Some new classes of extended generalized quadrangles of order $(q + 1, q - 1)$

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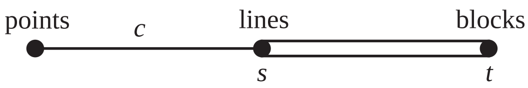
Abstract

In this paper three new classes of extended generalized quadrangles of order $(q + 1, q - 1)$ are constructed.

1 Introduction

An *extended generalized quadrangle* of *order* (s, t) (EGQ(s, t) for short) is a connected geometry with three types of elements, say *points, lines* and *blocks* (or *planes*) belonging to the following diagram:



This means that point-residues are generalized quadrangles of order (s, t), that block-residues are isomorphic to the complete graph K_{s+2} on $s + 2$ vertices, and that line-residues are generalized digons. The parameters are assumed to be finite. An EGQ(s, t) is said to satisfy *property* (LL) if any two distinct points are incident with at most one common line.

The residue of an element x of an extended generalized quadrangle Γ will be denoted by $\mathrm{Res}(x)_\Gamma$, or $\mathrm{Res}(x)$ for short. We say that Γ is an extension of the generalized quadrangle $\mathcal{S}$ if $\mathrm{Res}(x) \cong \mathcal{S}$ for every point x of Γ. Given two extended generalized quadrangles $\widetilde{\Gamma}$ and Γ, a *covering* from $\widetilde{\Gamma}$ to Γ is an incidence-preserving

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mapping σ from $\widetilde{\Gamma}$ to Γ inducing an isomorphism from $\mathrm{Res}(x)_{\widetilde{\Gamma}}$ to $\mathrm{Res}(x^\sigma)_\Gamma$, for every element x of $\widetilde{\Gamma}$. We say that $\widetilde{\Gamma}$ is a *cover* of Γ if there are coverings from $\widetilde{\Gamma}$ to Γ. The extended generalized quadrangle Γ is said to be *simply connected* if it does not admit a proper cover, that is, a cover for which σ is not an isomorphism. The extended generalized quadrangle $\widetilde{\Gamma}$ is called an $m-$*fold cover* of Γ (conversely, Γ is an $m-$*fold quotient* of $\widetilde{\Gamma}$) if all fibers of σ have size m.

Many infinite families and sporadic examples of extended generalized quadrangles are known, in particular there are many examples in which the point-residues are grids or dual grids. Examples with thick classical point-residues are quite rare. For a recent survey on extended generalized quadrangles of order $(q-1, q+1)$ and $(q+1, q-1)$ we refer to Del Fra and Pasini [8].

Here we construct some new classes of extended generalized quadrangles of order $(q+1, q-1)$.

2 The known extended generalized quadrangles of order $(q+1, q-1)$

2.1 Small examples

Three simply connected EGQ$(3,1)$s, with 70, 40 and 40 points respectively are due to Blokhuis and Brouwer [2]. The EGQ$(3,1)$ with 70 points admits a 2-fold quotient; one of the two EGQ$(3,1)$s with 40 points is a double cover of an EGQ$(3,1)$ due to Cameron and Fisher [5].

An EGQ$(4,2)$ with 126 points is due to Buekenhout and Hubaut [4]. It admits a 3-fold cover. Further, an EGQ$(4,2)$ with 162 points has been discovered by Pasechnik [12]; it is simply connected.

Two simply connected EGQ$(5,3)$s have been discovered by Yoshiara [15]; one of these geometries admits a 2-fold quotient.

2.2 An infinite family by Del Fra and Pasini

An infinite family of extended generalized quadrangles of order $(q+1, q-1)$, with q odd, has been discovered by Del Fra and Pasini [8]. The point-residues of such an EGQ$(q+1, q-1)$ are isomorphic to the dual of the Ahrens-Szekeres generalized quadrangle AS(q) (for a description of AS(q) see Payne and Thas [13]). Also, these EGQ$(q+1, q-1)$s are *flat*, that is, every point is incident with every block.

We remark that the (uniquely defined) EGQ$(4,2)$ of Del Fra and Pasini is not isomorphic to any one of the three EGQ$(4,2)$s of 2.1.

2.3 A construction by Yoshiara

In PG$(5,q)$, q even, let S be a set of $q+3$ planes $\pi_1, \pi_2, \ldots, \pi_{q+3}$, such that

(a) $\pi_i \cap \pi_j$ is a point for all $i, j = 1, 2, \ldots, q+3$ with $i \neq j$,

(b) the set $O_i = \{\pi_i \cap \pi_j \parallel j \in \{1, 2, \ldots, q+3\} \setminus \{i\}\}$ is a *hyperoval* of π_i, for all $i = 1, 2, \ldots, q+3$,

(c) S generates $\mathrm{PG}(5, q)$.

Then Yoshiara [14] constructs as follows an $\mathrm{EGQ}(q+1, q-1)$, which we will denote by $\mathcal{Y}_q(S)$. Embed $\mathrm{PG}(5, q)$ as a hyperplane in a projective space $\mathrm{PG}(6, q)$. Points of $\mathcal{Y}_q(S)$ are the 3-dimensional subspaces of $\mathrm{PG}(6, q)$ which contain an element of S, but are not contained in $\mathrm{PG}(5, q)$. Lines of $\mathcal{Y}_q(S)$ are the lines of $\mathrm{PG}(6, q)$ which contain one of the points $\pi_i \cap \pi_j, i \neq j$, but are not contained in $\mathrm{PG}(5, q)$. Blocks of $\mathcal{Y}_q(S)$ are the points of $\mathrm{PG}(6, q) \setminus \mathrm{PG}(5, q)$. If we take symmetrized inclusion as the incidence relation, then we obtain an $\mathrm{EGQ}(q+1, q-1)$, with $q^3(q+3)$ points and diameter 3. Yoshiara shows that $\mathcal{Y}_q(S)$ satisfies (LL) and that for any three distinct mutually collinear points of $\mathcal{Y}_q(S)$ there is a unique block incident with all of them.

Yoshiara [14] constructs as follows a set S of planes in $\mathrm{PG}(5, q)$, q even, satisfying (a), (b) and (c). Let O^* be a set of $q+2$ lines of $\mathrm{PG}(2, q)$, q even, no three of which are concurrent, that is, O^* is a *dual hyperoval* of $\mathrm{PG}(2, q)$. Further, let

$$\zeta : \mathrm{PG}(2, q) \to \mathrm{PG}(5, q); (x_0, x_1, x_2) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2).$$

Then ζ is a bijection from $\mathrm{PG}(2, q)$ onto the *Veronese surface* $\mathcal{V}_2^4$; see Hirschfeld and Thas [11] for a detailed description of $\mathcal{V}_2^4$. The $q+2$ lines of O^* are mapped by ζ onto $q+2$ conics $C_1, C_2, \ldots, C_{q+2}$ of $\mathcal{V}_2^4$. The set of the *nuclei* of these conics is a hyperoval O in the *nucleus* of $\mathcal{V}_2^4$. Let π_i be the plane of $C_i, i = 1, 2, \ldots, q+2$, and let π_{q+3} be the plane of O (π_{q+3} is the nucleus of $\mathcal{V}_2^4$). Then the set $S = \{\pi_1, \pi_2, \cdots, \pi_{q+3}\}$ satisfies conditions (a),(b) and (c). Let n_i be the nucleus of the conic $C_i, i = 1, 2, \ldots, q+2$. If Γ is the $\mathrm{EGQ}(q+1, q-1)$ corresponding to S, then the point-residue in Γ of a point Σ, with Σ a 3-dimensional space containing π_i, is isomorphic to the dual of the generalized quadrangle $T_2^*(C_i \cup \{n_i\})$ defined by the hyperoval $C_i \cup \{n_i\}, i = 1, 2, \ldots, q+2$; the point-residue in Γ of a point Σ, with Σ a 3-dimensional space containing the nucleus π_{q+3} of $\mathcal{V}_2^4$, is isomorphic to the dual of $T_2^*(O)$. Remark that the $q+2$ hyperovals $C_i \cup \{n_i\}, i = 1, 2, \ldots, q+2$, are *regular* (that is, contain a conic). If the dual hyperoval O^* is regular, then also O is regular, and in such a case Γ is an extension of the dual of $T_2^*(O)$.

For these examples of Yoshiara we will also use the notation $\mathcal{Y}_q(O^*)$. For $q = 2$ Yoshiara [14] shows that $\mathcal{Y}_2(O^*)$ is the double cover of the $\mathrm{EGQ}(3, 1)$ of Cameron and Fisher mentioned in 2.1; moreover, $\mathcal{Y}_4(O^*) = \mathrm{EGQ}(5, 3)$ is the 2-fold quotient mentioned in 2.1.

3 New sets of planes satisfying the conditions of Yoshiara

Let K be a $(q+1)$-*arc* of $\mathrm{PG}(3, q)$, q even and $q > 2$, that is, let K be a set of $q+1$ points no 4 of which are in a plane. In Casse and Glynn [6] it is proved that K is projectively equivalent to $\{(1, t, t^{2^m}, t^{2^m+1}) \parallel t \in \mathrm{GF}(q)\} \cup \{(0, 0, 0, 1)\}$, with $q = 2^h$, $1 \leq m \leq h-1$, and m coprime to h. The $(q+1)$-arc K is a twisted cubic if and only if $m = 1$ or $h-1$.

Let $K = \{p_1, p_2, \ldots, p_{q+1}\}$. Through each point p_i of K there pass exactly two lines L_i, M_i such that for each $j \neq i$ the plane L_ip_j respectively M_ip_j has just p_i and p_j in common with K, $i = 1, 2, \ldots, q+1$ (see Hirschfeld [10]). The lines L_i, M_i are called the *special unisecants* of K at p_i, $i = 1, 2, \ldots, q+1$. Also, the special unisecants $L_1, L_2, \ldots, L_{q+1}, M_1, M_2, \ldots, M_{q+1}$ are the generators of a hyperbolic quadric $\mathcal{H}$ (see Hirschfeld [10]). Notations will be chosen in such a way that $\{L_1, L_2, \cdots, L_{q+1}\}, \{M_1, M_2, \ldots, M_{q+1}\}$ are the systems of generators of $\mathcal{H}$.

Let θ be the *Klein mapping*, that is, θ maps the set $\mathcal{L}$ of lines of $\mathrm{PG}(3, q)$ bijectively onto the *Klein quadric* $\mathcal{K}$ (see Hirschfeld [10]); the quadric $\mathcal{K}$ is a non-singular hyperbolic quadric of $\mathrm{PG}(5, q)$. Now we put $\langle p_i, p_j\rangle^\theta = x_{ij}, L_i^\theta = l_i$ and $M_i^\theta = m_i$, $i, j = 1, 2, \ldots, q+1$ and $i \neq j$. Then for any $i \in \{1, 2, \ldots, q+1\}$ the points $x_{i1}, x_{i2}, \ldots, x_{i,i-1}, x_{i,i+1}, \ldots, x_{i,q+1}, l_i, m_i$ form a hyperoval O_i of a plane π_i of $\mathcal{K}$. As the subgroup of $\mathrm{PGL}(4, q)$ fixing K acts sharply 3-transitive on K, the hyperovals $O_1, O_2, \ldots, O_{q+1}$ are projectively equivalent, and so the hyperoval O_i is projectively equivalent to $\widetilde{O} = \{(1, t, t^{2^m}) \parallel t \in \mathrm{GF}(q)\} \cup \{(0, 0, 1), (0, 1, 0)\}$. Hence O_i is a *translation hyperoval* of π_i (see Hirschfeld [9]). Further, $\{l_1, l_2, \ldots, l_{q+1}\}$ is a conic C_1 of some plane π_{q+2} and $\{m_1, m_2, \ldots, m_{q+1}\}$ is a conic C_2 of some plane π_{q+3}. The planes π_{q+2}, π_{q+3} are polar with respect to the symplectic polarity defined by $\mathcal{K}$, and $\pi_{q+2} \cap \pi_{q+3} = \{n\}$, with n the common nucleus of C_1 and C_2.

It is an easy exercise to check that the set $S = \{\pi_1, \pi_2, \ldots, \pi_{q+3}\}$ satisfies conditions (a), (b) and (c). Hence there arises an $\mathrm{EGQ}(q+1, q-1)$. The point-residue in Γ of a point Σ, with Σ a 3-dimensional space containing $\pi_i, i \in \{1, 2, \ldots, q+1\}$ is isomorphic to the dual of the generalized quadrangle $T_2^*(\widetilde{O})$; the point-residue in Γ of a point Σ, with Σ a 3-dimensional space containing $C_i, i \in \{1, 2\}$, is isomorphic to the dual of the generalized quadrangle $T_2^*(O')$, with O' a regular hyperoval. If K is a twisted cubic, then Γ is an extension of the dual of $T_2^*(O')$, with O' a regular hyperoval.

For the $\mathrm{EGQ}(q+1, q-1)$ arising from the $(q+1)$-arc K we will also use the notation $\mathcal{Y}_q(K)$.

Theorem 1 *The extended generalized quadrangle $\mathcal{Y}_q(O^*)$ is isomorphic to the extended generalized quadrangle $\mathcal{Y}_q(K), q \neq 2$, if and only if O^* is regular and K is a twisted cubic.*

Proof. Assume that $\mathcal{Y}_q(O^*) \cong \mathcal{Y}_q(K)$, $q \neq 2$. The geometry $\mathcal{Y}_q(O^*)$ has at least $q^3(q+2)$ point-residues isomorphic to the dual of $T_2^*(O')$, with O' a regular hyperoval, and $\mathcal{Y}_q(K)$ has at least $q^3(q+1)$ point-residues isomorphic to the dual of $T_2^*(\widetilde{O})$, with $\widetilde{O} = \{(1, t, t^{2^m}) \parallel t \in \mathrm{GF}(q)\} \cup \{(0, 0, 1), (0, 1, 0)\}$. It follows that $T_2^*(\widetilde{O}) \cong T_2^*(O')$, and so, by Bichara, Mazzocca and Somma [1], $\widetilde{O}$ and O' are projectively equivalent. Hence $\widetilde{O}$ is regular, that is, $m \in \{1, h-1\}$. So K is a twisted cubic. In such a case $\mathcal{Y}_q(K)$ is an extension of the dual of $T_2^*(\widetilde{O})$, with $\widetilde{O}$ regular. It follows that $T_2^*(\widetilde{O}) \cong T_2^*(O)$, with O the dual of O^*. So the hyperovals $\widetilde{O}$ and O are projectively equivalent, that is, O is regular. Hence also O^* is regular.

Conversely, assume that K is a twisted cubic and that O^* is regular. By Cossidente, Hirschfeld and Storme [7] and with the notations of Section 3, the $(q+1)q/2$ points x_{ij} are points of a Veronese surface $\mathcal{V}_2^4 \subset \mathcal{K}$; also the tangent lines of (the algebraic curve) K are mapped by the Klein mapping onto points of $\mathcal{V}_2^4$. As the tangent lines are special unisecants of K, see Hirschfeld [10], one of the conics $C_i, i = 1, 2$, is contained in $\mathcal{V}_2^4$; say $C_1 \subset \mathcal{V}_2^4$. It is clear that $C_2 \cup \{n\}$ consists of the nuclei of the conics $C_1, O_1 \setminus \{m_1\}, \ldots, O_{q+1} \setminus \{m_{q+1}\}$. Further, $C_1^{\zeta^{-1}}, (O_1 \setminus \{m_1\})^{\zeta^{-1}}, \ldots, (O_{q+1} \setminus \{m_{q+1}\})^{\zeta^{-1}}$, with ζ the Veronese mapping of $\mathrm{PG}(2,q)$ onto $\mathcal{V}_2^4$, are lines of $\mathrm{PG}(2,q)$. As no three of the conics $C_1, O_1\setminus\{m_1\}, \cdots, O_{q+1}\setminus\{m_{q+1}\}$ have a point in common, no three of the corresponding lines are concurrent, so these lines form a dual hyperoval $\widetilde{O}^*$; as the nuclei of the $q+2$ conics $C_1, O_1\setminus\{m_1\}, \ldots, O_{q+1}\setminus\{m_{q+1}\}$ form a regular hyperoval $C_2 \cup \{n\}$, also $\widetilde{O}^*$ is regular. Now it is clear that $\mathcal{Y}_q(K) = \mathcal{Y}_q(\widetilde{O}^*) \cong \mathcal{Y}_q(O^*)$. ∎

4 A new construction of extended generalized quadrangles of order $(q+1, q-1)$

In $\mathrm{PG}(4,q)$, q even, let R be a set of $q+3$ planes $\alpha_1, \alpha_2, \ldots, \alpha_{q+3}$, such that

(i) $\alpha_i \cap \alpha_j$ is a point for all $i, j = 1, 2, \ldots, q+3$ with $i \neq j$,

(ii) the set $H_i = \{\alpha_i \cap \alpha_j \parallel j \in \{1, 2, \ldots, q+3\} \setminus \{i\}\}$, is a hyperoval of α_i, for all $i = 1, 2, \cdots, q+3$.

Then an $\mathrm{EGQ}(q+1, q-1)$, denoted $\mathcal{T}_q(R)$, can be constructed as follows. Embed $\mathrm{PG}(4,q)$ as a hyperplane in a projective space $\mathrm{PG}(5,q)$. Points of $\mathcal{T}_q(R)$ are the 3-dimensional subspaces of $\mathrm{PG}(5,q)$ which contain an element of R, but are not contained in $\mathrm{PG}(4,q)$. Lines of $\mathcal{T}_q(R)$ are the lines of $\mathrm{PG}(5,q)$ which contain one of the points $\alpha_i \cap \alpha_j, i \neq j$, but are not contained in $\mathrm{PG}(4,q)$. Blocks of $\mathcal{T}_q(R)$ are the points of $\mathrm{PG}(5,q) \setminus \mathrm{PG}(4,q)$. If we take symmetrized inclusion as the incidence relation, then we obtain an $\mathrm{EGQ}(q+1, q-1)$, with $q^2(q+3)$ points and diameter 2. Clearly $\mathcal{T}_q(R)$ satisfies (LL). However there are triples mutually collinear points in $\mathcal{T}_q(R)$ which are not incident with a common block.

Let $S = \{\pi_1, \pi_2, \ldots, \pi_{q+3}\}$ be a set of planes in some $\mathrm{PG}(5,q)$, q even, satisfying conditions (a), (b) and (c) of Yoshiara. Assume that y is a point of $\mathrm{PG}(5,q)$ not contained in any of the hyperplanes $\langle \pi_i, \pi_j \rangle, i \neq j$. Further, let $\mathrm{PG}(4,q)$ be a hyperplane of $\mathrm{PG}(5,q)$ not containing y. If α_i is the projection of π_i from y onto $\mathrm{PG}(4,q)$, $i = 1, 2, \ldots, q+3$, then $R = \{\alpha_1, \alpha_2, \ldots, \alpha_{q+3}\}$ satisfies conditions (i) and (ii) in $\mathrm{PG}(4,q)$. Consider a $\mathrm{PG}(6,q)$ containing $\mathrm{PG}(5,q)$, construct $\mathcal{Y}_q(S)$, consider a hyperplane $\overline{\mathrm{PG}(5,q)}$ of $\mathrm{PG}(6,q)$ containing $\mathrm{PG}(4,q)$ but not passing through y, and project $\mathcal{Y}_q(S)$ from y onto $\overline{\mathrm{PG}(5,q)}$. Then this projection of $\mathcal{Y}_q(S)$ is the extended generalized quadrangle $\mathcal{T}_q(R)$. Also, $\mathcal{Y}_q(S)$ is a q-fold cover of $\mathcal{T}_q(R)$.

Theorem 2 *The extended generalized quadrangle $\mathcal{Y}_q(O^*)$ of Yoshiara admits a q-fold quotient.*

Proof. Let $\mathcal{Y}_q(O^*) = \mathcal{Y}_q(S)$ and let y be one of the $(q^2-q)/2$ points of the Veronese surface $\mathcal{V}_2^4$ not contained in a plane of S. Let $S = \{\pi_1, \pi_2, \ldots, \pi_{q+3}\}$, with π_{q+3} the nucleus of $\mathcal{V}_2^4$. The hyperplane β of $\mathrm{PG}(5,q)$ generated by π_i and π_j, with $i, j \in \{1, 2, \ldots, q+2\}$ and $i \neq j$, intersects $\mathcal{V}_2^4$ in the conics $\pi_i \cap \mathcal{V}_2^4$ and $\pi_j \cap \mathcal{V}_2^4$. Hence $y \notin \beta$. The hyperplane γ of $\mathrm{PG}(5,q)$ generated by π_{q+3} and π_i, $i \in \{1, 2, \ldots, q+2\}$, intersects $\mathcal{V}_2^4$ in the conic $\pi_i \cap \mathcal{V}_2^4$. Hence $y \notin \gamma$. It follows that projection from y yields a q-fold quotient $\mathcal{T}_q(R)$ of $\mathcal{Y}_q(O^*)$. ∎

Corollary

The unique $\mathcal{Y}_4(O^*)$, which is an EGQ$(5,3)$, admits a 4-fold quotient.

Theorem 3 *The extended generalized quadrangle $\mathcal{Y}_q(K), q \neq 2$, admits a q-fold quotient.*

Proof. Let $\mathcal{Y}_q(K) = \mathcal{Y}_q(S)$ and let L be an *imaginary chord* of the $(q+1)$-arc K; for the definition of imaginary chord, see Bruen and Hirschfeld [3]. Further, let $y = L^\theta$, with θ the Klein mapping. We will use the notations of Section 3. Let δ be the 4-dimensional space generated by π_i and π_j, with $i, j \in \{1, 2, \ldots, q+1\}$ and $i \neq j$. Then $\delta \cap \mathcal{K}$ is singular, and so $(\delta \cap \mathcal{K})^{\theta^{-1}}$ consists of all lines of PG$(3,q)$ concurrent with the line $x_i x_j$. By Bruen and Hirschfeld [3] $x_i x_j \cap L = \emptyset$ and so $y \notin \delta \cap \mathcal{K}$, in particular $y \notin \delta$. The set of tangent lines of K is either $\{L_1, L_2, \cdots, L_{q+1}\}$ or $\{M_1, M_2, \ldots, M_{q+1}\}$, say $\{L_1, L_2, \ldots, L_{q+1}\}$. Next, let δ be the 4-dimensional space generated by π_i and π_{q+2}, with $i \in \{1, 2, \ldots, q+1\}$. Then $\delta \cap \mathcal{K}$ is singular, and $(\delta \cap \mathcal{K})^{\theta^{-1}}$ consists of all lines of PG$(3,q)$ concurrent with some line W through x_i. The nucleus m_i of the oval $O_i \setminus \{m_i\}$ is collinear on $\mathcal{K}$ with all points of C_1, and so m_i is the vertex of the cone $\delta \cap \mathcal{K}$, that is, $M_i = W$. By Bruen and Hirschfeld [3] $L \cap L_i = \emptyset$ for all $i = 1, 2, \ldots, q+1$, and as $L_1 \cup L_2 \cup \ldots \cup L_{q+1} = M_1 \cup M_2 \cup \ldots \cup M_{q+1} = \mathcal{H}$, we also have $L \cap M_i = \emptyset$ for all $i = 1, 2, \ldots, q+1$. So $y \notin \delta \cap \mathcal{K}$, in particular $y \notin \delta$. Next, let δ be the space generated by π_i and π_{q+3}, with $i \in \{1, 2, \ldots, q+1\}$. Then $\delta \cap \mathcal{K}$ is singular, and $(\delta \cap \mathcal{K})^{\theta^{-1}}$ consists of all lines of PG$(3,q)$ concurrent with some line through x_i. The nucleus l_i of the oval $O_i \setminus \{l_i\}$ is collinear on $\mathcal{K}$ with all points of C_2, and so l_i is the vertex of the cone $\delta \cap \mathcal{K}$, that is, $M = L_i$. By Bruen and Hirschfeld [3] $L \cap L_i = \emptyset$ for all $i = 1, 2, \ldots, q+1$, and so $y \notin \delta \cap \mathcal{K}$, in particular $y \notin \delta$. Finally, let δ be the space generated by π_{q+2} and π_{q+3}. Then $\delta \cap \mathcal{K}$ is non-singular and $(\delta \cap \mathcal{K})^{\theta^{-1}}$ consists of all lines of PG$(3,q)$ tangent to the hyperbolic quadric $\mathcal{H}$ with generators $L_1, \ldots, L_{q+1}, M_1, \ldots, M_{q+1}$. As $L \cap \mathcal{H} = \emptyset$, we have $y \notin \delta \cap \mathcal{K}$, and so $y \notin \delta$. Now it follows that projection from y yields a q-fold quotient $\mathcal{T}_q(R)$ of $\mathcal{Y}_q(K)$. ■

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