Multianisotropic Gevrey Regularity and Iterates of Operators with Constant Coefficients

Daniela Calvo Gagik H. Hakobyan*

Abstract

We consider linear partial differential operators with constant coefficients P and show that the inclusion of the Gevrey classes G_P^d defined by the iterates of P in some multianisotropic Gevrey classes implies a growth condition on the symbol of P. Under the hypothesis of hypoellipticity, the converse implication is also true. These results are also related to the regular weight of hypoellipticity, that gives a precise description of the growth of the symbol of P with respect to its derivatives.

1 Introduction

In literature, many results deal with the interplay between the analytic-Gevrey hypoellipticity and the problem of the iterates of an operator, cf. [1], [2], [14], [17], [18], [19], [21] and the bibliography of these works. More recent studies concern the problem of the iterates in relation with the anisotropic Gevrey classes (cf. Zanghirati [25, 24]) and the multianisotropic Gevrey classes (cf. Bouzar-Chaili [3, 4, 5] and Zanghirati [23]).To present our result, we begin by recalling some well known notions. L. Hörmander introduced the concept of hypoellipticity for an operator P, giving start to a wide study on the subject, cf. [15]. In our work we deal with the simpler

*The authors were partially supported by Programma Nazionale M.U.R.S.T. "Teoria e Applicazioni delle Equazioni Iperboliche Lineari e Nonlineari".

Received by the editors October 2003.

Communicated by P. Godin.

1991 *Mathematics Subject Classification :* 35H10, 35B65.

Key words and phrases : iterates of operators, generalized Gevrey classes, hypoelliptic operators.

case of operators with constant coefficients.
We say here that a differential operator with constant coefficients

$$P(D) = \sum_{|\alpha|\leq m} \gamma_\alpha D^\alpha$$

is hypoelliptic if all the distribution solutions of the equation $P(D)u = 0$ in any open set $\Omega \subset \mathbb{R}^n$ are there infinitely differentiable.
Different necessary and sufficient conditions for the hypoellipticity of $P(D)$ have been derived (cf. [15]), we recall in particular the following, that will be useful for our purposes.
An operator with constant coefficients $P(D)$ is hypoelliptic if and only if its symbol $P(\xi) = \sum_{|\alpha|\leq m} \gamma_\alpha \xi^\alpha$ satisfies the condition

$$\left|\frac{D^\alpha P(\xi)}{P(\xi)}\right| \to 0 \quad when \;\; |\xi| \to +\infty,$$

for all $\alpha \in \mathbb{N}^n$, $\alpha \neq 0$.
As a particular case, we start to consider the elliptic operators: namely, a differential operator $P(D)$ of order m is elliptic if its symbol $P(\xi)$ satisfies the growth condition

$$|\xi|^{2m} \leq C(1 + |P(\xi)|^2), \quad \forall \xi \in \mathbb{R}^n \tag{1.1}$$

for some constant $C > 0$. The estimate (1.1) implies the previous necessary and sufficient condition, and therefore elliptic operators are hypoelliptic. Moreover, they are analytic hypoelliptic, namely all the C^∞ (or distribution) solutions of $P(D)u = 0$ are analytic, cf. for instance [15].
Komatsu [17] and Kotake-Narasimhan [18] proved another important consequence of the ellipticity of $P(D)$, concerning the iterates property: we recall here their result.
Let Ω be an open nonempty set of $\mathbb{R}^n$; if an operator $P(D)$ of order m is elliptic, then any function $f \in C^\infty(\Omega)$ is analytic in Ω if and only if for any compact subset K of Ω there exists a constant $C = C(f, K) > 0$ for which it holds

$$\|P^j(D)f\|_K \leq C^{j+1}(j!)^m, \quad \forall j = 1, 2, \ldots, \tag{1.2}$$

where $P^j(D)$ denotes the j-th iterate of the operator $P(D)$ and $\|\cdot\|_K = \|\cdot\|_{L_2(K)}$.
This implies obviously the analytic hypoellipticity of the elliptic operators. The condition (1.2) can be generalized in order to define some Gevrey classes in terms of the operator $P(D)$. We start with the notion of standard Gevrey classes (for their properties and applications we can refer to [22]).
Let Ω be an open subset of $\mathbb{R}^n$ and let $s \in \mathbb{R}$, $s \geq 1$. We say that a function $f \in C^\infty(\Omega)$ belongs to the Gevrey class $G^s(\Omega)$ if for any compact subset $K \subset \Omega$ there is a constant $C > 0$ such that

$$\|D^\alpha f\|_K \leq C^{|\alpha|+1}\alpha!^s, \qquad \forall \alpha \in \mathbb{N}^n.$$

Then we can introduce the Gevrey classes $G^d_P(\Omega)$ defined by the iterates of an operator $P(D)$ of order m.

Let Ω be an open subset of $\mathbb{R}^n$ and $d \in \mathbb{R}$, $d > 0$. We say that a function $f \in C^\infty(\Omega)$ belongs to $G^d_P(\Omega)$ if for any compact subset K of Ω there is a constant $C > 0$ such that

$$\|P^j(D)f\|_K \leq C^{j+1}(j!)^d, \quad \forall j = 1, 2, \ldots.$$

The inclusion $G^s \subset G^{sm}_P$ is always satisfied by any operator of order m (cf. f.i. [2]) and the opposite inclusion is implied by the ellipticity (cf. Bolley-Camus [1]). Conversely, Metivier [19] proved that for any $s > 1$ the condition $G^{sm}_P(\Omega) = G^s(\Omega)$ is equivalent to the ellipticity of $P(D)$.

The inclusions of the Gevrey classes G^d_P in G^s for some s are also related to some growth conditions on the symbol of P. In particular, for hypoelliptic operators Neweberger-Zielezny [21] proved the equivalence of the inequality

$$|Q(\xi)| \leq C(1 + |P(\xi)|^{\frac{1}{d}}), \quad \forall \xi \in \mathbb{R}^n$$

(for a $d > 0$) and the inclusion $G^d_P \subset G_Q = G^1_Q$ and more generally $G^{sd}_P \subset G^s_Q$ for large s. In particular, if Q is elliptic of order m, the previous inequality reads

$$|\xi|^m \leq C(1 + |P(\xi)|^{\frac{1}{d}}), \quad \forall \xi \in \mathbb{R}^n,$$

and we have $G^{sd}_P(\Omega) \subset G^s_Q(\Omega) = G^{\frac{s}{m}}(\Omega)$ for large s.

It is also interesting to study the relation between the inclusion of G^d_P in some generalized Gevrey classes and growth conditions on $P(\xi)$: namely, we introduce the multianisotropic Gevrey classes (cf. also [6], [7], [8], [12], [11], [10]), that explain properly the regularity of the solutions of the hypoelliptic operators. They are related to completely regular polyhedra, of which we begin to give a rough idea. A completely regular polyhedron is a convex polyhedron $\mathcal{N}$ in $\mathbb{R}^n_+$ having vertices with rational coordinates and such that the outer normals of the faces of $\mathcal{N}$ have strictly positive components (cf. Definition 2.3). Now the multianisotropic Gevrey classes $G^{\mathcal{N}}(\Omega)$, for any open set $\Omega \subset \mathbb{R}^n$, are defined by the following condition (cf. Definition 2.10).

A function $f \in C^\infty(\Omega)$ belongs to the multianisotropic Gevrey class $G^{\mathcal{N}}(\Omega)$ if for any compact subset K of Ω there is a constant $C > 0$ such that

$$\|D^\alpha f\|_K \leq C^{j+1} j!, \quad \forall \alpha \in \mathcal{N}(j) \cap \mathbb{N}^n, j = 0, 1, \ldots,$$

where $\mathcal{N}(j) = \{\nu \in \mathbb{R}^n_+ : \frac{\nu}{j} \in \mathcal{N}\}$.

Then we will prove our main results (cf. Theorems 3.1 and 3.2).

Let $P(\xi)$ be a polynomial (or $P(D)$ an operator), $\mathcal{N}$ be a completely regular polyhedron. If there is a constant $d > 0$ such that $G^d_P(\Omega) \subset G^{\mathcal{N}}(\Omega)$, then for a constant $C > 0$ we have

$$h_{\mathcal{N}}(\xi) \leq C\left(|P(\xi)|^{\frac{1}{d}} + 1\right), \quad \forall \xi \in R^n, \tag{1.3}$$

where $h_{\mathcal{N}}(\xi) = \sum_{\alpha \in \mathcal{N}^0} |\xi^\alpha|$, the sum ranging over $\mathcal{N}^0$, the set of the vertices of $\mathcal{N}$. Conversely, the condition (1.3) *implies, under the hypothesis that P is hypoelliptic, the inclusion $G^d_P(\Omega) \subset G^{\mathcal{N}}(\Omega)$.*

To connect with the above mentioned result of Newberger-Zielezny [21], take as $\mathcal{N}$ the Newton polyhedron of an elliptic operator Q, for which $h_{\mathcal{N}} \sim 1 + |\xi|^m$; we recapture $G^{sd}_P(\Omega) \subset G^{\mathcal{N}(s)}(\Omega) = G^s_Q(\Omega) = G^{\frac{s}{m}}(\Omega)$ for large s.

Our results are also related to an important feature of hypoelliptic operators, represented by the regular weight of hypoellipticity (see Definition 2.6) introduced by Kazharyan [16] and studied also by Hakobyan-Markaryan [11, 12, 13], as expressed in Corollary 3.4.
In the particular case that $\mathcal{N}$ is the Newton polyhedron of P, this subject was studied by Zanghirati [23, 24, 25] and by Bouzar-Chaili [3, 4, 5].

2 Definitions and preliminary results

We shall use the subsets of $\mathbb{R}^n$ defined by $\mathbb{R}^n_0 = \{\xi \in \mathbb{R}^n : \xi_1 \dots \xi_n \neq 0\}$ and $\mathbb{R}^n_+ = \{\xi \in \mathbb{R}^n : \xi_j \geq 0, j = 1, \dots, n\}$. If $\mathbb{N}^n = \{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i \in \mathbb{N} \cup \{0\}, i = 1, \dots, n\}$ is the set of multiindeces, then we denote $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, for all $\xi \in \mathbb{R}^n$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ for any $\alpha \in \mathbb{N}^n$, where $D_j = -i\frac{\partial}{\partial \xi_j}$, $j = 1, \dots, n$.

Definition 2.1. *Let $A = \{\nu^k \in R^n_+, k = 0, \dots, m\}$ be a finite set of points of $\mathbb{R}^n_+$. The characteristic polyhedron (or Newton polyhedron) (C.P.) $\mathcal{N}_A$ of the set A is the smallest convex polyhedron in R^n_+ containing all the points $A \cup \{0\}$.*

Now let $P(D) = \sum_\alpha \gamma_\alpha D^\alpha$ be a linear differential operator with constant coefficients, and let $P(\xi)$ be its symbol. We denote $(P) = \{\alpha : \alpha \in \mathbb{N}^n, \gamma_\alpha \neq 0\}$.

Definition 2.2. *The characteristic polyhedron (or Newton polyhedron) (C.P.) $\mathcal{N} = \mathcal{N}_P$ of an operator $P(D)$ (or of a polynomial $P(\xi)$) is $\mathcal{N}_{(P)}$, i.e. the smallest convex polyhedron in $\mathbb{R}^n_+$ containing all the points $(P) \cup \{0\}$.*

Now we pass to consider an important class of convex polyhedra.

Definition 2.3. *A convex polyhedron $\mathcal{N} \subset \mathbb{R}^n_+$ is completely regular (C.R.) if it satisfies the following conditions:*

1. *all the vertices have rational coordinates;*
2. *the origin $(0, 0, \dots, 0)$ belongs to $\mathcal{N}$;*
3. *$dim(\mathcal{N}) = n$;*
4. *the outer normals to the non-coordinate $(n-1)$-dimensional faces of $\mathcal{N}$ have strictly positive components.*

It is well known that if $P(D)$ is a hypoelliptic operator, then its Newton polyhedron is completely regular, cf. Friberg [9].
Let $\eta \in \mathbb{R}^n_+$, we set

$$H(\eta) = \{\xi \in \mathbb{R}^n_+ : \xi \neq 0, \xi \neq \eta,\ \xi_j = \eta_j \text{ or } \xi_j = 0, \forall j = 1, \dots, n\}.$$

Definition 2.4. *A set $B \subset \mathbb{R}^n_+$ is completely regular (C.R.) if for any $\eta \in B$ there exists a neighborhood U of zero such that*

$$(\eta - \xi) + b \cdot sign(\eta - \xi) \in B, \quad \forall \xi \in H(\eta), \forall b \in U \cap \mathbb{R}^n_+, \tag{2.4}$$

where $b \cdot sign(\eta - \xi) = (b_1 sign(\eta_1 - \xi_1), \dots b_n sign(\eta_n - \xi_n))$.

A polyhedron $\mathcal{N} \subset \mathbb{R}^n_+$ is completely regular if and only if it satisfies condition 1 of Definition 2.3 and condition (2.4).

Definition 2.5. *A differential operator $P(D)$ is called regular if its symbol satisfies for a constant $C > 0$*

$$1 + |P(\xi)| \geq C \sum_{\alpha \in (P)} |\xi^\alpha|, \qquad \forall \xi \in \mathbb{R}^n.$$

If an operator has completely regular Newton polyhedron and is regular, then it is called multi-quasi-elliptic and is hypoelliptic (cf. Bouzar-Chaili [3, 4, 5]).
Let $\nu^k \in \mathbb{R}^n_+$, $k = 1, \ldots, m$, $\nu^0 = 0$. We set $h(\xi) = \sum\limits_{k=0}^{m} |\xi^{\nu^k}|$ and we denote by $\mathcal{N}_h$ the characteristic polyhedron of $\{\nu^k\}_{k=0}^m$.
According to Kazharyan [16], we can associate to a hypoelliptic operator $P(D)$ (or polynomial $P(\xi)$) a class of functions, called regular weights of hypoellipticity of P. They are related to important properties of P.

Definition 2.6. *Let $\nu^k \in \mathbb{R}^n_+$, $k = 1, \ldots, m$, $\nu^0 = 0$. A function $h(\xi) = \sum_{j=0}^{k} |\xi^{\nu^j}|$ is called regular weight of hypoellipticity of a polynomial $P(\xi)$ (of an operator $P(D)$) if there exists a constant $C > 0$ such that*

$$F_P(\xi) = \sum_{\alpha \neq 0} \left(\frac{|D^\alpha P(\xi)|}{|P(\xi)| + 1} \right)^{\frac{1}{|\alpha|}} \leq \frac{C}{h(\xi)}, \quad \forall \xi \in \mathbb{R}^n. \tag{2.5}$$

Definition 2.7. *A weight of hypoellipticity $h(\xi)$ of a polynomial $P(\xi)$ (of an operator $P(D)$) is called exact weight of hypoellipticity of $P(\xi)$ (or $P(D)$) if* (2.5) *is satisfied and if for any $\nu \in R^n_+ \backslash \mathcal{N}_h$ it holds*

$$\sup_{\xi \in \mathbb{R}^n} |\xi^\nu| F_P(\xi) = +\infty.$$

We denote

$$\mathcal{M}_P = \{\nu : \nu \in \mathbb{R}^n_+, |\xi|^\nu F_P(\xi) \leq const, \forall \xi \in \mathbb{R}^n\}. \tag{2.6}$$

Kazharyan [16] proved that if an operator $P(D)$ is regular hypoelliptic, then the set $\mathcal{M}_P$ is a completely regular polyhedron. For a class of nonregular hypoelliptic operators, Hakobyan-Markaryan [11] proved that the set $\mathcal{M}_P$ is a completely regular polyhedron. In the general case, we just know that $\mathcal{M}_P$ is a completely regular set.
Let $P(D)$ be a hypoelliptic operator with Newton polyhedron $\mathcal{N}_P$. We denote by $\mathcal{N}_P^0$ the set of the vertices of $\mathcal{N}_P$. For any $t > 0$ we set $\mathcal{N}_P(t) = \{\nu \in \mathbb{R}^n_+ : \frac{\nu}{t} \in \mathcal{N}_P\}$.

Proposition 2.8. *Let $\mathcal{N}$ be a completely regular polyhedron and let $d > 0$ satisfy $\mathcal{N}^0(d) \subset \mathbb{N}^n$. Then there exists a natural number j_0 such that for any $j \geq j_0$ and for any multi-index $\alpha \in \mathcal{N} + \mathcal{N}\left(\frac{j}{d}\right) = \mathcal{N}\left(\frac{d+j}{d}\right)$ there exists a multi-index $\beta \in \mathcal{N}$, $\beta \leq \alpha$, such that $\alpha - \beta \in \mathcal{N}\left(\frac{d}{j}\right)$.*

The proof is similar to Theorem 1.1 of [13].

Definition 2.9. *(cf. [21]) Let Ω be an open subset of $\mathbb{R}^n$ and let $d > 0$. For any differential operator $P(D)$ of order m on $\mathbb{R}^n$, we denote by $G_P^d(\Omega)$ the set of all functions $f \in C^\infty(\Omega)$ such that for every compact subset $K \subset \Omega$ there exists a constant $C > 0$ (depending on f, K and P) for which*

$$\|P^j(D)f\|_K \le C^{j+1}(j!)^d, \quad \forall j = 0, 1, \ldots .$$

Definition 2.10. *Let $\mathcal{N} \subset \mathbb{R}_+^n$ be a completely regular polyhedron. We denote by $G^{\mathcal{N}}(\Omega)$ the multianisotropic Gevrey class associated to $\mathcal{N}$, defined as the set of the functions $f \in C^\infty(\Omega)$ such that for every compact subset $K \subset \Omega$ there exists a constant $C > 0$ (depending on f and K) for which it holds*

$$\|D^\alpha f\|_K \le C^{j+1} j!, \quad \forall \alpha \in \mathcal{N}(j),\ j = 1, 2, \ldots .$$

Definition 2.11. *(cf. [15]) We say that the differential operator $P(D)$ (or the polynomial $P(\xi)$) is stronger than the differential operator $Q(D)$ (or the polynomial $Q(\xi)$) and write $Q \prec P$, if for some constant $C > 0$ it holds*

$$\widetilde{Q}(\xi) \le C\widetilde{P}(\xi), \quad \forall \xi \in \mathbb{R}^n,$$

where

$$\widetilde{R}(\xi) = \sqrt{\sum_{|\alpha|\ge 0} |D^\alpha R(\xi)|^2}$$

is the Hörmander function of the polynomial $R(\xi)$. If $Q \prec P$ and $P \prec Q$, then we write $Q \sim P$.

For any bounded set $\Omega \subset \mathbb{R}^n$ and $\varepsilon > 0$ we denote $\Omega_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > \varepsilon\}$, where ρ is the distance in $\mathbb{R}^n$.

Lemma 2.12. *Let $P(D)$ be a differential operator, $\Omega \subset \mathbb{R}^n$ an open set and l a natural number. Then for any $d > 0$ it is satisfied*

$$G_{P^l}^d(\Omega) = G_P^{\frac{d}{l}}(\Omega).$$

Proof. As the theorem has a local character, then it is possible to consider a bounded set $\Omega \subset \mathbb{R}^n$. Since $P^m \prec P^l$ for $l > m$, then according to Theorem 4.2 of Hörmander [14], there is a constant $\gamma > 0$, such that for every $s \ge 0$, $t > 0$ and for any $v \in C^\infty(\Omega_s)$ it holds

$$\sup_{0<\tau\le t} \tau^\gamma \|P^m(D)v\|_{\Omega_{s+\tau}} \le C\left(\sup_{0<\tau\le t} \tau^\gamma \|P^l(D)v\|_{\Omega_{s+\tau}} + \|v\|_{\Omega_s}\right), \tag{2.7}$$

where $C > 0$ is a constant depending only on P and the diameter of Ω.
From (2.7) it follows that

$$\|P^m(D)v\|_{\Omega_{s+t}} \le C_1(\|P^l(D)v\|_{\Omega_s} + t^{-\gamma}\|v\|_{\Omega_s}), \quad \forall v \in C^\infty(\Omega_s). \tag{2.8}$$

Substituting $j = lj_1 + r$, where $r \le l$, $r = m$, $s = t = \delta > 0$ and $v = P^{lj_1}(D)u$, in (2.8), we obtain

$$\|P^j(D)u\|_{\Omega_{2\delta}} \le C_1\left(\|P^{l(j_1+1)}(D)u\|_{\Omega_\delta} + \delta^{-\gamma}\|P^{lj_1}(D)u\|_{\Omega_\delta}\right). \tag{2.9}$$

If u belongs to $G^d_{P^l}(\Omega)$, then from Definition 2.9 it is satisfied

$$\begin{aligned}&\|P^{l(j_1+1)}(D)u\|_{\Omega_\delta} \le C_2^{j_1+1+1}(j_1+1)^{d(j_1+1)},\\ &\|P^{lj_1}(D)u\|_{\Omega_\delta} \le C_2^{j_1+1} j_1^{dj_1},\end{aligned}$$

where $C_2 > 0$ depends on u, δ and P. From (2.9) we can write

$$\|P^j(D)u\|_{\Omega_{2\delta}} \le C_3^{j_1+1} j_1^{dj_1} = C_3^{\frac{j-r}{l}+1}\left(\frac{j-r}{l}\right)^{d\frac{j-r}{l}} \le C_4^j j^{\frac{d}{l}j},$$

for suitable constants $C_3 = C_3(u,\delta) \ge 1$ and $C_4 = C_4(u,\delta) \ge 1$. Therefore u belongs to $G_P^{\frac{d}{l}}(\Omega)$. The inclusion $G^d_{P^l}(\Omega) \subset G_P^{\frac{d}{l}}(\Omega)$ is proved.
Let u belong to $G_P^{\frac{d}{l}}(\Omega)$. Then for any natural j and for any compact subset $K \subset \Omega$ there is a constant $C_5 = C_5(u, K, P) > 0$ for which it holds

$$\|P^{jl}(D)u\|_K \le C_5^{jl+1}(jl)^{\frac{d}{l}jl} = C_5^{jl+1}(jl)^{dj} \le C_6^{j+1} j^{dj},$$

therefore u belongs to $G^d_{P^l}(\Omega)$. The inclusion $G_P^{\frac{d}{l}}(\Omega) \subset G^d_{P^l}(\Omega)$ is proved. ∎

Lemma 2.13. *Let $P(D)$ be a regular operator with completely regular Newton polyhedron $\mathcal{N}_P$. Then for a sufficiently large $d > 0$ it is satisfied*

$$G^d_P(\Omega) \subset G^{\mathcal{N}_P\left(\frac{1}{d}\right)}(\Omega).$$

Proof. Because the lemma has a local character, then it is possible to consider a bounded open set $\Omega \subset \mathbb{R}^n$. Let u belong to $G^d_P(\Omega)$. Since $P(D)$ is regular, then for a constant $C > 0$ it is satisfied (cf. [20])

$$\sum_{\alpha \in \mathcal{N}_P^0} |\xi^\alpha| \le C(|P(\xi)| + 1), \quad \forall \xi \in \mathbb{R}^n. \tag{2.10}$$

Using Theorem 4.2 of [14], there is a constant $\gamma > 0$ such that for every $s \ge 0$, $t > 0$ and any $v \in C^\infty(\Omega'_s)$ the condition (2.10) can be rewritten in the form

$$\sum_{\alpha \in \mathcal{N}_P^0} \sup_{0<t\le\tau} \tau^\gamma \|D^\alpha v\|_{\Omega'_{s+\tau}} \le C_1 \left(\sup_{0<\tau\le t} \tau^\gamma \|P(D)v\|_{\Omega'_{s+\tau}} + \|v\|_{\Omega'_s}\right), \tag{2.11}$$

where $\Omega' \subset\subset \Omega$, and C_1 is a constant depending on P and the diameter of Ω'.
Hence

$$\sum_{\alpha \in \mathcal{N}_P^0} \|D^\alpha v\|_{\Omega'_{s+t}} \le C_2\left(\|P(D)v\|_{\Omega'_s} + t^{-\gamma}\|v\|_{\Omega'_s}\right), \quad \forall v \in C^\infty(\Omega'_s). \tag{2.12}$$

Since for any $f \in C^\infty(\Omega)$, any natural number r and any compact subset $K \subset \Omega$ there is a constant $C_{r,f,K} > 0$ such that

$$\sup_{x\in K} |D^\alpha f(x)| \le C_{r,f,K}, \quad \forall \alpha \in \mathbb{N}^n, |\alpha| \le r,$$

therefore to prove Lemma 2.13 it is sufficient to show that

$$\sup_{x\in K} |D^\beta f(x)| \le C^{j+1} j^j, \quad \forall \beta \in \mathcal{N}\left(\frac{j}{d}\right), \, j \ge j_0 \ge r,$$

where j_0 is as in Proposition 2.8.
Let $k = \left[\frac{j}{d}\right] + 1$. Let us prove the lemma for any $d \ge \gamma$. For $k > 1$ (i.e. $j \ge d$) from Proposition 2.8, for any $\beta \in \mathcal{N}_P(\frac{j}{d}) \cap \mathbb{N}^n$ there is a multi-index $\alpha^{(1)} \in \mathcal{N}_P$, $\alpha^{(1)} \le \beta$ such that $\beta^{(1)} = \beta - \alpha^{(1)} \in \mathcal{N}_P\left(\frac{j-d}{d}\right)$.
For $k = 1$ (i.e. $j < d$) we have $\mathcal{N}_P\left(\frac{j}{d}\right) \subset \mathcal{N}_P$, then instead of the multi-index $\alpha^{(1)}$ we can take $\alpha^{(1)} = \beta$ and $\beta^{(1)} = 0$.
Using (2.12) for $v = D^{\beta^{(1)}} u$ with $s = \delta - \frac{\delta}{k}$, $t = \frac{\delta}{k}$, $\delta > 0$, we get

$$\begin{aligned} \|D^\beta u\|_{\Omega'_\delta} &= \|D^\beta u\|_{\Omega'_{s+t}} = \|D^{\alpha^{(1)}}(D^{\beta^{(1)}} u)\|_{\Omega'_{s+t}} \\ &\le C_2 \left(\|P(D) D^{\beta^{(1)}} u\|_{\Omega'_s} + \left(\frac{K}{\delta}\right)^\gamma \|D^{\beta^{(1)}} u\|_{\Omega'_s} \right) \\ &= C_2 \left(\|D^{\beta^{(1)}} P(D) u\|_{\Omega'_s} + \left(\frac{K}{\delta}\right)^\gamma \|D^{\beta^{(1)}} u\|_{\Omega'_s} \right). \end{aligned} \tag{2.13}$$

If $k > 2$ (i.e. $\left[\frac{j}{d}\right] > 1$ and therefore $\beta^{(1)} \notin \mathcal{N}_P$), then from Proposition 2.8 it follows that there is a multi-index $\alpha^{(2)} \in \mathcal{N}_P(2)$, $\alpha^{(2)} \le \beta^{(1)}$ such that $\beta^{(2)} = \beta^{(1)} - \alpha^{(2)} \in \mathcal{N}_P\left(\frac{j-2d}{d}\right)$.
Applying (2.12) to both terms of the right-hand side of (2.13) and taking $v = D^{\beta^{(2)}} P(D) u$ for the first term and $v = D^{\beta^{(2)}} u$ for the second, after k steps we get

$$\|D^\beta u\|_{\Omega'_\delta} \le C_2^k \sum_{j=0}^{k} C_k^0 \left(\frac{k}{\delta}\right)^{j\gamma} \|P^{(k-j)}(D) u\|_{\Omega'}. \tag{2.14}$$

Since $\Omega' \subset\subset \Omega$ and $u \in G_P^d(\Omega)$, we obtain

$$\|P^j(D) u\|_{\Omega'} \le B^{j+1} j^{dj}, \quad \forall j = 1, 2, \ldots . \tag{2.15}$$

As $d \ge \gamma$, then from (2.15) we have

$$k^{j\gamma} \|P^{(k-j)}(D) u\|_{\Omega'} \le k^{j\gamma} B^{k-j+1} (k-j)^{(k-j)d} \le B_1^{k+1} k^{kd}. \tag{2.16}$$

Choosing $\delta > 0$ such that $A \subset \Omega'_\delta$, it follows from (2.14) and (2.16) that there exist $B_2, B_3 > 0$ such that

$$\|D^\beta u\|_A \le B_2^{k+1} k^{kd} = B_2^{\left[\frac{j}{d}\right]+1} \left(\left[\frac{j}{d}\right] + 1\right)^{\left(\left[\frac{j}{d}\right]+1\right)d} \le B_3^{j+1} j^j,$$

for all $\beta \in \mathcal{N}_P\left(\frac{j}{d}\right)$, $j = 1, 2, \ldots$. Thus u belongs to $G^{\mathcal{N}_P\left(\frac{1}{d}\right)}(\Omega)$. ■

An alternative proof of Lemma 2.13 can be found in Bouzar-Chaili [3]. Referring to [1], we have the following

Lemma 2.14. *Let a polynomial $P(\xi)$ have completely regular Newton polyhedron $\mathcal{N}_P$. Then for any $d > 0$*

$$G^{\mathcal{N}_P\left(\frac{1}{d}\right)}(\Omega) \subset G^d_P(\Omega).$$

Proof. Let u belong to $G^{\mathcal{N}_P\frac{1}{d}}(\Omega)$. For any compact subset $A \subset \Omega$ and any natural number j we have

$$\begin{aligned}\|P^j(D)u\|_A &\le L^j \max_{\alpha\in\mathcal{N}_P(j)} \|D^\alpha u\|_A \\ &\le L^j \max_{\alpha\in\mathcal{N}_P\left(\frac{[dj]+1}{d}\right)} \|D^\alpha u\|_A \\ &\le L^j C^{[dj]+1}\left([dj]+1\right)^{[dj]+1} \le C_1^{j+1} j^{dj},\end{aligned}$$

where L is the number of the multi-indeces $\alpha \in \mathcal{N}_P$. Thus u belongs to $G^d_P(\Omega)$. ∎

Remark 2.15. *Lemmas 2.13 and 2.14 hold in particular for hypoelliptic operators (or polynomials), as their Newton polyhedron is completely regular.*

Corollary 2.16. *If a polynomial $P(\xi)$ (or an operator $P(D)$) having completely regular Newton polyhedron is regular, then for a sufficiently large $d > 0$*

$$G^d_P(\Omega) = G^{\mathcal{N}_P\left(\frac{1}{d}\right)}(\Omega).$$

Lemma 2.17. *Let $P(\xi)$ be a hypoelliptic polynomial, $Q(\xi)$ be a regular polynomial having completely regular Newton polyhedron such that for a constant $C > 0$ they satisfy*

$$|Q(\xi)| \le C\left(|P(\xi)| + 1\right), \quad \forall \xi \in \mathbb{R}^n.$$

Then for $d > 0$ sufficiently large it holds

$$G^d_P(\Omega) \subset G^d_Q(\Omega) = G^{\mathcal{N}_Q\left(\frac{1}{d}\right)}(\Omega).$$

The proof follows by combining Theorem 1 of [21] and Corollary 2.16.
For a hypoelliptic polynomial $P(\xi)$ of order m we denote

$$\begin{aligned} D(P) &= \{\zeta \in \mathbb{C}^n : P(\zeta) = 0\}, \\ d_P(\xi) &= \inf_{\zeta\in D(P)} |\xi - \zeta|. \end{aligned}$$

Let $\mathcal{M}$ be a completely regular polyhedron, assume that $h_{\mathcal{M}}(\xi)$ is a regular weight of hypoellipticity of P, cf. Definition 2.6, and let r be a natural number such that $\frac{r}{m}$ is rational and $\mathcal{M}^0(r)$ is included in $\mathbb{N}^n$. We set

$$Q(\xi) = \sum_{\alpha\in\mathcal{M}^0(r)} \xi^{2\alpha}.$$

It is easy to see that

a) $\mathcal{M}(2r) = \mathcal{N}_Q$;

b) for a constant $C_1 > 0$ it is satisfied

$$C_1^{-1} h^{2r}_{\mathcal{M}}(\xi) \le Q(\xi) \le C_1 h^{2r}_{\mathcal{M}}(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

From Lemma 4.1.1 of [15] and the previous Definition 2.6, there are two constants $C_2, C_3 > 0$ such that

$$h_{\mathcal{M}}(\xi) \le C_2(d_P(\xi)+1) \le C_3(|P(\xi)|^{\frac{1}{m}}+1), \quad \forall \xi \in \mathbb{R}^n. \tag{2.17}$$

Taking into account b) and (2.17), we get for a constant $C > 0$

$$|Q(\xi)| \le C(|P(\xi)|^{\frac{2r}{m}}+1), \quad \forall \xi \in \mathbb{R}^n.$$

Proposition 2.18. *Let $P(\xi)$ be a hypoelliptic polynomial, $\mathcal{M}$ as before, corresponding to a regular weight of hypoellipticity of P, $\Omega \subset \mathbb{R}^n$ be an open nonempty set, then for sufficiently large $d > 0$ the following inclusion holds*

$$G_P^d(\Omega) \subset G^{\mathcal{M}\left(\frac{m}{d}\right)}(\Omega).$$

Proof. From Theorem 1 of [21] it follows that

$$G_P^d(\Omega) \subset G_Q^{\frac{2r}{m}d}(\Omega),$$

where $Q(\xi) = \sum\limits_{\alpha\in\mathcal{N}_P^0} \xi^{2\alpha}$. From Corollary 2.16 and a) it follows that for a sufficiently large $d > 0$ it holds

$$G_Q^{\frac{2r}{m}d}(\Omega) \subset G^{\mathcal{N}_Q\left(\frac{m}{2rd}\right)} = G^{\mathcal{M}\left(\frac{m}{d}\right)}(\Omega).$$

This completes the proof. ∎

3 Main Results

Theorem 3.1. *Let $P(\xi)$ be a polynomial (or an operator $P(D)$), $\mathcal{M}$ be a completely regular polyhedron. If there is a $d > 0$ such that $G_P^d(\Omega) \subset G^{\mathcal{M}}(\Omega)$, then for a constant $C > 0$ it is satisfied*

$$h_{\mathcal{M}}(\xi) \le C\left(|P(\xi)|^{\frac{1}{d}}+1\right), \quad \forall \xi \in \mathbb{R}^n. \tag{3.18}$$

Proof. Because the vertices of polyhedron $\mathcal{M}$ have rational coordinates, then for some natural number r we have $\mathcal{M}^0(r) \subset \mathbb{N}_0^n$. We set

$$Q(\xi) = \sum_{\alpha\in\mathcal{M}^0(r)} \xi^{2\alpha}.$$

The polynomial $Q(\xi)$ is obviously regular. From Corollary 2.16 it follows

$$G^{\mathcal{M}}(\Omega) \equiv G^{\mathcal{M}\left(2r\frac{1}{2r}\right)}(\Omega) = G_Q^{\frac{1}{2r}}(\Omega).$$

Hence from the conditions of Theorem 3.1 we get $G_P^d(\Omega) \subset G_Q^{\frac{1}{2r}}(\Omega)$. According to Theorem 2 of [21], we have for a constant $C > 0$

$$|Q(\xi)| \le C\left(|P(\xi)|^{\frac{2r}{d}}+1\right), \quad \forall \xi \in \mathbb{R}^n. \tag{3.19}$$

Since $h_{\mathcal{M}}^{2r}(\xi) \sim Q(\xi)$, the proof of the theorem follows from (3.19). ∎

Conversely, we have the following result

Theorem 3.2. *Let $P(\xi)$ be a hypoelliptic polynomial (or $P(D)$ a hypoelliptic operator), $\mathcal{M}$ a completely regular polyhedron. If the inequality* (3.18) *holds for some $d \geq m$, then the inclusion $G_P^d(\Omega) \subset G^{\mathcal{M}}(\Omega)$ is satisfied.*

The proof follows from Lemma 2.17 and the computations in the proof of Theorem 3.1.

Remark 3.3. *Theorem 3.2 implies that if P is a hypoelliptic operator, then we have $G_P^d(\Omega) \subset G^s(\Omega)$ for suitable $s, d \geq 1$, since the multianisotropic Gevrey classes $G^{\mathcal{M}}(\Omega)$ are always included in a standard Gevrey class $G^s(\Omega)$ for s sufficiently large, cf. [7].*

By taking $d = m$ in Proposition 2.18, we have the following result.

Corollary 3.4. *Let $P(D)$ be a hypoelliptic operator and let $h_{\mathcal{M}}(\xi) = \sum_{\alpha \in \mathcal{M}} |\xi^\alpha|$ be a regular weight of hypoellipticity of P associated to the completely regular polyhedron $\mathcal{M}$, then*

$$G_P^m(\Omega) \subset G^{\mathcal{M}}(\Omega).$$

Theorem 3.5. *Let a polynomial $P(\xi)$ (or an operator $P(D)$) have completely regular Newton polyhedron $\mathcal{N}_P$, $\Omega \subset \mathbb{R}^n$ be a nonempty open set. Then $P(\xi)$ (or $P(D)$) is regular if and only if there is $d > 0$ such that*

$$G_P^d(\Omega) \subset G^{\mathcal{N}_P\left(\frac{1}{d}\right)}(\Omega). \tag{3.20}$$

Remark 3.6. *Under the hypotheses of Theorem 3.5, the condition* (3.20) *implies also the hypoellipticity of P, as any regular polynomial (or operator) with completely regular Newton polyhedron is hypoelliptic.*

Proof. If P is regular, we apply Theorem 3.2 with $\mathcal{M} = \mathcal{N}_P(\frac{1}{d})$, to obtain (3.20). In the opposite direction, assume (3.20) is valid. It follows from Lemma 2.12 that

$$G_{P^2}^{2d} = G_P^d(\Omega) \subset G^{\mathcal{N}_P\left(\frac{1}{d}\right)}(\Omega).$$

From Corollary 2.16 it follows

$$G^{\mathcal{N}_P\left(\frac{1}{d}\right)}(\Omega) = G_{Q^2}^{2d}(\Omega),$$

where $Q(\xi) = \sum\limits_{\alpha \in \mathcal{N}_P^0} \xi^{2\alpha}$.

Using Theorem 2 of [21], for some constant $C > 0$ it is satisfied

$$\sum_{\alpha \in \mathcal{N}_P^0} |\xi^{2\alpha}| \leq C\left(|P(\xi)|^2 + 1\right), \quad \forall \xi \in \mathbb{R}^n.$$

Therefore from [20] the polynomial $P(\xi)$ is regular. ∎

Remark 3.7. *An alternative proof of Theorem 3.5 can be found in Bouzar-Chaili [3]; in [4, 5] the case of operators with variable coefficients in suitable Gevrey classes is also considered.*

Remark 3.8. *In the case that $\mathcal{M}$ is the Newton polyhedron of an elliptic operator, we recapture the results of [17] for elliptic operators, in the case of constant coefficients.*

We end by an example clarifying the results of Theorems 3.1 and 3.2. In particular, it is interesting to consider the case of nonregular operators. Let

$$P(\xi) = (\xi_1 - \xi_2)^6 + \xi_1^4 + \xi_2^4.$$

$P(\xi)$ is a non regular hypoelliptic polynomial. In this case the set $\mathcal{M}_P$ defined by (2.6) has the form

$$\mathcal{M}_P = \left\{ \nu \in \mathbb{R}^n_+ : (\nu, \lambda) \leq \frac{2}{3} \right\},$$

therefore the set $\mathcal{M}_P$ is a completely regular polyhedron, with $\lambda = (1, 1)$, and $\mathcal{N}_P$ is

$$\mathcal{N}_P = \left\{ \nu \in \mathbb{R}^n_+ (\nu, \lambda) \leq 6 \right\}.$$

It is easy to see that $9\mathcal{M}_P = \mathcal{N}_P$.

If we take $Q(\xi) = \xi_1^6 + \xi_2^6$ then $\mathcal{N}_P = \mathcal{N}_Q$, $9\mathcal{M}_P = \mathcal{N}_Q$ and

$$|Q(\xi)| \leq c(|P(\xi)|^{\frac{3}{2}} + 1).$$

We may therefore apply Corollary 3.4, or Theorem 3.2 with $d = 6$, and conclude that G^6_P is included in $G^{\mathcal{M}} = G^{\frac{3}{2}}$. In view of Theorem 3.1, this result is sharp in the frame of multianisotropic Gevrey classes.
For any $\varepsilon > 0$

$$|Q(\xi)| \not\leq c(|P(\xi)|^{\frac{3}{2} - \varepsilon} + 1).$$

Then, the estimate

$$|Q(\xi)|^r \leq c(|P(\xi)| + 1)$$

is a sufficient and necessary condition in order that $G^d_P(\Omega) \subset G^{\frac{d}{r}}_Q(\Omega)$ for some large $d > 0$. It holds

$$\begin{aligned} G^d_P(\Omega) \subset G^{\frac{\frac{d}{2}}{3}}_Q(\Omega) = G^{\frac{3d}{2}}_Q(\Omega) = G^{\mathcal{N}_Q(\frac{2}{3d})}(\Omega) = G^{\mathcal{N}_P(\frac{2}{3d})}(\Omega) \\ = G^{\mathcal{M}_P(9\frac{2}{3d})}(\Omega) = G^{\mathcal{M}_P(6d)}(\Omega) \end{aligned}$$

and

$$G^d_P(\Omega) \not\subset G^{\frac{\frac{d}{2}}{3-\varepsilon}}_Q(\Omega) = G^{\mathcal{N}_P(\frac{2}{(3-\varepsilon)d})}(\Omega).$$

In the other hand

$$G^{\mathcal{N}_P(\frac{1}{d})}(\Omega) \subset G^{\mathcal{N}_P(\frac{2}{(3-\varepsilon)d})}(\Omega)$$

for $\frac{2}{(3-\varepsilon)} < 1$, or $0 < \varepsilon < 1$.
So, for the nonregular polynomial $P(\xi)$ the inclusion

$$G^d_P(\Omega) \subset G^{\mathcal{N}_P(\frac{1}{d})}(\Omega)$$

is false.

References

[1] P. Bolley - J. Camus, *Power and Gevrey's regularity for systems of differential operators*, Czechoslovak Math. J. 29 (1979), 649-661.

[2] P. Bolley - J. Camus - L. Rodino, *Hypoellipticité analytique et itérés d'operateurs*, Rend. Sem. Mat. Univ. Pol. Torino 453 (1988), 1-61.

[3] C. Bouzar - R. Chaili, *Une généralisation du problème des itérés*, Arch. Math. (Basel) 76 (1) (2001), 57-66.

[4] C. Bouzar - R. Chaili, *Vecteurs Gevrey d' operateurs differentiels quasihomogenes*, Bull. Belg. Math. Soc. Simon Stevin 9 (2002), 299-310.

[5] C. Bouzar - R. Chaili, *Gevrey vectors of multi-quasi-elliptic systems*, Proc. Amer. Math. Soc. 131 (5) (2003), 1565-1572.

[6] D. Calvo, *Generalized Gevrey classes and multi-quasi-hyperbolic operators*, Rend. Sem. Mat. Univ. Pol. Torino 60 (2) (2002), 73-100.

[7] D. Calvo, *Cauchy problem in multianisotropic Gevrey classes for weakly hyperbolic operators*, to appear in Boll. U.M.I., Sez. B.

[8] A. Corli, *Un Teorema di rappresentazione per certe classi generalizzate di Gevrey*, Boll. U.M.I. Serie VI, 4-C (1) (1985), 245-257.

[9] J. Friberg, *Multi-quasielliptic polynomials*, Ann. Sc.Norm. Sup. Pisa, Cl. di Sc. 21 (1967), 239-260.

[10] G. Gindikin - L. R. Volevich, *The method of Newton's polyhedron in the theory of partial differential equations*, Mathematics and its Applications, Soviet Series, 86, 1992.

[11] G. H. Hakobyan - V. N. Markaryan, *The growth of hypoelliptic polynomials*, Math. 4, Yerevan, (1986), 108-123.

[12] G. H. Hakobyan - V. N. Markaryan, *On Gevrey type solution of hypoelliptic equations*, IZV. Nat. Acad. Nauk Arm., Math. 31 (2) (1996), 33-47.

[13] G. H. Hakobyan - V. N. Markaryan, *On Gevrey class solutions of hypoelliptic equations*, IZV. Nat.Acad. Nauk Arm., Math. 1 (1998), 1-13.

[14] L. Hörmander, *On interior regularity of the solutions of partial differential equations*, Comm. Pure Appl. Math. 11 (1958), 197-218.

[15] L. Hörmander, *Linear partial differential operators*, Springer-Verlag, Berlin, 1963.

[16] H. G. Kazharyan, *On a functional index of hypoellipticity*, Mat. Sbor. 128 (170) (1985), 339-353.

[17] H. Komatsu, *A characterization of real analytic functions*, Proc. Japan Acad. 36 (1960), 90-93.

[18] T. Kotake - M. S. Narasimhan, *Regularity theorems for fractional powers of a linear elliptic operator*, Bull. Soc. Math. France, 90 (1962), 449-471.

[19] G. Metivier, *Propriété des itérés et ellipticité*, Comm. P.D.E. 3 (9) (1978), 827-876.

[20] V. P. Mikhailov, *On the behavior at infinity of a class of polynomials*, Trudy Mat. Inst. Steklov. S.1, 59-80; English trans. in Proc. Steklov Inst. Math. 91 (1967), 59-80.

[21] E. Newberger - Z. Zielezny, *The growth of hypoelliptic polynomials and Gevrey classes*, Proc. Amer. Math. Soc. 39 (3) (1973), 547-552.

[22] L. Rodino, *Linear partial Differential Operators in Gevrey Spaces*, World Scientific, Singapore, 1993.

[23] L. Zanghirati, *Iterati di una classe di operatori ipoellittici e classi generalizzate di Gevrey*, Suppl. Boll. U.M.I. 1 (1980), 177-195.

[24] L. Zanghirati, *Iterates of quasielliptic operators, and Gevrey classes*, Boll. U.M.I. 18-B (1981), 411-428.

[25] L. Zanghirati, *Iterati di operatori e regolarità Gevrey microlocale anisotropa*, Rend. Sem. Mat. Univ. Padova 67 (1982), 85-104.

Dipartimento di Matematica, Università di Torino,
Via Carlo Alberto, 10, 10123 Torino, Italy
calvo@dm.unito.it

Chair of High Math., Department of Physics,
Yerevan State University,
375025, Al Manoogian Str. 1, Armenia
Gaghakob@ysu.am