An alternate proof of Hall's theorem on a conformal mapping inequality

R. Balasubramanian S. Ponnusamy*

In this note we give a different and direct proof of the following result of Hall [2], which actually implies the conjecture of Sheil-Small [3]. For details about the related problems we refer to [1, 3].

THEOREM. *Let f be regular for $|z| < 1$ and $f(0) = 0$. Further, let f be starlike of order $1/2$. Then*

$$\int_0^r |f'(\rho e^{i\theta})| d\rho < \frac{\pi}{2} |f(re^{i\theta})|$$

for every $r < 1$ and real θ.

Proof. As in [2, p.125] (see also [1]), to prove our result it suffices to show that

$$J = I(t,\tau) + I(\tau,t) < \pi - 2 \quad \text{for} \quad 0 < t < \tau < \pi \tag{1}$$

where

$$I(t,\tau) = \int_0^1 \frac{2|\sin(t/2)|}{\sqrt{1 - 2\rho\cos t + \rho^2}} \left\{ \frac{1}{\sqrt{1 - 2\rho\cos\tau + \rho^2}} - \frac{1 - \rho\cos\tau}{1 - 2\rho\cos\tau + \rho^2} \right\} d\rho.$$

To evaluate these integrals we define k by

$$k^2 = \frac{\sin^2(\tau/2) - \sin^2(t/2)}{\cos^2(t/2)\sin^2(\tau/2)}$$

*This work has been done with the support of National Board for Higher Mathematics.

Received by the editors May 1995.
Communicated by R. Delanghe.
1991 *Mathematics Subject Classification* : Primary:30C45; Secondary:30C35.
Key words and phrases : Starlike and conformal mappings.

so that

$$\cos t = \frac{\cos\tau + k^2\sin^2(\tau/2)}{1 - k^2\sin^2(\tau/2)} \tag{2}$$

and

$$\sin^2(t/2) = \frac{(1-k^2)\sin^2(\tau/2)}{1-k^2\sin^2(\tau/2)}.$$

Further we let

$$\rho = \frac{\sin\theta}{\sin(\theta+\tau)}, \quad 0 \le \theta \le \frac{\pi-\tau}{2}. \tag{3}$$

(The idea of change of variables already occurs in [2, 3]). Then, from (2) and (3), we easily have

$$d\rho = \frac{\sin\tau}{\sin^2(\theta+\tau)}d\theta$$

$$|1-\rho e^{i\tau}|^2 = \frac{\sin^2\tau}{\sin^2(\theta+\tau)} = 1 - 2\rho\cos\tau + \rho^2$$

$$|1-\rho e^{it}|^2 = \frac{\sin^2\tau[1-k^2\sin^2(\theta+\tau/2)]}{\sin^2(\theta+\tau)[1-k^2\sin^2(\tau/2)]}$$

and

$$1-\rho\cos t = \frac{\sin\tau[\cos\theta - k^2\sin(\tau/2)\sin(\theta+\tau/2)]}{\sin(\theta+\tau)[1-k^2\sin^2(\tau/2)]}.$$

After some work we find that

$$J = I(t,\tau) + I(\tau,t) = \int_0^{(\pi-\tau)/2} H(k,\tau,\theta)d\theta \tag{4}$$

where

$$H(k,\tau,\theta) = \frac{1}{\cos(\tau/2)}\left[\sqrt{\frac{1-k^2\sin^2(\tau/2)}{1-k^2\sin^2(\theta+\tau/2)}} - \frac{\cos\theta - k^2\sin(\tau/2)\sin(\theta+\tau/2)}{1-k^2\sin^2(\theta+\tau/2)} + \right.$$

$$\left. + \frac{\sqrt{(1-k^2)}(1-\cos\theta)}{\sqrt{1-k^2\sin^2(\theta+\tau/2)}}\right].$$

We put

$$\frac{\pi-\tau}{2} = \lambda$$

to obtain

$$\begin{aligned} J &= \int_0^{(\pi-\tau)/2} H(k,\tau,\frac{\pi-\tau}{2}-\theta)d\theta \\ &= \int_0^{\lambda}[F(k,\lambda,\theta) + G(k,\lambda,\theta)]d\theta \end{aligned} \tag{5}$$

where F and G are defined by
$F(k,\lambda,\theta)$

$$= \frac{1}{\sin\lambda}\left[\sqrt{\frac{1-k^2\cos^2\lambda}{1-k^2\cos^2\theta}} - \frac{\cos(\lambda-\theta)-k^2\cos\lambda\cos\theta}{1-k^2\cos^2\theta}\right]$$

$$= \frac{1}{\sin\lambda}\left[\frac{(1-k^2)\sin^2(\lambda-\theta)}{(1-k^2\cos^2\theta)[\sqrt{1-k^2\cos^2\lambda}\sqrt{1-k^2\cos^2\theta}+\cos(\lambda-\theta)-k^2\cos\lambda\cos\theta]}\right]$$

and

$$G(k,\lambda,\theta) = \frac{\sqrt{1-k^2}(1-\cos(\lambda-\theta))}{\sin\lambda\sqrt{1-k^2\cos^2\theta}}.$$

Therefore

$$\frac{\partial J}{\partial\lambda} = \int_0^\lambda \frac{\partial F(k,\lambda,\theta)}{\partial\lambda}d\theta + \int_0^\lambda \frac{\partial G(k,\lambda,\theta)}{\partial\lambda}d\theta + F(k,\lambda,\lambda) + G(k,\lambda,\lambda).$$

Since $F(k,\lambda,\lambda) = 0 = G(k,\lambda,\lambda)$ the above becomes

$$\frac{\partial J}{\partial\lambda} = \int_0^\lambda \frac{\partial F}{\partial\lambda}d\theta + \int_0^\lambda \frac{\partial G}{\partial\lambda}d\theta.$$

A simple calculation shows that

$$\frac{\partial F}{\partial\lambda} = \frac{(1-k^2)\sin(\lambda-\theta)}{(1-k^2\cos^2\theta)X^2}\Bigg[\sqrt{\frac{1-k^2\cos^2\theta}{1-k^2\cos^2\lambda}}\{\cos(\lambda-\theta)\sin\lambda+\sin\theta$$
$$-k^2\cos\lambda\sin(\lambda+\theta)\Big\} - k^2\cos\theta\sin(\lambda+\theta)+\cos(\lambda-\theta)\sin\theta+\sin\lambda\Bigg]$$

where X is the denominator of the second expression in $F(k,\lambda,\theta)$.

Since the right hand side of the above expression is a decreasing function of k for each k in $(0,1)$ and since the square bracketed term in this expression for $k=1$ is positive, we have

$$\frac{\partial F}{\partial\lambda} \geq \frac{\partial F}{\partial\lambda}\bigg|_{k=1} = 0. \tag{6}$$

Similarly we have

$$\frac{\partial G}{\partial\lambda} \geq \frac{\partial G}{\partial\lambda}\bigg|_{k=1} = 0.$$

Thus to prove (1), by (5), (6) and the above, it is sufficient to prove that

$$\int_0^{\pi/2}[F(k,\pi/2,\theta)+G(k,\pi/2,\theta)]d\theta \leq \pi-2,$$

or equivalently

$$\int_0^{\pi/2}\frac{\sqrt{1-k^2\cos^2\theta}-\sin\theta}{1-k^2\cos^2\theta}d\theta + \sqrt{1-k^2}\int_0^{\pi/2}\frac{1-\sin\theta}{\sqrt{1-k^2\cos^2\theta}}d\theta \leq \pi-2. \tag{7}$$

(Note that this corresponds to $\tau=0$ in (4) or (5)). In the first of the above integrals we put $\tan\theta = y\sqrt{1-k^2}$ so that it becomes

$$\int_0^\infty \frac{\sqrt{1+y^2}-y}{(1+y^2)\sqrt{1+(1-k^2)y^2}}dy$$

which, by substituting $y = \tan\theta$, yields

$$\int_0^{\pi/2} \frac{1-\sin\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta.$$

Put

$$L(k,\theta) = (1-\sin\theta)\left[\frac{1}{\sqrt{1-k^2\sin^2\theta}} + \frac{\sqrt{1-k^2}}{\sqrt{1-k^2\cos^2\theta}}\right].$$

Therefore (7) is now equivalent to

$$\begin{aligned}\int_0^{\pi/2} L(k,\theta)d\theta &= \int_0^{\pi/4} L(k,\theta)d\theta + \int_{\pi/4}^{\pi/2} L(k,\theta)d\theta \\ &= \int_0^{\pi/4}[L(k,\theta) + L(k,\pi/2-\theta)]d\theta \\ &\leq \pi - 2.\end{aligned}$$

Note that

$$\int_0^{\pi/2} L(0,\theta)d\theta = \pi - 2.$$

It is therefore suffices to prove that

$$\frac{\partial}{\partial k}L(k,\theta) + \frac{\partial}{\partial k}L(k,\pi/2-\theta) \leq 0, \tag{8}$$

for all $0 \leq \theta \leq \pi/4$. For this we easily find that

$$\begin{aligned}&\frac{\partial L(k,\theta)}{\partial k} + \frac{\partial L(k,\pi/2-\theta)}{\partial k} \\ &\quad = k(1-\sin\theta)(1-\cos\theta)[(1-k^2\sin^2\theta)^{-3/2}\{\sqrt{1-k^2}(1+\cos\theta)-(1+\sin\theta)\} \\ &\qquad + (1-k^2\cos^2\theta)^{-3/2}\{\sqrt{1-k^2}(1+\sin\theta)-(1+\cos\theta)\}](1-k^2)^{-1/2}.\end{aligned}$$

Since the inequalities

$$(1-k^2\cos^2\theta)^{-3/2} \geq (1-k^2\sin^2\theta)^{-3/2}$$

and

$$1+\cos\theta - \sqrt{1-k^2}(1+\sin\theta) \geq -(1+\sin\theta) + \sqrt{1-k^2}(1+\cos\theta)$$

hold for $0 \leq \theta \leq \pi/4$, (8) follows easily. This finishes the proof of the theorem.

References

[1] R. BALASUBRAMANIAN, V. KARUNAKARAN and S. PONNUSAMY, A proof of Hall's conjecture on starlike mappings, *J. London Math. Soc.* (2) **48**(1993), 278-288.

[2] R.R. HALL, A conformal mapping inequality for starlike functions of order 1/2, *Bull. London Math. Soc.* 12 (1980) 119-126.

[3] T. SHEIL-SMALL, Some conformal mapping inequalities for starlike and convex functions, *J. London Math. Soc.* (2) 1 (1969) 577-587.

R. Balasubramanian
Institute of Mathematical Sciences
C.I.T. Campus, Madras -600 113, INDIA.

S. Ponnusamy
Department of Mathematics
PO. BOX 4 (Yliopistonkatu 5)
00014 University of Helsinki, Finland.
e-mail:samy@geom.helsinki.fi