All 2-(21,7,3) designs are residual

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Abstract

In a previous classification of symmetric 2-$(31, 10, 3)$ designs it was discovered that the 151 pairwise non-isomorphic designs found yielded a total of 3809 residual 2-$(21, 7, 3)$ designs that were pairwise non-isomorphic. Here we report on a computer search for all 2-$(21, 7, 3)$ designs which showed that the 3809 obtained above constitute the complete set.

1 Introduction

By a 2-(v, k, λ) design we mean a pair $\mathcal{D} = (\mathcal{X}, \mathcal{B})$, where $\mathcal{X}$ is a set of v 'points' and $\mathcal{B}$ is a collection of b 'blocks' together with an incidence relation that satisfies the following conditions: each block is incident with k points and each pair of distinct points is incident with λ blocks. For more details and basic facts concerning these 2-(v, k, λ) designs see [1] and [5]. From a given symmetric ($b = v$) 2-(v, k, λ) design $\mathcal{D} = (\mathcal{X}, \mathcal{B})$ there is a way of constructing its *residual* design. This is obtained by fixing a block $B \in \mathcal{B}$ and taking $\mathcal{D}' = (\mathcal{X} \setminus B, \mathcal{B}')$, where $\mathcal{B}' = \{B' \setminus B : B' \in \mathcal{B}, B' \neq B\}$, and the incidence relation is that induced from $\mathcal{D}$. The parameters of the residual design are $(v - k, k - \lambda, \lambda)$. Any design with the parameters of a residual design is called *quasi-residual*. It is well-known [5, Theorem 16.1.3] that any quasi-residual design with $\lambda = 1$ or 2 is in fact residual, but when $\lambda > 2$ the situation is somewhat different. There is a 2-$(16, 6, 3)$ design, whose construction is due to Bhattacharya [2], and which is not the residual of a 2-$(25, 9, 3)$ design since it has two blocks that intersect in four points. In the Tables of [7] the three 'smallest' sets of parameters of 2-designs with $\lambda = 3$ that are quasi-residual designs are 2-$(8, 4, 3)$ (number 15), 2-$(16, 6, 3)$ (number 35)

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and 2-$(21,7,3)$ (number 49). In the first of these cases all designs are in fact residual, and this is perhaps not surprising since they are relatively few in number. A computer investigation by the author in 1994 (unpublished) showed that the number of non-isomorphic 2-$(16,6,3)$ designs is $18,920$ and of these only 1305 are the residuals of the 78 symmetric 2-$(25,9,3)$ designs found by Denniston [4]. It turns out that $5,397$ of the $18,920$ designs discovered have two blocks that meet in four points, a property shared by the design discovered by Bhattacharya. The remaining $13,523$ designs all have maximum intersection number 3, and as we have pointed out, the majority of these are non-embeddable.

Without going into details we simply note that the method that the author used successfully on several different occasions [8],[9], [10] was able to cope with the 2-$(21,7,3)$ case, and yielded the astonishing (to the author) result that the figure of 3809 mentioned above was in fact the correct number. *All quasi-residual* 2-$(21,7,3)$ *designs are residual.* The figure of 3809 given in [7] was, at the time it was printed, not known to be true. It was taken from the author's paper [8] where it was quoted as a lower bound. It was the total number of residual designs that came from the complete classification of symmetric 2-$(31,10,3)$ designs.

With the knowledge of this discovery it surely would not be too long before a computer-free proof would be obtained, or so the author thought. However, despite spending a considerable amount of time on the problem he has been unable to establish a proof. He hopes that by bringing this problem to the attention of others, one of the readers might discover a solution to the problem.

In the next section we list a few of the elementary results that the author has been able to establish and which might be of use.

2 Some properties of 2-$(21,7,3)$ designs

The aim of this section is to prove that two distinct blocks of a 2-$(21,7,3)$ design meet in at most three points. As a first step in this direction we use the following result.

Proposition 1
Two distinct blocks of a 2-$(21,7,3)$ *design meet in at most four points.*

Proof. Following Connor [3, Cor. 3.1], if two distinct blocks of a 2-(v,k,λ) design meet in μ points then, in the usual notation,

$$\mu \leq \left(2\lambda k + r(r-\lambda-k)\right)/r.$$

From this it is seen that $\mu \leq (2\times 3\times 7 + 10\times 0)/10 = 4.2$, and the stated result immediately follows. ■

Proposition 2
Let B be a fixed block of a 2-$(21,7,3)$ *design and for $i = 0,1,\ldots,4$ let n_i denote the number of other blocks that meet B in i points. Then the intersection numbers (n_0,n_1,n_2,n_3,n_4) take one of the four possible sets of values shown in* TABLE I.

Proof. A simple counting argument gives

$$\sum_{i=0}^{4} n_i = 29,\ \sum_{i=0}^{4} i n_i = 63,\ \sum_{i=0}^{4} \binom{i}{2} n_i = 42,$$

and combining these suitably yields $3n_0 + n_1 + n_4 = 3$. Thus $n_0 = 0$ or 1 and the stated result follows immediately. ■

TABLE I

n_0	n_1	n_2	n_3	n_4
0	1	24	2	2
0	2	21	5	1
0	3	18	8	0
1	0	21	7	0

Although Proposition 1 allows the possibility that $n_4 \neq 0$, we can show quite simply that this cannot in fact happen. For this we follow an argument of Hamada and Kobayashi [6].

Let $B_1, B_2, \ldots, B_b$ be the blocks of a 2-(v,k,λ) design and let S denote the incidence matrix of these blocks. Then S is a $(0,1)$ matrix of size $v \times b$ whose (i,j)th entry is 1 if the ith element is in the block B_j and 0 otherwise. It is clearly the case that $SS^t = (r-\lambda)I + \lambda J$, where, as usual, I and J are the identity matrix and the all-one matrix, respectively, of order v. Now define $C = S^t S$, so that C is of size $b \times b$ and satisfies the relations $C\mathbf{j} = rk\mathbf{j}$ ($\mathbf{j}$ is the all-one vector) and $C^2 = (r-\lambda)C + \lambda k^2 J$. Since $C_{rs} = |B_r \cap B_s|$, the following identity is easily established.

$$\sum_{i \neq r,s} (C_{ir}-2)(C_{is}-2) = \lambda k^2 + 4k + 4b - 4rk - 8 - (2k + \lambda - r - 4)C_{rs}.$$

Suppose now that the blocks B_r and B_s of a 2-$(21,7,3)$ design meet in four points. For these two blocks we should then have

$$\sum_{i \neq r,s} (C_{ir}-2)(C_{is}-2) = -5,$$

but examination of the entries in TABLE I shows that in the two possible cases in question, $\sum_{i \neq r,s}(C_{ir}-2)(C_{is}-2) \geq -4$. Thus we have proved:

Theorem 1
Two distinct blocks of a 2-$(21,7,3)$ *design can have at most three points in common.*

2.1 Case $(n_0, n_1, n_2, n_3) \equiv (1, 0, 21, 7)$

Consider a fixed block, B_0 say, of a 2-$(21,7,3)$ design having intersection numbers $(1,0,21,7)$. This induces a sub-design on the seven points of B_0 in which there are 21 blocks of size 2 and 7 blocks of size 3, and each pair of points occurs twice among the blocks. An easy counting argument shows that each point lies in six of the blocks of size 2 and one of the blocks of size 3. Thus the blocks of each size form 1-designs. The same argument can be repeated for the set of seven points belonging to the (unique) block, B_1 say, disjoint from B_0. We immediately see that the intersection numbers of the 28 blocks (all blocks except B_0 and B_1) with the fourteen points belonging to the union of B_0 and B_1, must be 4, 5 or 6. However, closer examination along the lines of Proposition 2 shows that only 4 and 6 are possible. It follows that the intersections of the same 28 blocks with the seven points in neither B_0 nor B_1 are 3 and 1. It is clear that the 21 blocks of size three on these seven points form a 2-$(7,3,3)$ design, of which there are 10 [7]. Thus the points and blocks of the 2-$(21,7,3)$ design can be permuted so that the incidence matrix takes the form

$$\begin{bmatrix} \mathbf{j} & \mathbf{0} & A & B \\ \mathbf{0} & \mathbf{j} & C & D \\ \mathbf{0} & \mathbf{0} & I & E \end{bmatrix}, \tag{1}$$

where $\mathbf{j}$ and $\mathbf{0}$ are the all-one vector and the all-zero vector of size 7 respectively, and A, B, C, D are the incidence matrices of one-designs on 7 points, with the respective block sizes $3, 2, 3, 2$. Further, E is the incidence matrix of a 2-$(7,3,3)$ design and I is the identity matrix of order 7. It would seem plausible that the one-designs above are in fact 2-designs, and indeed this is sometimes so, as the example below shows.

Example Let B_1, B_2, B_3 be the cyclic zero-one matrices of order 7 which are defined in terms of their first rows: $B_1 = \text{cycl}(0001100), B_2 = \text{cycl}(0010010)$, $B_3 = \text{cycl}(0100001)$. Then B_1, B_2, B_3 are symmetric, commute in pairs and satisfy $B_1 + B_2 + B_3 = J - I$. Also, the matrix $B = \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}$ is the incidence matrix of a 2-$(7,2,2)$ design. Further, let A denote the cyclic incidence matrix of a finite projective plane of order 2. It is now a straightforward matter to verify that the matrix

$$\begin{bmatrix} \mathbf{j} & \mathbf{0} & A & B_1 & B_2 & B_3 \\ \mathbf{0} & \mathbf{j} & A & B_3 & B_1 & B_2 \\ \mathbf{0} & \mathbf{0} & I & A^t & A^t & A^t \end{bmatrix}$$

is the incidence matrix of a 2-$(21,7,3)$ design. Its full automorphism group has order 21. Moreover, it is also easy to see that it is residual, as the following matrix shows.

$$\begin{bmatrix} \mathbf{j} & \mathbf{0} & A & B_1 & B_2 & B_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{j} & A & B_3 & B_1 & B_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & A^t & A^t & A^t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A & B_2 & B_3 & B_1 & \mathbf{j} \\ 1 & 1 & \mathbf{0} & \mathbf{j}^t & \mathbf{0} & \mathbf{0} & 1 \\ 1 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{j}^t & \mathbf{0} & 1 \\ 1 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{j}^t & 1 \end{bmatrix}.$$

This illustration is by no means typical. In fact, there are 854 2-$(21,7,3)$ designs that have a pair of disjoint blocks and in 755 of these none of the one-designs mentioned above is a 2-design. The matrices A and C referred to above in (1) have row and column sums 3 and have the property that the inner product of any two distinct rows is $0, 1$ or 2. There are exactly

10 such one-designs that are pairwise non-isomorphic and all but one of them appear amongst the 854 designs. Moreover, the matrices B and D are uniquely determined up to column permutations by A and C, respectively. It is perhaps also worthwhile pointing out that all ten 2-$(7, 3, 3)$ designs do in fact occur as sub-designs with incidence matrix E.

2.2 Case $(n_0, n_1, n_2, n_3) \equiv (0, 3, 18, 8)$

If the design does not have a pair of disjoint blocks, then clearly all blocks have the same intersection array, namely $(0, 3, 18, 8)$. Thus we may assume that the design has just three intersection numbers, $1, 2$ or 3. In the literature there seems to be very little known about such designs unless one of the intersection numbers is $k - r + \lambda$, which is not the case here.

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