathcal{S}'$ of order (s, q), $s > 1$, fixed pointwise by a nontrivial automorphism θ of $\mathcal{S}$. If $H^2 \leq Z(H)$, and if for all LIp and $xILIp \neq x$, we have that $|H(x, L, p)| = q^2$, then there is an automorphism group H' of $\mathcal{S}$ such that $H' \ncong H$ and $(\mathcal{S}^{(p)}, H')$ is an EGQ.*

Proof. By K. Thas and S. E. Payne [16], we have that $\mathcal{S}' \cong W(q)$ (so that in particular $s = q$), and that θ is an involution. The rest follows from the part of this section occuring before this theorem. ■

Corollary 4.3. *Let $\mathcal{S} = (\mathcal{S}^{(p)}, H)$ be an EGQ of order (q^2, q), where q is even, which contains a subGQ $\mathcal{S}'$ of order (s, q), $s > 1$, which is fixed pointwise by a nontrivial automorphism θ of $\mathcal{S}$. Let $z \not\sim p$ and suppose $z \sim z_i \sim p$ for $i = 0, 1, \ldots, q$. If all groups $H(p, pz_i, z_i) \leq H$ are elementary abelian and have size q^2, then there is an automorphism group H' of $\mathcal{S}$ such that $H' \ncong H$ and $(\mathcal{S}^{(p)}, H')$ is an EGQ.*

Proof. The root-groups are elementary abelian if and only if $H^2 \in Z(H)$ (easy exercise). ■

In the next section, we will show that $H(3, q^2)$ with q even satisfies the assumptions of Theorem 4.2, therefore providing a "concrete" answer to the question of S. E. Payne.

5 An Example of Theorem 4.2: $H(3, q^2)$, q even

Consider $\mathcal{S} \cong H(3, q^2)$, q even, and suppose p is a point of $\mathcal{S}$. We will show that all the assumptions of Theorem 4.2 are satisfied.

Suppose LIp, and let $xILIp \neq x$; then the group of all root-elations $H(x, L, p)$ has size q^2, and is isomorphic to the additive group of $\mathbf{GF}(q^2)$. By putting H equal to the group generated by all such root-elations (so that $(\mathcal{S}^{(p)}, H)$ is an EGQ), the assumptions of Theorem 4.2 are satisfied ($H^2 = Z(H)$ for this H).

Remark 5.1. The previous result was independently obtained by R. Rostermundt [11] in an entirely different fashion. He represents $H(3, q^2)$ (q even) as a group coset geometry in the extra-special group $K = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbf{GF}(q^2), c \in \mathbf{GF}(q)\}$, where the group operation is given by

$$(\alpha, c, \beta) \circ (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta\alpha'^T, \beta + \beta').$$

He then constructs $q^2 - 1$ distinct elation groups $K_i = 1, 2, \ldots, q^2 - 1$ of size q^5, and shows that all K_i are mutually isomorphic. The K_i's have nilpotency class 3, while K has nilpotency class 2, so that $K \ncong K_i$ for all i. The proofs are long and technical. For details and several other results, see R. Rostermundt [11].

6 Group and GQ Automorphisms

Suppose $(\mathcal{S}^{(p)}, G)$ is a thick EGQ. Then there is associated a 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ to $\mathcal{S}^{(p)}$, and, conversely, each 4-gonal family yields an EGQ. It is clear that any automorphism of G that fixes $\mathcal{J}$ as a set — and then also $\mathcal{J}^*$ — induces in a natural way an automorphism of $\mathcal{S}^{(p)}$ fixing p. It is therefore a basic question whether the converse holds:

QUESTION. *Is any automorphism of $\mathcal{S}^{(p)}$ fixing p induced by such a group automorphism of G?*

In this section, we answer this question by constructing a class of counter examples.

For translation generalized quadrangles, an answer to a stronger version of this question is known. Recall first from Chapter 8 of [9] that a *generalized ovoid* $\mathcal{O}(n, m, q)$ of $\mathbf{PG}(2n+m-1, q)$ is a set of $q^m + 1$ $(n-1)$-dimensional spaces, denoted $\mathbf{PG}^{(i)}(n-1, q)$, $i \in \{0, 1, \ldots, q^m\}$, so that

(i) every three generate a $\mathbf{PG}(3n-1, q)$,

(ii) for every $i \in \{0, 1, \ldots, q^m\}$ there is a subspace $\mathbf{PG}^{(i)}(n+m-1, q)$ of $\mathbf{PG}(2n+m-1, q)$ of dimension $n+m-1$, which contains $\mathbf{PG}^{(i)}(n-1, q)$ and which is disjoint from each $\mathbf{PG}^{(j)}(n-1, q)$ if $j \neq i$.

In [9] it is shown that from $\mathcal{O}(n, m, q) = \mathcal{O}$ can be constructed a TGQ $T(\mathcal{O})$ of order (q^n, q^m), and given a TGQ $\mathcal{S}$ there is an $\mathcal{O}(n, m, q) = \mathcal{O}$ so that $\mathcal{S} \cong T(\mathcal{O})$. Whence *any TGQ can be represented in a projective space.* The following theorem was independently obtained in L. Bader, G. Lunardon and I. Pinneri [1] and J. A. Thas and K. Thas [13].

Theorem 6.1 ([1]; [13]). *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q^n, q^m) with translation point (∞), and let $\mathbf{GF}(q)$ be a subfield of the kernel $\mathbf{GF}(q')$ of $T(\mathcal{O})$, where $\mathcal{O}$ is a generalized ovoid in $\mathbf{PG}(2n+m-1, q) \subseteq \mathbf{PG}(2n+m, q)$. Then every automorphism of $\mathcal{S}$ which fixes (∞) is induced by an automorphism of $\mathbf{PG}(2n+m, q)$ which fixes $\mathcal{O}$, and conversely.* ■

Whence there is a very satisfactory treatment for TGQ's.

We now return to the original problem posed in the beginning of this section. First of all, we note that if ϕ is an element of $Aut(\mathcal{S})_p$, then ϕ is induced by an automorphism of G if and only if ϕ fixes G under conjugation in $Aut(\mathcal{S})$, that is, *if and only if $G^\phi = G$.* Now suppose $\mathcal{S} = (\mathcal{S}^{(p)}, H)$ satisfies the hypotheses of §4, and suppose S_2 is the Sylow 2-subgroup of W which is generated by H and H^-. Suppose $\alpha \neq \mathbf{1}$ is an involution in W that fixes a subGQ of order q pointwise, and that is not contained in S_2 (this is an extra hypothesis!). Then $S_2^W \neq S_2$, and as $H^W = H$, it follows that $H^{-W} \neq H^-$. So there are elements in $W \leq Aut(\mathcal{S})_p$ which are not induced by automorphisms of H^-.

Again, $H(3, q^2)$ with q even is an example.

7 Final Remark (On Property (F), and Kantor's Conjecture)

Let $\mathcal{S} = (\mathcal{S}^{(p)}, H) = \mathcal{S}(G, \mathcal{J})$ be an EGQ (using the notation of §2) of order (s, t), $s, t > 1$. We introduce Property (F) as follows:

PROPERTY (F). *For each $H_i^* \in \mathcal{J}^*$ we have $H_i^* \trianglelefteq H$.*

Each known EGQ $(\mathcal{S}^{(p)}, H)$ (up to now) satisfies this property. We now show that the example $(\mathcal{S}^{(p)}, H^-)$ constructed in §4, and in particular $(H(3, q^2)^{(p)}, H^-)$ with q even, does not have Property (F).

Proof. First note that $(H(3, q^2)^{(p)}, H)$ has (F). Property (F) is satisfied if and only if for each $x \sim p \neq x$, H_x fixes px pointwise. Clearly an element of the form $\theta\phi \in H^-$, with $\phi \in H_z \setminus H_z^-$, $z \sim p \neq z$, does not have this property. ■

CONJECTURE. *If Property (F) does not hold for $(\mathcal{S}^{(x)}, G)$, then $\mathcal{S}$ (which has order (s, t)) has non-isomorphic (full) elation groups, and $\mathcal{S}$ has a subGQ of order $(s/t, t)$ fixed pointwise by some nontrivial collineation (possibly under some mild extra assumption).*

This conjecture is closely related to Kantor's fundamental conjecture which states that a group admitting a 4-gonal family necessarily is a p-group. The author is working on both conjectures at present [15].

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Ghent University, Department of Pure Mathematics
and Computer Algebra, Krijgslaan 281, S22, B-9000 Ghent,
Belgium, E-mail: `kthas@cage.UGent.be`