Combinatorial and geometrical properties of a class of tilings

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Abstract

In this paper, we consider a tiling generated by a Pisot unit number of degree $d \geq 3$ which has a finite expansible property. We compute the states of a finite automaton which recognizes the boundary of the central tile. We also prove in the case $d = 3$ that the interior of each tile is simply connected.

1 Introduction

Let $\beta > 1$ be a real number. A *β-representation* of a real number $x \geq 0$ is an infinite sequence $(a_i)_{k \geq i > -\infty}$, $a_i \in \mathbb{N}$, such that

$$x = a_k\beta^k + a_{k-1}\beta^{k-1} + \cdots + a_1\beta + a_0 + a_{-1}\beta^{-1} + a_{-2}\beta^{-2} + \cdots$$

for a certain integer $k \geq 0$. It is denoted by

$$x = a_k a_{k-1} \ldots a_1 a_0 . a_{-1} a_{-2} \ldots$$

A particular β-representation, called *β-expansion*, is computed by the "greedy algorithm" (see [4] and [5]): denote by $\lfloor y \rfloor$ and $\{y\}$ respectively the integer part and the fractional part of a number y. There exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$. Then for $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$ and $r_i = \{\beta r_{i+1}\}$. We get

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$$

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If $k < 0$ (i.e., if $x < 1$), we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. If an expansion ends with infinitely many zeros, it is said to be *finite*, and the ending zeros are omitted.

The digits x_i's computed by the previous algorithm belong to the set $A = \{0, \ldots, \beta - 1\}$ if β is an integer, or to the set $A = \{0, \ldots, \lfloor\beta\rfloor\}$ if β is not an integer. We will sometimes omit the splitting point between the integer part and the fractional part of the β-expansion; then the infinite sequence is just an element of $A^{\mathbb{N}}$.

For the numbers $0 \leq x < 1$, the expansion defined above coincides with the β-representation of Rényi [10], which can be defined by means of the β transformation of the unit interval

$$T_\beta(x) = \{\beta x\},\ x \in [0, 1].$$

For $x \in [0, 1)$, we have $x_{-j} = \lfloor \beta T_\beta^{j-1}(x) \rfloor$ for $j = 1, 2, \ldots$.

Remark 1.1. *For $x = 1$ the two algorithms differ. The β-expansion of 1 is just $1 = 1.0000\ldots$, while the Rényi β-representation of 1 is*

$$d(1, \beta) = .t_{-1}t_{-2}\ldots,$$

where

$$t_{-j} = \lfloor \beta T_\beta^{j-1}(1) \rfloor,\ \forall j \geq 1.$$

Let $Fin(\beta)$ be the set of nonnegative real numbers which have a finite β-expansion. We will sometimes denote a finite β-expansion $x_n \ldots x_k$, $k \leq n$, by $(x_i)_{n \geq i \geq k}$. We denote the set of finite β-expansions by F_β. We put

$$E_\beta = \{(x_i)_{i \geq k},\ k \in \mathbb{Z} \mid \forall n \geq k, (x_i)_{n \geq i \geq k} \in F_\beta\}.$$

We say that β has a finite expansible property and we denote this by (F) if

$$\mathbb{Z}[\beta] \cap [0, +\infty) = Fin(\beta),$$

where $\mathbb{Z}[\beta]$ is the ring generated by $\mathbb{Z}$ and β. If β satisfies the property (F), then $d(1, \beta)$ is finite, because $\beta - \lfloor\beta\rfloor \in Fin(\beta)$.

We say that β is a *Pisot number* if β is an algebraic integer number whose all Galois conjugates have modulus less than one. Moreover, if

$$X^d + b_{d-1}X^{d-1} + \cdots + b_0$$

is the minimal polynomial of β then β is said to be a *unit* Pisot number if $b_0 = \pm 1$. In the following, we assume that $\beta = \beta_1$ is a Pisot unit number of degree $d \geq 3$. We denote by $\beta_2, \ldots, \beta_r$ the real Galois conjugates of β and by $\beta_{r+1}, \ldots, \beta_{r+s}, \beta_{r+s+1} = \overline{\beta_{r+1}}, \ldots, \beta_{r+2s} = \overline{\beta_{r+s}}$ its complex Galois conjugates. We also assume that β has the property (F) and that $d(1, \beta) = .a_{-1} \ldots a_{-t}$, where $a_{-t} \neq 0$. We have $a_{-1} = \lfloor\beta\rfloor$.

Let $\psi = (\beta_2, \ldots, \beta_{r+s})$. We denote $(a\beta_2^i, \ldots, a\beta_{r+s}^i)$ by $a\psi^i$ for all $i \in \mathbb{Z}$ and $a \in \mathbb{Z}$. If B is a subset of $\mathbb{Z}$ and $i \in \mathbb{Z}$, then we denote by $B\psi^i$ the set $\{b\psi^i \mid b \in B\}$.

If $.x_{-1} \ldots x_{-N}$ is a finite β-expansion, we put

$$\mathcal{K}_{\cdot x_{-1} \ldots x_{-N}} = \{\sum_{i=-N}^{+\infty} d_i\psi^i \mid (d_i)_{i \geq -N} \in E_\beta,\ d_i = x_i,\ \forall i = -N, -N+1, \ldots, -1\}$$

and call it a *tile*. We denote $\psi^N \mathcal{K}_{\cdot x_{-1} \ldots x_{-N}}$ by $\mathcal{K}_{x_{-1} \ldots x_{-N}.}$ and $\mathcal{K}_{\cdot 0}$ by $\mathcal{K}$. We call $\mathcal{K}$ the *central tile*. It is known that the central tile $\mathcal{K}$ induces a periodic tiling of $\mathbb{R}^{r-1} \times \mathbb{C}^s$.

Proposition 1. *(see [1],[2], [3] and [8]) The tiles are compact sets of* $\mathbb{R}^{r-1} \times \mathbb{C}^s$ *and satisfy the following properties.*

1. *Every tile intersects a finite number of different tiles.*

2. *The Lebesgue measure of the intersection of two different tiles is zero.*

3. *The intersection of a tile with the interior of another tile is empty.*

4. *If* $x = \sum_{i=-N}^{M} d_i\psi^i$ *is an element of a tile* $\mathcal{K}_{\cdot x_{-1}\ldots x_{-N}}$*, then* x *is an interior point of this tile. In particular,* 0 *is an interior point of the central tile.*

5. *If* $a_{-t} = 1$*, then the tiles are arcwise connected sets.*

In this paper we study the boundary of the tiles. In particular, we compute the states of a finite automaton that recognizes the boundary of the central tile. We also prove that in the case $d = 3$ the interior of each tile is simply connected. This generalizes a result of Rauzy (see [9]) which was done in the case of β satisfying the relation $\beta^3 - \beta^2 - \beta - 1 = 0$.

2 Notations and definitions

We denote by $|| \ ||$ the norm in $\mathbb{R}^{r-1} \times \mathbb{C}^s$ defined by

$$||(x_1, \ldots, x_{r-1}, z_1, \ldots, z_s)|| = max\{|x_i|,\ |z_j| \mid i = 1, \ldots, r-1,\ j = 1, \ldots, s\}$$

where $|x_i|$ is the absolute value of x_i and $|z_j|$ is the modulus of z_j.
Let $z = (z_2, \ldots, z_{r+s}) \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ and $i \in \mathbb{Z}$, we denote $(z_2\beta_2^i, \ldots, z_{r+s}\beta_{r+s}^i)$ by $z\psi^i$.
Let $\mathcal{Z}$ be a subset of $\mathbb{R}^{r-1} \times \mathbb{C}^s$. We denote by $diam(\mathcal{Z})$ the diameter of $\mathcal{Z}$, by $int(\mathcal{Z})$ the interior of $\mathcal{Z}$, by $\partial(\mathcal{Z})$ the boundary of $\mathcal{Z}$ and by $\psi^i\mathcal{Z}$ the set $\{z\psi^i \mid z \in \mathcal{Z}\}$ for all $i \in \mathbb{Z}$.

Let X be a finite and non-empty set. Let $X^{\mathbb{N}}$ be the set of infinite sequences on X. An *automaton* over X is an oriented graph denoted by $\mathcal{A} = (V, X, E, I, T)$ with edges labelled by the elements of X where V is the set of vertices, called states, $I \subset V$ is the set of initial states, $T \subset V$ is the set of terminal states, and $E \subset V \times X \times V$ is the set of labelled edges. The automaton is said to be finite if V is a finite set. All states of the automata considered in this paper are final. Let $(a_n)_{n\geq 0} \in X^{\mathbb{N}}$; we say that the automaton $\mathcal{A}$ *recognizes* $(a_n)_{n\geq 0}$ if there exists a sequence of states $(q_n)_{n\geq 0}$ such that q_0 is an initial state and for all $n \geq 1$, q_n is a final state satisfying $(q_{n-1}, a_{n-1}, q_n) \in E$. For more information about automata, see [11].

A subset Y of $X^{\mathbb{N}}$ is said to be *recognized* by a finite automaton if there exists a finite automaton such that Y is exactly the set of sequences recognized by the automaton.

A subset $\mathcal{C}$ of $\mathcal{K}$ is said to be *recognized* by a finite automaton if the set $\{(d_i)_{i\geq 0} \in E_\beta \mid \sum_{i=0}^{+\infty} d_i\psi^i \in \mathcal{C}\}$ is recognized by a finite automaton.

3 Boundary of $\mathcal{K}$

Proposition 2. *Let $x = \sum_{i=0}^{+\infty} \varepsilon_i \psi^i$ and $y = \sum_{i=0}^{+\infty} \varepsilon'_i \psi^i$ where $(\varepsilon_i)_{i\geq 0}$, $(\varepsilon'_i)_{i\geq 0} \in E_\beta$. Then $x = y$ if and only if there exists $M = M(\beta) \in \mathbb{N}$ such that the set $\{\sum_{i=0}^{k}(\varepsilon_i - \varepsilon'_i)\psi^{i-k} \mid k \geq 0\}$ is included in the set $\{\pm \sum_{i=-M}^{0} c_i\psi^i \mid (c_i)_{0\geq i\geq -M} \in F_\beta\}$.*

Lemma 1. *Let $x_0.x_{-1}\ldots x_{-n}$ be a finite β-expansion. Then $x_0 + x_{-1}/\beta + \cdots + x_{-n}/\beta^n < \beta$.*

Proof. The proof is a direct consequence of the greedy algorithm (see [7]). ■

Proof of Proposition 2. Assume that $x = y$ and put $A_k = \sum_{i=0}^{k}(\varepsilon_i - \varepsilon'_i)\psi^{i-k}$. Assume that $A_k \neq 0$. Since β satisfies the property (F), there exists a finite β-expansion $(c_i)_{L\geq i\geq -M}$ with $c_L \neq 0$ such that $\sum_{i=0}^{k}(\varepsilon_i - \varepsilon'_i)\beta^{i-k} = \pm\sum_{i=-M}^{L} c_i\beta^i$. Now assume without loss of generality that $\sum_{i=0}^{k}(\varepsilon_i - \varepsilon'_i)\beta^{i-k} = \sum_{i=-M}^{L} c_i\beta^i$. Let h be an integer such that $h > max(k, M)$. Put $P(x) = x^h(\sum_{i=0}^{k}(\varepsilon_i - \varepsilon'_i)x^{i-k} - \sum_{i=-M}^{L} c_i x^i)$. Then $P(x)$ is a polynomial with integer coefficients satisfying $P(\beta) = 0$. Then for all Galois conjugates γ of β we have $P(\gamma) = 0$. Hence

$$A_k = \sum_{i=-M}^{L} c_i\psi^i.$$

Since

$$\sum_{i=0}^{k} \varepsilon_i\beta^{i-k} = \sum_{i=0}^{k} \varepsilon'_i\beta^{i-k} + \sum_{i=-M}^{L} c_i\beta^i \tag{1}$$

we should have $\sum_{i=0}^{k} \varepsilon_i\beta^{i-k} \geq \beta^L$. Therefore $L \leq 0$, otherwise we have

$$\sum_{i=0}^{k} \varepsilon_i\beta^{i-k} \geq \beta.$$

This latter inequality contradicts Lemma 1, because $\varepsilon_k\ldots\varepsilon_0$ is a finite β-expansion. On the other hand, since $x = y$, $A_k = \sum_{i=k+1}^{+\infty}(\varepsilon'_i - \varepsilon_i)\psi^{i-k} = \sum_{i=1}^{+\infty} \varepsilon'_{k+i}\psi^i - \sum_{i=1}^{+\infty} \varepsilon_{k+i}\psi^i$; then there exists a fixed constant $c(\beta) = c > 0$ such that $||A_k|| < c$. Hence

$$|\sum_{i=0}^{k}(\varepsilon_i - \varepsilon'_i)\beta_j^{i-k}| < c,\ \forall j = 2,\ldots,r+2s,\ \forall k \geq 0. \tag{2}$$

Now put for all $k \geq 0$, $z_k = \sum_{i=0}^{k}(\varepsilon_i - \varepsilon'_i)\beta^{i-k}$. Since β is a Pisot unit number, $1/\beta$ is an algebraic integer, hence for all $k \geq 0$, z_k is an algebraic integer of $\mathbb{Q}(\beta)$. The Galois conjugates of z_k are contained in the set $\{\sum_{i=0}^{k}(\varepsilon_i - \varepsilon'_i)\beta_j^{i-k} \mid j = 2,\ldots,r+2s\}$. By (2) and the fact that z_k is bounded by a fixed constant independent of k, we deduce that the set Γ constituted of all z_k and their Galois conjugates is bounded independently of k. This implies that there exists a fixed constant l independent of k such that for all $k \geq 0$, the coefficients of the minimal polynomial of z_k are bounded by l. Since these coefficients are integer numbers, we deduce that the set of minimal polynomial of all z_k is finite. Hence the set Γ is finite. Thus the set

$\{A_k \mid k \geq 0\}$ is finite. Then M is a finite integer independent of k. This ends the proof of the direct implication.

Now assume that the set $\{\sum_{i=0}^{k}(\varepsilon_i - \varepsilon_i')\psi^{i-k} \mid k \geq 0\}$ is included in the set $\{\pm\sum_{i=-M}^{0} c_i\psi^i \mid (c_i)_{0\geq i\geq -M} \in F_\beta\}$, then there exists $d > 0$ such that $||\sum_{i=0}^{k}(\varepsilon_i - \varepsilon_i')\psi^{i-k}|| < d$ for all $k \geq 0$. Hence for all $k \geq 0$, $||\sum_{i=0}^{k}(\varepsilon_i - \varepsilon_i')\psi^i|| < d||\psi||^k$. Since $||\psi|| < 1$, we obtain $\sum_{i=0}^{+\infty}\varepsilon_i\psi^i = \sum_{i=0}^{+\infty}\varepsilon_i'\psi^i$.

■

Theorem 1. *The boundary of $\mathcal{K}$ is recognized by a finite automaton whose set of states is contained in the product set $\{\pm\sum_{i=-M-1}^{-1} c_i\psi^i \mid (c_i)_{-1\geq i\geq -M-1} \in F_\beta\}\times A\times A$ where M is a fixed nonnegative integer number and $A = \{0,\dots,\lfloor\beta\rfloor\}$.*

Lemma 2. *Let $x \in \mathbb{R}^{r-1}\times\mathbb{C}^s$, then $x \in \partial(\mathcal{K})$ if and only if there exist $N = N(\beta) < 0$ and $l \in \{N,\dots,-1\}$ such that $x = \sum_{i=0}^{+\infty}\varepsilon_i\psi^i = \sum_{i=l}^{+\infty}\varepsilon_i'\psi^i$, where $(\varepsilon_i)_{i\geq 0}$, $(\varepsilon_i')_{i\geq l} \in E_\beta$ and $\varepsilon_l' \neq 0$.*

Proof. Assume that $x \in \partial(\mathcal{K})$. Since $x \notin int(\mathcal{K})$, for all $\tau > 0$ there exists $y \notin \mathcal{K}$ such that $||x - y|| < \tau$. Hence there exists a sequence $(y_n)_{n\geq 1}$ such that for all $n \geq 1$, $||x - y_n|| < 1/n$ and $y_n \notin \mathcal{K}$. Since there exists a finite number of tiles intersecting $\mathcal{K}$ (item 1 of Proposition 1), we deduce that there exists a tile $\mathcal{K}' = \mathcal{K}_{.\varepsilon_{-1}'\dots\varepsilon_l'} \neq \mathcal{K}$ and a subsequence $(y_{p_n})_{n\geq 0}$ such that for all $n \geq 0$, $||x - y_{p_n}|| < 1/p_n$ and $y_{p_n} \in \mathcal{K}'$. Hence $\lim_{n\mapsto+\infty} y_{p_n} = x$. Since $\mathcal{K}'$ is a compact set, $x \in \mathcal{K}'$, hence $x \in \mathcal{K}\cap\mathcal{K}'$. The number l is limited independently of x because of item 1 of Proposition 1. This proves the direct implication.

Assume that $x = \sum_{i=0}^{+\infty}\varepsilon_i\psi^i = \sum_{i=l}^{+\infty}\varepsilon_i'\psi^i$ where $l < 0$, then $x \in \mathcal{K}\cap\mathcal{K}_{.\varepsilon_{-1}'\dots\varepsilon_l'}$. If $x \in int(\mathcal{K})$, then there exists a real number $r_1 > 0$ such that $B(x, r_1) = \{z \in \mathbb{R}^{r-1}\times\mathbb{C}^s \mid ||z - x|| < r_1\} \subset \mathcal{K}$. Put for all $n \in \mathbb{N}$, $z_n = \sum_{i=l}^{n}\varepsilon_i'\psi^i$. Since the sequence z_n converges to x and z_n is an interior point of $\mathcal{K}_{.\varepsilon_{-1}'\dots\varepsilon_l'}$ (item 4 of Proposition 1), there exists a positive integer n and a real number $r_2 > 0$ such that

$$z_n \in B(x, r_1) \text{ and } B(z_n, r_2) \subset \mathcal{K}_{.\varepsilon_{-1}'\dots\varepsilon_l'}.$$

Then there exists $\delta > 0$ such that $B(z_n, \delta) \subset \mathcal{K}\cap\mathcal{K}_{.\varepsilon_{-1}'\dots\varepsilon_l'}$; this is a contradiction because the Lebesgue measure of $\mathcal{K}\cap\mathcal{K}_{.\varepsilon_{-1}'\dots\varepsilon_l'}$ is zero (item 2 of Proposition 1). This ends the proof. ■

Beginning of the proof of Theorem 1. Let l be a negative integer. Put

$$D_l = \{(\varepsilon_i)_{i\geq 0} \in E_\beta \mid \exists(\varepsilon_i')_{i\geq l} \in E_\beta\ ;\ \varepsilon_l' \neq 0,\ \sum_{i=0}^{+\infty}\varepsilon_i\psi^i = \sum_{i=l}^{+\infty}\varepsilon_i'\psi^i\},$$

$E_l = \{(\varepsilon_i, \varepsilon_i')_{i\geq l} \mid (\varepsilon_i)_{i\geq l}, (\varepsilon_i')_{i\geq l} \in E_\beta,\ \varepsilon_i = 0,\ \forall l \leq i \leq -1\ , \varepsilon_l' \neq 0\ , \sum_{i=0}^{\infty}\varepsilon_i\psi^i = \sum_{i=l}^{\infty}\varepsilon_i'\psi^i\}$ and

$$V_l = \{\sum_{i=l}^{k}(\varepsilon_i - \varepsilon_i')\psi^{i-k} \mid k \geq l,\ (\varepsilon_i, \varepsilon_i')_{i\geq l} \in E_l\}.$$

By Proposition 2, the set V_l is finite. First, put $l = -1$ and assume that $V_{-1}\cap A\psi^0 \neq \emptyset$. Let $y_{-1} \in A$ such that $y_{-1}\psi^0 \in V_{-1}\cap A\psi^0$. Put $x_{-1} = 0$ and $A_{-1} = y_{-1}\psi^0$. Hence

$$A_{-1} = 0\psi^{-1} + (y_{-1} - x_{-1})\psi^0.$$

Now, consider the equation in $(X, a, b) \in V_{-1} \times A \times A$ defined by:

$$X = A_{-1}\psi^{-1} + (b-a)\psi^0 \tag{3}$$

Let $(A_0, x_0, y_0) \in V_{-1} \times A \times A$. If (A_0, x_0, y_0) is a solution of (3) and

$$x_0/\beta + x_{-1}/\beta^2 < 1,\ y_0/\beta + y_{-1}/\beta^2 < 1 \tag{4}$$

then we put an edge from (A_{-1}, x_{-1}, y_{-1}) to (A_0, x_0, y_0) and label it by x_0. The relation (4) guarantees that the words x_0x_{-1} and y_0y_{-1} are finite β-expansions. Now assume that we have constructed the sequence $(A_i, x_i, y_i),\ -1 \le i \le m$. If $(A_{m+1}, x_{m+1}, y_{m+1})$ is a solution of the equation $X = A_m\psi^{-1} + (b-a)\psi^0$ and $x_{m+1}/\beta + \cdots + x_{-1}/\beta^{m+3} < 1,\ y_{m+1}/\beta + \cdots + y_{-1}/\beta^{m+3} < 1$, then we put an edge from (A_m, x_m, y_m) to $(A_{m+1}, x_{m+1}, y_{m+1})$ and label it by x_{m+1}. If we continue on, we obtain an automaton that we denote by $\mathcal{A}_{-1}$. Since V_{-1} is a finite set, the automaton $\mathcal{A}_{-1}$ is finite.

Lemma 3. *The automaton $\mathcal{A}_{-1}$ recognizes the set D_{-1}.*

Proof. Let $(\varepsilon_i)_{i\ge 0} \in D_{-1}$; then there exists $(\varepsilon'_i)_{i\ge -1} \in E_\beta$ such that $\varepsilon'_{-1} \neq 0$ and $\sum_{i=0}^{\infty} \varepsilon_i\psi^i = \sum_{i=-1}^{\infty} \varepsilon'_i\psi^i$. Put $\varepsilon_{-1} = 0$ and $B_k = \sum_{i=-1}^{k} (\varepsilon'_i - \varepsilon_i)\psi^{i-k}$ for all $k \ge -1$. We have $B_{-1} = \varepsilon'_{-1}\psi^0$ and by induction $B_k = B_{k-1}\psi^{-1} + (\varepsilon'_k - \varepsilon_k)\psi^0,\ \forall k \in \mathbb{N}$. Hence the sequence $(\varepsilon_i)_{i\ge 0}$ is recognized by the automaton $\mathcal{A}_{-1}$.

Now let $(x_i)_{i\ge 0}$ be a sequence recognized by the automaton $\mathcal{A}_{-1}$; then there exists a sequence $(y_i)_{i\ge -1} \in E_\beta$ such that $A_{-1} = y_{-1}\psi^0$ and $A_k = A_{k-1}\psi^{-1} + (x_k - y_k)\psi^0$ for all $k \in \mathbb{N}$. Hence for all $k \in \mathbb{N}$,

$$A_k\psi^{k+1} = y_{-1}\psi^0 + \sum_{i=0}^{k} (x_i - y_i)\psi^{i+1}.$$

Since $||\psi|| < 1$ and for all $k \in \mathbb{N},\ A_k \in V_{-1}$ (finite set), we have

$$\lim_{k \mapsto +\infty} A_k\psi^{k+1} = 0.$$

Therefore $\sum_{i=0}^{\infty} x_i\psi^i = \sum_{i=-1}^{\infty} y_i\psi^i$. Thus $(x_i)_{i\ge 0} \in D_{-1}$. This ends the proof of the lemma.

■

End of the proof of Theorem 1. Consider $l < -1$. It is easy to see that a sequence $(x_i)_{i\ge 0}$ belongs to D_l if and only if the associated sequence $(y_i)_{i\ge 0}$, defined by $y_i = 0$ for $i = 0, \ldots, -l-2$ and $y_i = x_{i+l+1}$ for all $i \ge -l-1$, belongs to D_{-1}. Hence we construct an automaton $\mathcal{A}_l$ which recognizes D_l in the following manner: denote by I_l the set of all states $v^{(j)}_{-l-2}$ of $\mathcal{A}_{-1}$ such that there exist an initial state $v^{(j)}_{-1}$ of $\mathcal{A}_{-1}$ and final states $v^{(j)}_0, \ldots, v^{(j)}_{-l-3}$ such that for all $i = -1, \ldots, -l-3$, the edge between $v^{(j)}_i$ and $v^{(j)}_{i+1}$ is labeled by 0. Then if we denote $\mathcal{A}_{-1}$ by $(S, A, E_{-1}, I_{-1}, T_{-1})$, we have $\mathcal{A}_l = (S, A, E_l, I_l, T_l)$ where $E_l = E_{-1}\backslash R_l$ where $R_l = \{(v^{(j)}_i, 0, v^{(j)}_{i+1}) \in E_{-1} \mid -1 \le i \le -l-3\ ,\ v^{(j)}_{-l-2} \in I_l\}$ and T_l is the set of $v \in T_{-1}$ such that there exist $k+1$ states of $\mathcal{A}_{-1}:\ v_{-l-2}, v_{-l-1}, \ldots, v_{-l-2+k} = v$ such that $v_{-l-2} \in I_l$ and for all $i = -l-1, \ldots, -l-2+k,\ (v_{i-1}, a_i, v_i) \in E_l$ for some sequence $(a_i)_{-l-1\le i\le -l-2+k}$ of elements of A.

Let $\mathcal{C} = \{(\varepsilon_i)_{i\geq 0} \in E_\beta \mid \sum_{i=0}^{\infty} \varepsilon_i \psi^i \in \partial(\mathcal{K})\}$. By Lemma 2, we have $\mathcal{C} = \bigcup_{l=N}^{-1} D_l$ where N is the integer given in Lemma 2. Then an automaton which recognizes $\mathcal{C}$ is $\mathcal{L} = (S, A, E, I, T)$ where $I = \bigcup_{l=N}^{-1} I_l,\ T = T_{-1},\ E = E_{-1}$ and $S \subset V_{-1} \times A \times A$. ■

Remark 3.1. *By using the same approach, we can prove that the boundary of every tile is recognized by a finite automaton.*

Remark 3.2. *The interest of automata remains in the fact that they give information for the boundary of compact sets given by numeration systems. For example in the case of β satisfying the relation $\beta^3 - \beta^2 - \beta - 1 = 0$, the central tile (Rauzy fractal) (see [9]) is the set $\mathcal{K} = \{\sum_{i=0}^{+\infty} \varepsilon_i \alpha^i \mid \forall i : \varepsilon_i = 0, 1 \wedge \varepsilon_i \varepsilon_{i+1} \varepsilon_{i+2} = 0\}$, where α is one of the two complex roots of the polynomial $x^3 - x^2 - x - 1$. The automaton which recognizes the boundary of $\mathcal{K}$ helps us to show that this boundary is a Jordan curve and that it is a quasi-circle (image of a circle by a quasi-conformal map) with Hausdorff dimension* 1.0645 *(see [6]). It will be interesting to try to extend these results to other Pisot unit numbers.*

Theorem 2. *If β is a cubic Pisot unit number with the property (F), then the interior of each tile is simply connected.*

Remark 3.3. *The class of β cubic Pisot unit numbers with the property (F) is equal to the class of numbers $\beta > 1$ with minimal polynomial $x^3 - ax^2 - bx - 1 = 0$ where a, b are integer numbers satisfying the property $-1 \leq b \leq a + 1$ and $a + b \geq 1$ (see [2]). This class of real numbers β satisfies also $d(1, \beta) = .a_{-1} \dots a_{-t}$, where $a_{-t} = 1$. Then for this class the tiles are arcwise connected sets (see item 5 of Proposition 1).*

Proof. It suffices to prove the result for the central tile $\mathcal{K}$. We notice that in the case of β cubic Pisot unit number, we have $\psi = \beta_2$ if β is not totally real, and otherwise $\psi = (\beta_2, \beta_3)$.

Let Γ be a Jordan simple and closed curve contained in $int(\mathcal{K})$. Let C be the connected bounded component of Γ (C is the open set delimited by Γ) and C' be the connected unbounded component of Γ. Let us prove that $C \subset int(\mathcal{K})$.
First we shall show that $\psi C \cap \mathcal{K} \subset \psi\mathcal{K}$. Let $z_0 \in \psi C \cap \mathcal{K}$. Assume that $z_0 \notin \psi\mathcal{K}$. Since

$$\mathcal{K} = \bigcup_{i=0}^{\lfloor\beta\rfloor} \mathcal{K}_{i.} \text{ and } \psi\mathcal{K} = \mathcal{K}_{0.},$$

there exists $i_0 \in \{1, \dots, \lfloor\beta\rfloor\}\}$ such that $z_0 \in \mathcal{K}_{i_0.}$. Put

$$r = d(\psi\Gamma, K_2 \backslash int(\psi\mathcal{K})),$$

where

$$d(\mathcal{X}, \mathcal{Y}) = inf\{||x - y|| \mid x \in \mathcal{X},\ y \in \mathcal{Y}\}$$

for every $\mathcal{X}$ and $\mathcal{Y}$ subsets of K_2, where $K_2 = \mathbb{C}$ if β is not totally real, and otherwise $K_2 = \mathbb{R}^2$. Since the set $\psi\Gamma$ is contained in $int(\psi\mathcal{K})$, we have $r > 0$.

Since $\mathcal{K}_{i_0.} \cap int(\psi\mathcal{K}) = \emptyset$, $d(\mathcal{K}_{i_0.}, \psi\Gamma) \geq r$. Since $\mathcal{K}_{i_0.}$ is connected (item 5 of Proposition 1) and $\mathcal{K}_{i_0.} \cap \psi C \neq \emptyset$, we have $\mathcal{K}_{i_0.} \subset \psi C$. Since $\mathcal{K}$ is connected (item 5 of Proposition 1), for all $\varepsilon > 0$ there exist $x_1, \ldots, x_n \in \mathcal{K}$ such that $x_1 = x$, $x_n = y$ and $||x_i - x_{i+1}|| < \varepsilon$ for all $1 \leq i \leq n-1$.

Let $1 \leq j \leq \lfloor\beta\rfloor$, $j \neq i_0$, and $\delta = min\{d(\mathcal{K}_{i.}, \mathcal{K}_{j.}) \mid \mathcal{K}_{i.} \cap \mathcal{K}_{j.} = \emptyset\}$. In both cases $\delta = 0$ and $\delta > 0$, we deduce by taking $\varepsilon = \delta$ (in the second case) that there exist k integers $n_1, \ldots, n_k \in \{1, \ldots, \lfloor\beta\rfloor\}$ such that $n_1 = i_0$, $n_k = j$ and $\mathcal{K}_{n_i.} \cap \mathcal{K}_{n_{i+1}.} \neq \emptyset$ for all $1 \leq i \leq k-1$. Therefore $\mathcal{K}_{j.}$ contains a point of $\psi C \cap \mathcal{K}$. Hence by using the same argument used for $\mathcal{K}_{i_0.}$, we deduce that $\mathcal{K}_{j.} \subset \psi C$, $\forall 1 \leq j \leq \lfloor\beta\rfloor$. Then $\psi C' \cap \mathcal{K} \subset \psi\mathcal{K}$. Let x be an element of $\mathcal{K}$ such that $||x|| = max\{||z|| \mid z \in \mathcal{K}\}$. Then $x \in \psi C'$. Thus $x \in \psi\mathcal{K}$. This is impossible, because in this case we have $x\psi^{-1} \in \mathcal{K}$ and $||x\psi^{-1}|| > ||x||$. Therefore

$$\psi C \cap \mathcal{K} \subset \psi\mathcal{K}. \tag{5}$$

The relation (5) implies that $\psi C \cap \mathcal{K} = \psi C \cap \psi\mathcal{K}$. If we apply the same argument to the curve $\psi^{n-1}\Gamma$, we obtain

$$\forall n \in \mathbb{N}\backslash\{0\},\ \psi^n C \cap \mathcal{K} = \psi^n C \cap \psi\mathcal{K}.$$

Then by induction we have

$$\psi^n C \cap \mathcal{K} = \psi^n C \cap \psi^n\mathcal{K},\ \forall n \in \mathbb{N}\backslash\{0\}.$$

Let $z \in C$. Since $0 \in int(\mathcal{K})$ and $|\psi| < 1$, there exists $n \in \mathbb{N}$ such that $z\psi^n \in \mathcal{K}$. Then $z\psi^n \in \psi^n C \cap \mathcal{K} = \psi^n C \cap \psi^n\mathcal{K}$. Hence $z \in \mathcal{K}$. This implies that $C \subset \mathcal{K}$. Since C is an open set, we have $C \subset int(\mathcal{K})$. ∎

Remark 3.4. *The proof cannot be extended to β with $deg(\beta) = d > 3$, because a Jordan simple and closed curve Γ does not separate the $d-1$ dimensional space $\mathbb{R}^{r-1} \times \mathbb{C}^s$ into two connected components. However if we take Γ as a $d-1$ sphere, using the same proof of Theorem 2, we can show that $C \subset int(\mathcal{K})$.*

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