Discrete Vekua equations with constant coefficients in the complex and quaternionic case

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Abstract

A discrete version of complex Vekua type equations is considered. We construct a representation formula for the solution of the homogeneous complex equation and investigate the inhomogeneous equation. Similar to the continuous case this representation formula is a product of two functions. We also factorize the solution of a homogeneous Vekua type equation in the quaternionic case. In the complex plane we analyse both factors in detail and study the relation between the non-holomorphic factors in the discrete and continuous case.

1 Introduction

Vekua equations play an important role because a lot of partial differential equations can be transformed into this type of equations. The theory of generalized analytic functions by Vekua (see [22]) is used in areas like analysis, geometry and mechanics. We mention only quasiconformal mappings and the theory of gas dynamics.
In this paper we study a discrete version of Vekua type equations. The ideas are inspired by many results from discrete potential theory ([17], [3] and [16]) and discrete function theory ([5], [23], [15], [7], [14], [13] and [9]). The analogy to function theoretic methods becomes obviously if we split the solution of our homogeneous

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equation into two factors such that one factor is discrete holomorphic. This factorization is demonstrated in [11] in the complex case. In the following we analyse the structure of both factors and describe the relation to the exponential function in the continuous case. Furthermore we investigate the inhomogeneous discrete equation. The solution of this problem is constructed by the help of an right inverse operator to the difference operator.
Generalizations of discrete Cauchy-Riemann operators are already studied in [12] and [13]. By the help of these difference operators a discrete version of Vekua equations can be studied also in the quaternionic case. We look at the homogeneous equation and show that also in the quaternionic case a factorization of the solution is possible.

2 The solution of the homogeneous Vekua equation in the complex case

2.1 A summary of classical and discrete results

Let $w(z)$ be a complex valued function with $z = x + iy$ and $G \subset \mathbb{R}^2$ be a bounded domain. Furthermore let $G^* \subset G$ be a set of isolated points with respect to G. A differential equation of the form

$$\partial_{\bar{z}} w + A\, w + B\, \bar{w} = 0 \quad \text{with} \quad \partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{1}$$

is called homogeneous Vekua equation. We denote $w(z)$ as *generalized solution* if $w(z)$ fulfils the differential equation at each point $G\backslash G^*$. Let E be the unit disc and $L_{p,2}(\mathbb{R}^2)$ be the set of functions with $f(z) \in L_p(E)$ and $|z|^{-2} f(\frac{1}{z}) \in L_p(E)$, $p \geq 1$. If $A, B \in L_{p,2}(\mathbb{R}^2)$ for $p > 2$ then each generalized solution of the homogeneous Vekua equation can be written in the form

$$w(z) = \Phi(z)\, e^{v(z)}, \tag{2}$$

where Φ is holomorphic in G, the function $v(z) = \frac{1}{\pi} \int\limits_G \frac{g(\zeta)\, dG}{\zeta - z}$ belongs to $C_{\frac{p-2}{2}}$ and $g(z) = \begin{cases} A(z) + B(z)\overline{\frac{w(z)}{w(z)}} & \text{for } w(z) \neq 0, \quad z \in G \\ A(z) + B(z) & \text{for } w(z) = 0, \quad z \in G. \end{cases}$

By the help of formula (2) a lot of function theoretic properties can be carried over to the theory of generalized analytic functions. Basic ideas to obtain such a formula as main tool in a discrete theory are presented in [11]. We give an overview of these results and extend the theory in view of some interesting properties of both factors in the discrete product.

An equidistant lattice with the mesh width $h > 0$ is defined by $\mathbb{R}^2_h = \{mh = (m_1 h, m_2 h) : m_1, m_2 \in \mathbb{Z}\}$. We denote by $G_h = G \cap \mathbb{R}^2_h$ the discrete domain and look at complex valued functions $w(mh) = (w_0(mh), w_1(mh)) = (\mathrm{Re}\ w(mh), \mathrm{Im}\ w(mh))$. For $j \in \{1, 2\}$ and $k \in \{0, 1\}$ we introduce forward differences $D_h^j w_k(mh) = h^{-1}(w_k(mh + he_j) - w_k(mh))$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$ and backward differences $D_h^{-j} w_k(mh) = h^{-1}(w_k(mh) - w_k(mh - he_j))$. In order to simplify the problem we write $A = a_1 + ia_2$ and $B = b_1 + ib_2$ and require that the real coefficients a_1, a_2, b_1 and b_2 are constant for all $mh \in G_h$ and zero for all $mh \in \mathbb{R}^2_h \setminus G_h$. Using the group homomorphism between complex numbers and matrices we can approximate equation (1) by

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix}\begin{pmatrix} w_0 \\ -w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3)$$

We remark that the operators $D^{1h} = \begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}$ and $D^{2h} = \begin{pmatrix} D_h^1 & D_h^2 \\ -D_h^{-2} & D_h^{-1} \end{pmatrix}$ factorize the discrete Laplacian and $-D^{2h}$ is the adjoint operator to D^{1h}. A function $w(mh)$ with $D^{1h}D^{2h}w(mh) = 0$ is called *discrete harmonic.* If $D^{1h}w(mh) = 0$ then $w(mh)$ is said to be *discrete holomorphic.* More details about these operators are contained in [16], [7] and [9].

In the following we restrict us to the case $b_1 = b_2 = 0$ and consider the system

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} w_0 & -w_1 \\ w_1 & w_0 \end{pmatrix} = \begin{pmatrix} -a_1 & a_2 \\ -a_2 & -a_1 \end{pmatrix}\begin{pmatrix} w_0 & -w_1 \\ w_1 & w_0 \end{pmatrix} \quad (4)$$

instead of (3). For each fixed mesh width h this system consists of four difference equations. But if h tends to zero it approximates only the two equations

$$\frac{1}{2}\left(\frac{\partial}{\partial x}w_0 - \frac{\partial}{\partial y}w_1\right) = -a_1 w_0 + a_2 w_1 \quad \text{and} \quad \frac{1}{2}\left(\frac{\partial}{\partial y}w_0 + \frac{\partial}{\partial x}w_1\right) = -a_2 w_0 - a_1 w_1.$$

In order to get more compact formulas we write in the following (m_1, m_2) instead of $(m_1 h, m_2 h)$. The following theorem we take from the main result in [11]:

Theorem 2.1. *Let $w(mh)$ be an arbitrary solution of (4) and $u(mh)$ be a solution of the problem*

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} u_0(m_1+1, m_2+1) & -u_1(m_1+1, m_2+1) \\ u_1(m_1, m_2) & u_0(m_1, m_2) \end{pmatrix} =$$

$$\begin{pmatrix} a_1\, u_0(m_1, m_2+1) - a_2\, u_1(m_1, m_2+1) & -a_2\, u_0(m_1, m_2+1) - a_1\, u_1(m_1, m_2+1) \\ a_2\, u_0(m_1+1, m_2) + a_1\, u_1(m_1+1, m_2) & a_1\, u_0(m_1+1, m_2) - a_2\, u_1(m_1+1, m_2) \end{pmatrix}$$

then we obtain for all $mh \in G_h$

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\left[\begin{pmatrix} u_0(m_1+1, m_2+1) & -u_1(m_1+1, m_2+1) \\ u_1(m_1, m_2) & u_0(m_1, m_2) \end{pmatrix}\begin{pmatrix} w_0(m_1, m_2) & -w_1(m_1, m_2) \\ w_1(m_1, m_2) & w_0(m_1, m_2) \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For the proof we refer to [11]. A nontrivial solution $u(mh)$ of the problem in Theorem 2.1 we obtain for arbitrary (m_1, m_2) by using the ansatz

$$\begin{aligned} u_0(m_1, m_2) &= \frac{1}{2}(\alpha - i\beta)^{m_1}(\gamma + i\delta)^{m_2} + \frac{1}{2}(\alpha + i\beta)^{m_1}(\gamma - i\delta)^{m_2} \\ u_1(m_1, m_2) &= \frac{1}{2i}(\alpha - i\beta)^{m_1}(\gamma + i\delta)^{m_2} - \frac{1}{2i}(\alpha + i\beta)^{m_1}(\gamma - i\delta)^{m_2} \end{aligned}$$

with the unknowns α, β, γ and δ, where $\alpha^2 + \beta^2 \neq 0$ and $\gamma^2 + \delta^2 \neq 0$. We remark that these unknowns depend on h. If we substitute $s_1 = 1 + 2a_1 h$, $s_2 = 1 + 2a_2 h$, $s_3 = 1 - 2a_2 h$ and $s_4 = 1 - 2a_1 h$ then we calculate $\alpha + i\beta$, $\alpha - i\beta$, $\gamma + i\delta$ and $\gamma - i\delta$ as square root with the smallest argument of the equations

$$\left((\alpha \pm i\beta) - \left[\frac{1}{s_4 \pm is_2} + \frac{s_1 \pm is_3}{2}\right]\right)^2 = \frac{s_4^2 - s_2^2}{(s_4^2 + s_2^2)^2} + \frac{s_1^2 - s_3^2}{4} \mp \frac{2is_2 s_4}{(s_4^2 + s_2^2)^2} \pm \frac{2is_1 s_3}{4}$$
$$\left((\gamma \pm i\delta) - \left[\frac{1}{s_3 \pm is_1} + \frac{s_2 \pm is_4}{2}\right]\right)^2 = \frac{s_3^2 - s_1^2}{(s_3^2 + s_1^2)^2} + \frac{s_2^2 - s_4^2}{4} \mp \frac{2is_1 s_3}{(s_3^2 + s_1^2)^2} \pm \frac{2is_2 s_4}{4}.$$

From these equations we conclude $\lim_{h\to 0} \alpha = \lim_{h\to 0} \gamma = 1$ and $\lim_{h\to 0} \beta = \lim_{h\to 0} \delta = 0$. Based on Theorem 2.1 the following theorem can be proved:

Theorem 2.2. *If the mesh width h is sufficiently small such that $1 + 4a_1 a_2 h^2 \neq 0$ then each solution of the problem (4) can be written in the form*

$$\begin{pmatrix} w_0(m_1,m_2) & -w_1(m_1,m_2) \\ w_1(m_1,m_2) & w_0(m_1,m_2) \end{pmatrix} \tag{5}$$
$$= \frac{1}{\det u(m_1,m_2)} \begin{pmatrix} u_0(m_1,m_2) & u_1(m_1{+}1,m_2{+}1) \\ -u_1(m_1,m_2) & u_0(m_1{+}1,m_2{+}1) \end{pmatrix} \begin{pmatrix} \Phi_1(m_1,m_2) & \Phi_2(m_1,m_2) \\ \Phi_3(m_1,m_2) & \Phi_4(m_1,m_2) \end{pmatrix}$$

with $\det u(m_1,m_2) = u_0(m_1,m_2)u_0(m_1+1,m_2+1) + u_1(m_1,m_2)u_1(m_1+1,m_2+1)$. *The matrix $\Phi(mh)$ has the property*

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix} \begin{pmatrix} \Phi_1(m_1,m_2) & \Phi_2(m_1,m_2) \\ \Phi_3(m_1,m_2) & \Phi_4(m_1,m_2) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \forall mh \in G_h. \tag{6}$$

We have already mentioned in [11] that there is a relation between Φ_1, Φ_2, Φ_3 and Φ_4 because there are two equations for the real part w_0 and two equations for the imaginary part w_1. On the other hand we know that the matrix $\Phi(mh)$ is a solution of (6). Is this equation automatically fulfilled if we use the relation between the $\Phi_i, i = 1,2,3,4$? In order to find an answer to this question it is necessary to look at the elements in more detail. In the following section we use a new possibility to show how the elements of $\Phi(mh)$ depend from each other.

2.2 The elements of the matrix $\Phi(mh)$

From (6) it follows that (Φ_1, Φ_3) as well as (Φ_2, Φ_4) are discrete holomorphic functions. We express Φ_2 and Φ_4 by the help of Φ_1 and Φ_3 and ask if the pair of both expressions is automatically discrete holomorphic or if we have to fulfil an additional condition.

If we eliminate w_0 and w_1 in (5) we obtain

$$\Phi_2 = -\frac{u_0(m_1,m_2)}{u_1(m_1,m_2)}\Phi_1 - \frac{u_1(m_1+1,m_2+1)}{u_1(m_1,m_2)}\Phi_3 + \frac{u_0(m_1+1,m_2+1)}{u_1(m_1,m_2)}\Phi_4$$
$$\Phi_4 = \frac{u_1^2(m_1,m_2) + u_0^2(m_1,m_2)}{\det u(m_1,m_2)}\Phi_1 + \frac{u_0(m_1,m_2)u_1(m_1+1,m_2+1) - u_0(m_1+1,m_2+1)u_1(m_1,m_2)}{\det u(m_1,m_2)}\Phi_3.$$

We use now the ansatz for $u_0(m_1,m_2)$ and $u_1(m_1,m_2)$ with α,β,γ and δ. It follows

$$\begin{aligned}
& u_0(m_1,m_2)u_1(m_1+1,m_2+1) - u_0(m_1+1,m_2+1)u_1(m_1,m_2)\\
= & \left(\frac{1}{2}(\alpha-i\beta)^{m_1}(\gamma+i\delta)^{m_2} + \frac{1}{2}(\alpha+i\beta)^{m_1}(\gamma-i\delta)^{m_2}\right)\\
& \cdot\left(\frac{1}{2i}(\alpha-i\beta)^{m_1+1}(\gamma+i\delta)^{m_2+1} - \frac{1}{2i}(\alpha+i\beta)^{m_1+1}(\gamma-i\delta)^{m_2+1}\right)\\
& -\left(\frac{1}{2}(\alpha-i\beta)^{m_1+1}(\gamma+i\delta)^{m_2+1} + \frac{1}{2}(\alpha+i\beta)^{m_1+1}(\gamma-i\delta)^{m_2+1}\right)\\
& \cdot\left(\frac{1}{2i}(\alpha-i\beta)^{m_1}(\gamma+i\delta)^{m_2} - \frac{1}{2i}(\alpha+i\beta)^{m_1}(\gamma-i\delta)^{m_2}\right)\\
= & (\alpha^2+\beta^2)^{m_1}(\gamma^2+\delta^2)^{m_2}\left(-\frac{1}{2i}(\alpha+i\beta)(\gamma-i\delta) + \frac{1}{2i}(\alpha-i\beta)(\gamma+i\delta)\right)\\
= & (\alpha^2+\beta^2)^{m_1}(\gamma^2+\delta^2)^{m_2}(\alpha\delta-\beta\gamma).
\end{aligned}$$

By the same way we prove that $\det u(m_1,m_2) = (\alpha^2+\beta^2)^{m_1}(\gamma^2+\delta^2)^{m_2}(\alpha\gamma+\beta\delta)$ and $u_1^2(m_1,m_2)+u_0^2(m_1,m_2) = (\alpha^2+\beta^2)^{m_1}(\gamma^2+\delta^2)^{m_2}$ and obtain

$$\Phi_4 = \frac{\Phi_1}{\alpha\gamma+\beta\delta} + \frac{\alpha\delta-\beta\gamma}{\alpha\gamma+\beta\delta}\,\Phi_3.$$

We remark that Φ_4 converges to Φ_1 if h tends to zero. We consider now the expression

$$\begin{aligned}
\Phi_2 = & \frac{\Phi_1}{u_1(m_1,m_2)}\left(-u_0(m_1,m_2) + \frac{u_0(m_1+1,m_2+1)}{\alpha\gamma+\beta\delta}\right)\\
& +\frac{\Phi_3}{u_1(m_1,m_2)}\left(-u_1(m_1+1,m_2+1) + u_0(m_1+1,m_2+1)\left(\frac{\alpha\delta-\beta\gamma}{\alpha\gamma+\beta\delta}\right)\right)
\end{aligned}$$

and write

$$\begin{aligned}
& -u_0(m_1,m_2)(\alpha\gamma+\beta\delta) + u_0(m_1+1,m_2+1)\\
= & \left(-\frac{1}{2}(\alpha-i\beta)^{m_1}(\gamma+i\delta)^{m_2} - \frac{1}{2}(\alpha+i\beta)^{m_1}(\gamma-i\delta)^{m_2}\right)\\
& \cdot\left(\frac{1}{2}(\alpha-i\beta)(\gamma+i\delta) + \frac{1}{2}(\alpha+i\beta)(\gamma-i\delta)\right)\\
& +\frac{1}{2}(\alpha-i\beta)^{m_1+1}(\gamma+i\delta)^{m_2+1} + \frac{1}{2}(\alpha+i\beta)^{m_1+1}(\gamma-i\delta)^{m_2+1}\\
= & \left(\frac{1}{2}(\alpha-i\beta)^{m_1}(\gamma+i\delta)^{m_2} - \frac{1}{2}(\alpha+i\beta)^{m_1}(\gamma-i\delta)^{m_2}\right)\\
& \cdot\left(\frac{1}{2}(\alpha-i\beta)(\gamma+i\delta) - \frac{1}{2}(\alpha+i\beta)(\gamma-i\delta)\right)\\
= & -u_1(m_1,m_2)\,(\alpha\delta-\beta\gamma).
\end{aligned}$$

In analogy we can show

$$\begin{aligned} & -u_1(m_1+1,m_2+1)(\alpha\gamma+\beta\delta)+u_0(m_1+1,m_2+1)(\alpha\delta-\beta\gamma) \\ = & -u_1(m_1,m_2)(\alpha^2+\beta^2)(\gamma^2+\delta^2). \end{aligned}$$

These calculations lead to the expression

$$\Phi_2 = -\frac{\alpha\delta-\beta\gamma}{\alpha\gamma+\beta\delta}\,\Phi_1 - \frac{(\alpha^2+\beta^2)(\gamma^2+\delta^2)}{\alpha\gamma+\beta\delta}\,\Phi_3$$

and it is easy to see that Φ_2 converges to $-\Phi_3$ if h tends to zero. If $\alpha\gamma+\beta\delta \neq 0$ it follows from (6)

$$\begin{aligned} & D_h^{-1}[(\alpha^2+\beta^2)(\gamma^2+\delta^2)\Phi_3] + D_h^2\Phi_1 + (\alpha\delta-\beta\gamma)[D_h^{-1}\Phi_1 + D_h^2\Phi_3] = 0 \quad \text{and} \\ & D_h^{-2}[-(\alpha^2+\beta^2)(\gamma^2+\delta^2)\Phi_3] + D_h^1\Phi_1 + (\alpha\delta-\beta\gamma)[-D_h^{-2}\Phi_1 + D_h^1\Phi_3] = 0. \end{aligned}$$

We remark that for $h \to 0$ these difference equations approximate the equations

$$\frac{\partial}{\partial x}\Phi_3 + \frac{\partial}{\partial y}\Phi_1 = 0 \quad \text{and} \quad -\frac{\partial}{\partial y}\Phi_3 + \frac{\partial}{\partial x}\Phi_1 = 0$$

such that in the continuous case (Φ_1, Φ_3) is automatically holomorphic. Different from the continuous case we understand the above difference equations as additional compatibility conditions for the discrete holomorphic functions $(\Phi_1(mh), \Phi_3(mh))$.

2.3 The approximation of the exponential function

It was already proved in [11] that for $h \to 0$ the solution of (4) tends to

$$\lim_{h\to 0} w_0(m_1,m_2) = \frac{\cos(x_1\,L2 - x_2\,L3)\Phi_1(x_1,x_2)}{e^{x_1\,(L3+2a_1)}e^{x_2\,(L2+2a_2)}} - \frac{\sin(x_1\,L2 - x_2\,L3)\Phi_3(x_1,x_2)}{e^{x_1\,(L3+2a_1)}e^{x_2\,(L2+2a_2)}}$$

and

$$\lim_{h\to 0} w_1(m_1,m_2) = \frac{\sin(x_1\,L2 - x_2\,L3)\Phi_1(x_1,x_2)}{e^{x_1\,(L3+2a_1)}e^{x_2\,(L2+2a_2)}} + \frac{\cos(x_1\,L2 - x_2\,L3)\Phi_3(x_1,x_2)}{e^{x_1\,(L3+2a_1)}e^{x_2\,(L2+2a_2)}},$$

where $L2 = \lim_{h\to 0}(h^{-1}\arctan\frac{\beta}{\alpha}) = \sqrt{2}a_1 - a_1 - a_2$ and $L3 = \lim_{h\to 0}(h^{-1}\arctan\frac{\delta}{\gamma}) = \sqrt{2}a_2 - a_2 - a_1$. We compare these expressions with the classical product

$$\begin{aligned} & e^{-(a_1+ia_2)(x_1-ix_2)}(\Phi_1+i\Phi_3) = e^{-a_1x_1}\,e^{-a_2x_2}\,e^{i(a_1x_2-a_2x_1)}(\Phi_1+i\Phi_3) \\ = \; & e^{-a_1x_1}\,e^{-a_2x_2}\Big(\cos(a_1x_2-a_2x_1)\Phi_1 - \sin(a_1x_2-a_2x_1)\Phi_3 \\ & \qquad + i[\sin(a_1x_2-a_2x_1)\Phi_1 + \cos(a_1x_2-a_2x_1)\Phi_3]\Big). \end{aligned}$$

In detail we have to study the expression $e^{-x_1(L3+2a_1)}\,e^{-x_2(L2+2a_2)}\,e^{i(x_1L2-x_2L3)}$. We substitute $L3 = L4 - 2a_1$ and $L2 = L5 - 2a_2$ and write the above expression in the following form

$$\begin{aligned} & e^{-x_1(L3+2a_1)}\,e^{-x_2(L2+2a_2)}\,e^{i(x_1L2-x_2L3)} \\ = \; & e^{\frac{1}{2}(-L3+iL2)(x_1+ix_2)}\,e^{\frac{1}{2}(-L4+2a_1+i(L5-2a_2))(x_1+ix_2)}\,e^{-2x_1a_1}\,e^{-2x_2a_2} \\ = \; & e^{\frac{1}{2}(-L3+iL2)(x_1+ix_2)}\,e^{\frac{1}{2}(-L4+iL5)(x_1+ix_2)}\,e^{(a_1-ia_2)(x_1+ix_2)}\,e^{-2x_1a_1}\,e^{-2x_2a_2} \\ = \; & e^{\frac{1}{2}(-L3-L4+i(L2+L5))(x_1+ix_2)}\,e^{-(a_1+ia_2)(x_1-ix_2)}. \end{aligned}$$

It is easy to check that the first factor is holomorphic. Because in the continuous case the product of two holomorphic functions is also holomorphic we can substitute

$$\Phi_1^* + i\Phi_3^* = e^{\frac{1}{2}(-L3-L4+i(L2+L5))(x_1+ix_2)}\,(\Phi_1 + i\Phi_3).$$

Consequently we approximate the classical exponential function up to a holomorphic factor. From this point of view we hope that in future it is possible to eliminate a part of the solution u.

2.4 Another representation formula for the solution $w(mh)$

If the mesh width h is small enough we can use Theorem 2.2 and write the solution of the problem (4) in the form

$$\begin{aligned} w_0(m_1,m_2) &= \frac{u_0(m_1,m_2)\Phi_1(m_1,m_2) + u_1(m_1+1,m_2+1)\Phi_3(m_1,m_2)}{\det\ u(m_1,m_2)} \\ w_1(m_1,m_2) &= \frac{-u_1(m_1,m_2)\Phi_1(m_1,m_2) + u_0(m_1+1,m_2+1)\Phi_3(m_1,m_2)}{\det\ u(m_1,m_2)}. \end{aligned}$$

We show that it is possible to write this solution also in another form in which no mesh points from the neighbourhood are included:

Based on the ansatz for $u_0(m_1,m_2)$ and $u_1(m_1,m_2)$ with the coefficients α, β, γ and δ we obtain

$$\det u(m_1,m_2) = (\alpha^2+\beta^2)^{m_1}\,(\gamma^2+\delta^2)^{m_2}\,(\alpha\gamma+\beta\delta).$$

Furthermore we have

$$\begin{aligned} \frac{u_0(m_1,m_2)}{(\alpha^2+\beta^2)^{m_1}(\gamma^2+\delta^2)^{m_2}} &= \frac{1}{2}\left(\frac{\alpha-i\beta}{\alpha^2+\beta^2}\right)^{m_1}\left(\frac{\gamma+i\delta}{\gamma^2+\delta^2}\right)^{m_2} + \frac{1}{2}\left(\frac{\alpha+i\beta}{\alpha^2+\beta^2}\right)^{m_1}\left(\frac{\gamma-i\delta}{\gamma^2+\delta^2}\right)^{m_2} \\ &= \frac{1}{2}(\alpha+i\beta)^{-m_1}(\gamma-i\delta)^{-m_2} + \frac{1}{2}(\alpha-i\beta)^{-m_1}(\gamma+i\delta)^{-m_2} \\ &= u_0(-m_1,-m_2). \end{aligned}$$

Using the same idea we can show that

$$\begin{aligned} \frac{u_1(m_1+1,m_2+1)}{(\alpha^2+\beta^2)^{m_1}(\gamma^2+\delta^2)^{m_2}} &= -(\alpha\gamma+\beta\delta)u_1(-m_1,-m_2) + (\alpha\delta-\beta\gamma)u_0(-m_1,-m_2) \\ \frac{-u_1(m_1,m_2)}{(\alpha^2+\beta^2)^{m_1}(\gamma^2+\delta^2)^{m_2}} &= u_1(-m_1,-m_2) \quad \text{and} \\ \frac{u_0(m_1+1,m_2+1)}{(\alpha^2+\beta^2)^{m_1}(\gamma^2+\delta^2)^{m_2}} &= (\alpha\gamma+\beta\delta)u_0(-m_1,-m_2) + (\alpha\delta-\beta\gamma)u_1(-m_1,-m_2). \end{aligned}$$

Consequently we proved that

$$\begin{aligned} w_0(m_1,m_2) &= \frac{u_0(-m_1,-m_2)\Phi_1(m_1,m_2)}{\alpha\gamma+\beta\delta} - u_1(-m_1,-m_2)\Phi_3(m_1,m_2) \\ &\quad + \frac{\alpha\delta-\beta\gamma}{\alpha\gamma+\beta\delta}u_0(-m_1,-m_2)\Phi_3(m_1,m_2) \end{aligned}$$

and

$$\begin{aligned} w_1(m_1,m_2) &= \frac{u_1(-m_1,-m_2)\Phi_1(m_1,m_2)}{\alpha\gamma+\beta\delta} + u_0(-m_1,-m_2)\Phi_3(m_1,m_2) \\ &\quad + \frac{\alpha\delta-\beta\gamma}{\alpha\gamma+\beta\delta} u_1(-m_1,-m_2)\Phi_3(m_1,m_2). \end{aligned}$$

We remark that we obtain from formula (5) a second equation for $w_0(m_1,m_2)$ as well as for $w_1(m_1,m_2)$. In these cases we end up with the same representation formulas if we use the relations between Φ_1, Φ_2, Φ_3 and Φ_4 from Section 2.2.

3 The inhomogeneous Vekua equation

Starting from (4) we look now at the system

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} w_0 & -w_1 \\ w_1 & w_0 \end{pmatrix} + \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} w_0 & -w_1 \\ w_1 & w_0 \end{pmatrix} = \begin{pmatrix} f_0 & -\tilde{f}_1 \\ f_1 & \tilde{f}_0 \end{pmatrix}. \quad (7)$$

We require that $\tilde{f}_0$ converges to f_0 and $\tilde{f}_1$ converges to f_1 if the mesh width h tends to zero. Consequently we approximate with the system (7) the two differential equations

$$\frac{1}{2}\left(\frac{\partial}{\partial x}w_0 - \frac{\partial}{\partial y}w_1\right) + a_1 w_0 - a_2 w_1 = f_0 \quad \text{and} \quad \frac{1}{2}\left(\frac{\partial}{\partial y}w_0 + \frac{\partial}{\partial x}w_1\right) + a_2 w_0 + a_1 w_1 = f_1.$$

In the following we describe the solution of the inhomogeneous equation (7) by using a difference operator which is right inverse to the operator $\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}$. In order to define this right inverse operator we look for an representation formula of the fundamental solution $E_h(mh)$ which solves the system

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} E_h^{11}(mh) & E_h^{12}(mh) \\ E_h^{21}(mh) & E_h^{22}(mh) \end{pmatrix} = \begin{pmatrix} \delta_h(mh) & 0 \\ 0 & \delta_h(mh) \end{pmatrix}$$

with $\delta_h(mh) = \begin{cases} 1/h^2 & mh = (0,0) \\ 0 & mh \neq (0,0). \end{cases}$ We calculate this fundamental solution by the help of the discrete Fourier transform (see [20] and [7] for the details of the calculation). This discrete Fourier transform F_h is a transform from the lattice into the square $Q_h = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : -\pi/h < \xi_i < \pi/h, \ i = 1,2\}$. The inverse transform is defined by $F_h^{-1} = R_h F$ where F is the classical Fourier transform and $R_h u$ denotes the restriction of the function $u(x)$ to the lattice $\mathbb{R}_h^2$. As representation formulas we obtain

$$E_h^{11} = \frac{1}{\pi}R_h F\left(\frac{\xi_{-1}^h}{d^2}\right) \quad E_h^{12} = \frac{1}{\pi}R_h F\left(\frac{\xi_{-2}^h}{d^2}\right) \quad E_h^{21} = \frac{1}{\pi}R_h F\left(\frac{\xi_2^h}{d^2}\right) \quad E_h^{22} = \frac{1}{\pi}R_h F\left(-\frac{\xi_1^h}{d^2}\right)$$

with $\xi_{-j}^h = h^{-1}(1 - e^{-ih\xi_j})$, $\xi_j^h = h^{-1}(1 - e^{ih\xi_j})$ and $d^2 = 4h^{-2}(\sin^2\frac{h\xi_1}{2} + \sin^2\frac{h\xi_2}{2})$ for $j \in \{1,2\}$ and $-\pi/h < \xi_j < \pi/h$. Based on the discrete fundamental solution we construct the right inverse operator $T_h = (T_h^1, T_h^2)$ by

$$(T_h^k[v_0,v_1])(mh) = \sum_{lh \in G_h} h^2 \begin{pmatrix} E_h^{k1}(mh-lh) \\ E_h^{k2}(mh-lh) \end{pmatrix}^T \begin{pmatrix} v_0(lh) \\ v_1(lh) \end{pmatrix}.$$

Theorem 3.1. *The operator T_h has the property*

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} (T_h^1[v_0,v_1])(mh) \\ (T_h^2[v_0,v_1])(mh) \end{pmatrix} = \begin{pmatrix} v_0(mh) \\ v_1(mh) \end{pmatrix}\chi_{G_h} \quad \textit{with} \quad \chi_{G_h} = \begin{cases} 1 & mh \in G_h \\ 0 & \textit{else.} \end{cases}$$

Proof: From the definition of the right inverse operator it follows

$$\begin{aligned}
&\frac{1}{2}D_h^{-1}(T_h^1[v_0,v_1])(mh) - \frac{1}{2}D_h^2(T_h^2[v_0,v_1])(mh) \\
= &\sum_{lh\in G_h} h^2\left\{\frac{1}{2}D_h^{-1}\left[\begin{pmatrix} E_h^{11}(mh-lh) \\ E_h^{12}(mh-lh) \end{pmatrix}^T\begin{pmatrix} v_0(lh) \\ v_1(lh) \end{pmatrix}\right] - \frac{1}{2}D_h^2\left[\begin{pmatrix} E_h^{21}(mh-lh) \\ E_h^{22}(mh-lh) \end{pmatrix}^T\begin{pmatrix} v_0(lh) \\ v_1(lh) \end{pmatrix}\right]\right\} \\
= &\sum_{lh\in G_h} h^2\begin{pmatrix} \frac{1}{2}D_h^{-1} \\ -\frac{1}{2}D_h^2 \end{pmatrix}^T\left[\begin{pmatrix} E_h^{11}(mh-lh) & E_h^{12}(mh-lh) \\ E_h^{21}(mh-lh) & E_h^{22}(mh-lh) \end{pmatrix}\begin{pmatrix} v_0(lh) \\ v_1(lh) \end{pmatrix}\right] \\
= &\sum_{lh\in G_h} h^2\left[\begin{pmatrix} \frac{1}{2}D_h^{-1} \\ -\frac{1}{2}D_h^2 \end{pmatrix}^T\begin{pmatrix} E_h^{11}(mh-lh) & E_h^{12}(mh-lh) \\ E_h^{21}(mh-lh) & E_h^{22}(mh-lh) \end{pmatrix}\right]\begin{pmatrix} v_0(lh) \\ v_1(lh) \end{pmatrix} \\
= &\sum_{lh\in G_h} h^2\begin{pmatrix} \delta_h(mh-lh) \\ 0 \end{pmatrix}^T\begin{pmatrix} v_0(lh) \\ v_1(lh) \end{pmatrix} = \begin{cases} v_0(mh) & \forall\, mh \in G_h \\ 0 & \textit{else.} \end{cases}
\end{aligned}$$

In analogy we get

$$\begin{aligned}
&\frac{1}{2}D_h^{-2}(T_h^1[v_0,v_1])(mh) + \frac{1}{2}D_h^1(T_h^2[v_0,v_1])(mh) \\
= &\sum_{lh\in G_h} h^2\begin{pmatrix} 0 \\ \delta_h(mh-lh) \end{pmatrix}^T\begin{pmatrix} v_0(lh) \\ v_1(lh) \end{pmatrix} = \begin{cases} v_1(mh) & \forall\, mh \in G_h \\ 0 & else \end{cases}
\end{aligned}$$

■

A straightforward calculation shows that we obtain a solution of the inhomogeneous problem (7) inside the domain G_h by

$$\begin{pmatrix} w_0 & -w_1 \\ w_1 & w_0 \end{pmatrix} = \begin{pmatrix} w_0^{hom} & -w_1^{hom} \\ w_1^{hom} & w_0^{hom} \end{pmatrix} + \begin{pmatrix} T_h^1[f_0,f_1] & T_h^1[-\tilde{f}_1,\tilde{f}_0] \\ T_h^2[f_0,f_1] & T_h^2[-\tilde{f}_1,\tilde{f}_0] \end{pmatrix}$$

where $\begin{pmatrix} w_0^{hom} & -w_1^{hom} \\ w_1^{hom} & w_0^{hom} \end{pmatrix}$ is a solution of the homogeneous problem (4). We remark that we have two equations for w_0 and two equations for w_1. In order to get the same expressions we require

$$T_h^1[-\tilde{f}_1,\tilde{f}_0] = -T_h^2[f_0,f_1] \quad \text{and} \quad T_h^2[-\tilde{f}_1,\tilde{f}_0] = T_h^1[f_0,f_1]$$

for all $mh \in G_h$. These equations are fulfilled if we define

$$\begin{pmatrix} -\tilde{f}_1 \\ \tilde{f}_0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} T_h^1[-\tilde{f}_1,\tilde{f}_0] \\ T_h^2[-\tilde{f}_1,\tilde{f}_0] \end{pmatrix} = \frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} -T_h^2[f_0,f_1]\,\chi_{G_h} \\ T_h^1[f_0,f_1]\,\chi_{G_h} \end{pmatrix}.$$

For the proof we use the discrete Borel-Pompeiu formula

$$T_h^k\left[\frac{1}{2}D_h^{-1}u_1 - \frac{1}{2}D_h^2u_2, \frac{1}{2}D_h^{-2}u_1 + \frac{1}{2}D_h^1u_2\right] + F_h^k[u_1,u_2] = u_k\,\chi_{G_h}, \qquad k = 1,2$$

and the property that the operator $\tilde{F}_h = (F_h^1, F_h^2)$ acts only from the boundary into the domain G_h. With the above definition we have the possibility to calculate $\tilde{f}_0$ and $\tilde{f}_1$ for all arbitrary chosen functions f_0 and f_1. Indeed we can use this formula in order to determine $\tilde{f}_0$ and $\tilde{f}_1$ because we can show that $\tilde{f}_0$ converges to f_0 and $\tilde{f}_1$ converges to f_1 if h tends to zero: From

$$E_h^{11} \to E^{11} = \frac{1}{\pi} F\left(\frac{i\xi_1}{|\xi|^2}\right), \qquad E_h^{12} \to E^{12} = \frac{1}{\pi} F\left(\frac{i\xi_2}{|\xi|^2}\right),$$
$$E_h^{21} \to E^{21} = \frac{1}{\pi} F\left(\frac{-i\xi_2}{|\xi|^2}\right), \qquad E_h^{22} \to E^{22} = \frac{1}{\pi} F\left(\frac{i\xi_1}{|\xi|^2}\right)$$

and the properties

$$-\int_{y\in G} \begin{pmatrix} E^{21}(x-y) \\ E^{22}(x-y) \end{pmatrix}^T \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} dG = \int_{y\in G} \begin{pmatrix} E^{12}(x-y) \\ E^{11}(x-y) \end{pmatrix}^T \begin{pmatrix} f_0 \\ -f_1 \end{pmatrix} dG$$

and

$$\int_{y\in G} \begin{pmatrix} E^{11}(x-y) \\ E^{12}(x-y) \end{pmatrix}^T \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} dG = \int_{y\in G} \begin{pmatrix} E^{22}(x-y) \\ E^{21}(x-y) \end{pmatrix}^T \begin{pmatrix} f_0 \\ -f_1 \end{pmatrix} dG$$

we follow that $T_h^1[-\tilde{f}_1, \tilde{f}_0] = -T_h^2[f_0, f_1] \to T_h^1[-f_1, f_0]$ and $T_h^2[-\tilde{f}_1, \tilde{f}_0] = T_h^1[f_0, f_1] \to T_h^2[-f_1, f_0]$. We apply now the difference operator D^{1h} and use Theorem 3.1.

4 Generalization of Theorem 2.1 to the quaternionic case

Let $\mathbb{R}^4$ be the 4-dimensional Euclidean vector space. We choose the orthogonal basis $e_0 = (1,0,0,0)$, $e_1 = (0,1,0,0)$, $e_2 = (0,0,1,0)$ and $e_3 = (0,0,0,1)$. Because of the multiplication rules

$$\begin{aligned} &e_0^2 = e_0, \quad e_i^2 = -e_0, \quad i = 1,2,3 \\ &e_i e_j + e_j e_i = 0, \quad i \neq j, \quad i,j = 1,2,3 \\ &e_0 e_i = e_i e_0 = e_i, \quad i = 0,1,2,3 \\ &e_1 e_2 = e_3, \quad e_2 e_3 = e_1 \quad \text{and} \quad e_3 e_1 = e_2 \end{aligned}$$

the algebra of the quaternions $a = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ is noncommutative. Quaternions can be identified with a special kind of real 4×4 matrices which have the form

$$a = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}.$$

For more details we refer to [13]. We prove now a similar result to the assertion in Theorem 2.1 in the quaternionic case. In the following it is very important at which mesh point of the space $\mathbb{R}^3$ we consider our functions. In order to simplify the notation we use no symbol for the mesh points $(m_1 h, m_2 h, m_3 h)$. All other symbols are explained in Tabular 1:

symbol	mesh point
[1]	$(m_1h,(m_2+1)h,(m_3+1)h)$
[2]	$((m_1+1)h,m_2h,(m_3+1)h)$
[3]	$((m_1+1)h,(m_2+1)h,m_3h)$
[4]	$((m_1+1)h,(m_2+1)h,(m_3+1)h)$
[5]	$((m_1+1)h,m_2h,m_3h)$
[6]	$(m_1h,(m_2+1)h,m_3h)$
[7]	$(m_1h,m_2h,(m_3+1)h)$

Tabular 1

Theorem 4.1. *Let (w_0,w_1,w_2,w_3) be an arbitrary solution of the problem*

$$\begin{pmatrix} 0 & -D_h^1 & -D_h^2 & -D_h^3 \\ D_h^1 & 0 & -D_h^{-3} & D_h^{-2} \\ D_h^2 & D_h^{-3} & 0 & -D_h^{-1} \\ D_h^3 & -D_h^{-2} & D_h^{-1} & 0 \end{pmatrix} \begin{pmatrix} w_0 & -w_1 & -w_2 & -w_3 \\ w_1 & w_0 & -w_3 & w_2 \\ w_2 & w_3 & w_0 & -w_1 \\ w_3 & -w_2 & w_1 & w_0 \end{pmatrix}$$
$$= \begin{pmatrix} -a_0 & a_1 & a_2 & a_3 \\ -a_1 & -a_0 & a_3 & -a_2 \\ -a_2 & -a_3 & -a_0 & a_1 \\ -a_3 & a_2 & -a_1 & -a_0 \end{pmatrix} \begin{pmatrix} w_0 & -w_1 & -w_2 & -w_3 \\ w_1 & w_0 & -w_3 & w_2 \\ w_2 & w_3 & w_0 & -w_1 \\ w_3 & -w_2 & w_1 & w_0 \end{pmatrix}$$

and (u_0,u_1,u_2,u_3) be a solution of

$$\begin{pmatrix} 0 & -D_h^1 & -D_h^2 & -D_h^3 \\ D_h^1 & 0 & -D_h^{-3} & D_h^{-2} \\ D_h^2 & D_h^{-3} & 0 & -D_h^{-1} \\ D_h^3 & -D_h^{-2} & D_h^{-1} & 0 \end{pmatrix} \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ -u_1^{[1]} & u_0^{[1]} & -u_3^{[1]} & u_2^{[1]} \\ -u_2^{[2]} & u_3^{[2]} & u_0^{[2]} & -u_1^{[2]} \\ -u_3^{[3]} & -u_2^{[3]} & u_1^{[3]} & u_0^{[3]} \end{pmatrix}$$
$$= \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

with

$$\begin{aligned} A_{11} &= a_0u_0^{[4]} + a_1u_1^{[4]} + a_2u_2^{[4]} + a_3u_3^{[4]} \\ A_{12} &= -a_1u_0^{[4]} + a_0u_1^{[4]} + a_3u_2^{[4]} - a_2u_3^{[4]} \\ A_{13} &= -a_2u_0^{[4]} - a_3u_1^{[4]} + a_0u_2^{[4]} + a_1u_3^{[4]} \\ A_{14} &= -a_3u_0^{[4]} + a_2u_1^{[4]} - a_1u_2^{[4]} + a_0u_3^{[4]} \end{aligned}$$

$$\begin{aligned} A_{21} &= a_1u_0^{[5]} - a_0u_1^{[5]} + a_3u_2^{[5]} - a_2u_3^{[5]} \\ A_{22} &= a_0u_0^{[5]} + a_1u_1^{[5]} - a_2u_2^{[5]} - a_3u_3^{[5]} \\ A_{23} &= -a_3u_0^{[5]} + a_2u_1^{[5]} + a_1u_2^{[5]} - a_0u_3^{[5]} \\ A_{24} &= a_2u_0^{[5]} + a_3u_1^{[5]} + a_0u_2^{[5]} + a_1u_3^{[5]} \end{aligned}$$

$$\begin{aligned}
A_{31} &= a_2 u_0^{[6]} - a_3 u_1^{[6]} - a_0 u_2^{[6]} + a_1 u_3^{[6]} \\
A_{32} &= a_3 u_0^{[6]} + a_2 u_1^{[6]} + a_1 u_2^{[6]} + a_0 u_3^{[6]} \\
A_{33} &= a_0 u_0^{[6]} - a_1 u_1^{[6]} + a_2 u_2^{[6]} - a_3 u_3^{[6]} \\
A_{34} &= -a_1 u_0^{[6]} - a_0 u_1^{[6]} + a_3 u_2^{[6]} + a_2 u_3^{[6]} \\
A_{41} &= a_3 u_0^{[7]} + a_2 u_1^{[7]} - a_1 u_2^{[7]} - a_0 u_3^{[7]} \\
A_{42} &= -a_2 u_0^{[7]} + a_3 u_1^{[7]} - a_0 u_2^{[7]} + a_1 u_3^{[7]} \\
A_{43} &= a_1 u_0^{[7]} + a_0 u_1^{[7]} + a_3 u_2^{[7]} + a_2 u_3^{[7]} \\
A_{44} &= a_0 u_0^{[7]} - a_1 u_1^{[7]} - a_2 u_2^{[7]} + a_3 u_3^{[7]}.
\end{aligned}$$

We obtain

$$\begin{pmatrix} 0 & -D_h^1 & -D_h^2 & -D_h^3 \\ D_h^1 & 0 & -D_h^{-3} & D_h^{-2} \\ D_h^2 & D_h^{-3} & 0 & -D_h^{-1} \\ D_h^3 & -D_h^{-2} & D_h^{-1} & 0 \end{pmatrix} \left[\begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ -u_1^{[1]} & u_0^{[1]} & -u_3^{[1]} & u_2^{[1]} \\ -u_2^{[2]} & u_3^{[2]} & u_0^{[2]} & -u_1^{[2]} \\ -u_3^{[3]} & -u_2^{[3]} & u_1^{[3]} & u_0^{[3]} \end{pmatrix} \right.$$
$$\left. \cdot \begin{pmatrix} w_0 & -w_1 & -w_2 & -w_3 \\ w_1 & w_0 & -w_3 & w_2 \\ w_2 & w_3 & w_0 & -w_1 \\ w_3 & -w_2 & w_1 & w_0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof: The next steps are quite similar to the proof of the analogous theorem in the complex case (see [11]). We consider here only the first matrix element on the left-hand side. For all other matrix elements we have to repeat the calculations. In order to determine this element we have to add the summands

$$\begin{aligned}
S_1 &= -D_h^1[-u_1^{[1]} w_0 + u_0^{[1]} w_1 - u_3^{[1]} w_2 + u_2^{[1]} w_3] \\
&= \frac{1}{h}[-u_1^{[1]} w_0 + u_0^{[1]} w_1 - u_3^{[1]} w_2 + u_2^{[1]} w_3 + u_1^{[4]} w_0^{[5]} - u_0^{[4]} w_1^{[5]} + u_3^{[4]} w_2^{[5]} - u_2^{[4]} w_3^{[5]} \\
&\qquad -u_1^{[4]} w_0 + u_0^{[4]} w_1 - u_3^{[4]} w_2 + u_2^{[4]} w_3 + u_1^{[4]} w_0 - u_0^{[4]} w_1 + u_3^{[4]} w_2 - u_2^{[4]} w_3] \\
&= u_1^{[4]}(D_h^1 w_0) + u_0^{[4]}(-D_h^1 w_1) + u_3^{[4]}(D_h^1 w_2) + u_2^{[4]}(-D_h^1 w_3) \\
&\quad + w_0(D_h^1 u_1^{[1]}) + w_1(-D_h^1 u_0^{[1]}) + w_2(D_h^1 u_3^{[1]}) + w_3(-D_h^1 u_2^{[1]}),
\end{aligned}$$

$$\begin{aligned}
S_2 &= -D_h^2[-u_2^{[2]} w_0 + u_3^{[2]} w_1 + u_0^{[2]} w_2 - u_1^{[2]} w_3] \\
&= u_2^{[4]}(D_h^2 w_0) + u_3^{[4]}(-D_h^2 w_1) + u_0^{[4]}(-D_h^2 w_2) + u_1^{[4]}(D_h^2 w_3) \\
&\quad + w_0(D_h^2 u_2^{[2]}) + w_1(-D_h^2 u_3^{[2]}) + w_2(-D_h^2 u_0^{[2]}) + w_3(D_h^2 u_1^{[2]}) \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
S_3 &= -D_h^3[-u_3^{[3]} w_0 - u_2^{[3]} w_1 + u_1^{[3]} w_2 + u_0^{[3]} w_3] \\
&= u_3^{[4]}(D_h^3 w_0) + u_2^{[4]}(D_h^3 w_1) + u_1^{[4]}(-D_h^3 w_2) + u_0^{[4]}(-D_h^3 w_3) \\
&\quad + w_0(D_h^3 u_3^{[3]}) + w_1(D_h^3 u_2^{[3]}) + w_2(-D_h^3 u_1^{[3]}) + w_3(-D_h^3 u_0^{[3]}).
\end{aligned}$$

From our assertion we obtain

$$
\begin{aligned}
& S_1+S_2+S_3 \\
= \; & u_0^{[4]}(-D_h^1 w_1 - D_h^2 w_2 - D_h^3 w_3) + u_1^{[4]}(D_h^1 w_0 + D_h^2 w_3 - D_h^3 w_2) \\
& + u_2^{[4]}(-D_h^1 w_3 + D_h^2 w_0 + D_h^3 w_1) + u_3^{[4]}(D_h^1 w_2 - D_h^2 w_1 + D_h^3 w_0) \\
& + w_0(D_h^1 u_1^{[1]} + D_h^2 u_2^{[2]} + D_h^3 u_3^{[3]}) + w_1(-D_h^1 u_0^{[1]} - D_h^2 u_3^{[2]} + D_h^3 u_2^{[3]}) \\
& + w_2(D_h^1 u_3^{[1]} - D_h^2 u_0^{[2]} - D_h^3 u_1^{[3]}) + w_3(-D_h^1 u_2^{[1]} + D_h^2 u_1^{[2]} - D_h^3 u_0^{[3]}) \\
= \; & u_0^{[4]}(-a_0 w_0 + a_1 w_1 + a_2 w_2 + a_3 w_3) + u_1^{[4]}(-a_0 w_1 - a_1 w_0 - a_2 w_3 + a_3 w_2) \\
& + u_2^{[4]}(-a_0 w_2 + a_1 w_3 - a_2 w_0 - a_3 w_1) + u_3^{[4]}(-a_0 w_3 - a_1 w_2 + a_2 w_1 - a_3 w_0) \\
& + w_0(a_0 u_0^{[4]} + a_1 u_1^{[4]} + a_2 u_2^{[4]} + a_3 u_3^{[4]}) + w_1(-a_1 u_0^{[4]} + a_0 u_1^{[4]} + a_3 u_2^{[4]} - a_2 u_3^{[4]}) \\
& + w_2(-a_2 u_0^{[4]} - a_3 u_1^{[4]} + a_0 u_2^{[4]} + a_1 u_3^{[4]}) + w_3(-a_3 u_0^{[4]} + a_2 u_1^{[4]} - a_1 u_2^{[4]} + a_0 u_3^{[4]}) \\
= \; & 0
\end{aligned}
$$

■

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