Group invariants of certain Burn loop classes

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Abstract

In this paper, we determine the collineation groups generated by Bol reflections, the core, the automorphism groups and the full direction preserving collineation groups of the loops B_{4n} and C_{4n} given by R.P. Burn [6]. These are infinite classes of Bol loops, whose left section $S(L) = \{\lambda_x : x \in L\}$ is invariant under conjugation with the left translations. We also prove some lemmas and use new methods in order to simplify calculations in these groups.

1 Introduction

With any loop $(L, \cdot)$, one can associate several groups, for example its multiplication groups $G_{\text{left}}(L)$ and $G_{\text{right}}(L)$ and $M(L) = \langle G_{\text{left}}(L), G_{\text{right}}(L)\rangle$, the groups of (left or right) pseudo-automorphisms, the group of automorphisms, or the group of collineations of the associated 3-net. Groups which are isotope invariants are of special interest. For example, the groups $G_{\text{left}}(L)$, $G_{\text{right}}(L)$ and $M(L)$ are isotope invariant for any loop L. These groups contain many information about the loop L, the standard references on this field are [2], [4], [11].

For some special loop classes, other isotope invariant groups can be defined. For Bol loops, M. Funk and P.T. Nagy [7] investigated *the collineation group generated by the Bol reflections*. The notion of the *core* was first studied by R.H. Bruck [4] for

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Moufang loops and by V.D. Belousov [3] for Bol loops. Recently, this concept was intensively used by P.T. Nagy and K. Strambach [9].

In the paper [6], R.P. Burn defined two infinite classes of Bol loops, namely the loops B_{4n} for $n \geq 2$ and C_{4n} for $n \geq 2$, n even. These examples satisfy the left conjugacy closed property, that is, their section

$$S(L) = \{\lambda_x : x \in L\}$$

is invariant under conjugation with elements of the group $G_{\text{left}}(L) = \langle \lambda_x | x \in L \rangle$ generated by the (left) translations $\lambda_x : y \mapsto xy$.

2 Basic concepts

A loop L is said to be a *Bol loop*, if

$$x \cdot (y \cdot xz) = (x \cdot yx) \cdot z$$

holds for all $x, y, z \in L$. This is equivalent with $\lambda_x \lambda_y \lambda_x \in S(L)$ for all $x, y \in L$. In any Bol loop, the group

$$N = \langle (\lambda_x^{-1} \rho_x^{-1}, \lambda_x) | x \in L \rangle \tag{1}$$

is a normal subgroup of the directions preserving collineation group of the 3-net belonging to the loop L, cf. [7], [8]. Actually, the fact that $(\lambda_x^{-1} \rho_x^{-1}, \lambda_x)$ is a direction preserving collineation for all $x \in L$ is equivalent with the Bol property for the coordinate loop. As in [7], we define the epimorphism Φ by

$$\Phi : \begin{cases} N \to G(L) = G_{\text{left}}(L) \\ (\lambda_x^{-1} \rho_x^{-1}, \lambda_x) \mapsto \lambda_x. \end{cases} \tag{2}$$

This map Φ will help us to determine the group N, which acts transitively on the set of horizontal lines and, in this way plays an important role in the description of the full collineation group of the 3-net. In general, the only known fact about the kernel of Φ is that it is isomorphic to a subgroup of the left nucleus of L (see [7], Theorem 3.1).

The *core* of a Bol loop $(L, \cdot)$ is the groupoid $(L, +)$, where the binary operation "+" is defined by

$$x + y = x \cdot y^{-1} x, \qquad x, y \in L.$$

This groupoid satisfies the following identities:

$$\begin{aligned} & x + x = x \\ & x + (x + y) = y \qquad\qquad \forall x, y, z \in L \\ & x + (y + z) = (x + y) + (x + z) \end{aligned}$$

An altarnative way to define the core is via the action of the Bol reflections on the set of vertical lines of the associated 3-net. In this way, the core turns out to be strongly related to the group N.

We say that the loop L is *left conjugacy closed*, if $S(L)$ is invariant under the conjugation with the elements of $G(L)$. This concept was introduced in the paper [10] by P.T. Nagy and K. Strambach. They also defined the notion of *Burn loop*, which is a left conjugacy closed Bol loop. Examples for such loops are the following constructions due to R.P. Burn [6].

The section $S(L)$ of a loop L is a sharply transitive set of permutations. For any $x \in L$, there is a uniquely defined λ_x mapping the unit element 1 to x. Thus, by $x \cdot y = y^{\lambda_x}$, the multiplication of L is given by the set $S(L)$ and the choice of some unit element 1. Theorem 7 in [5] says that if the set $S(L)$ is invariant under conjugation with its own elements, different choices of the unit element still give isomorphic loops, hence a Burn loop is completely determined by its section $S(L)$ (up to isomorphism).

The loop B_{4n} for $n \geq 2$: Let the group G_{8n} be generated by the elements α, β, γ with the relations $\alpha^{2n} = \beta^2 = \gamma^2 = (\alpha\beta)^2 = id$, $\alpha\gamma = \gamma\alpha$ and $\beta\gamma = \gamma\beta$. Clearly, G_{8n} is isomorphic to $D_{4n} \times Z_2$, where $D_{4n} = \langle \alpha, \beta \rangle$ and $Z_2 = \langle \gamma \rangle$. Denote by B_{4n} the set of right cosets of $\langle \beta \rangle$ in G_{8n} and define the section $S(B_{4n})$ by

$$S(B_{4n}) = \{\alpha^{2i}, \alpha^{2j+1}\beta, \alpha^k\beta\gamma : i, j \in \{1, \ldots, n\}, k \in \{1, \ldots, 2n\}\}.$$

Then, the action of $S(B_{4n})$ on B_{4n} via right multiplication represents a Burn loop which is non-Moufang, $n \geq 2$. Even if slightly different in construction, it is easy to verify that these loops are isomorphic to the loops in [6].

The loop C_{4n} for $n \geq 2$, n even: Let the group H_{8n} be

$$H_{8n} = \langle \alpha, \beta, \gamma : \alpha^{2n} = \beta^2 = \gamma^2 = (\alpha\beta)^2 = id, \alpha\gamma = \gamma\alpha, \beta\gamma = \gamma\beta\alpha^n \rangle.$$

Similarly to the previous construction, we denote by C_{4n} the set of right cosets of $\langle \beta \rangle$ in H_{8n} and define the section $S(C_{4n})$ by

$$S(C_{4n}) = \{\alpha^{2i}, \alpha^{2j+1}\beta, \alpha^k\beta\gamma : i, j \in \{1, \ldots, n\}, k \in \{1, \ldots, 2n\}\}.$$

Again, the action of $S(C_{4n})$ on C_{4n} via right multiplication represents a Burn loop which is non-Moufang, $n \geq 2$, n even (cf. [6]).

In [10], the authors showed that the square of any element of a Burn loop belongs to the intersection of the left and middle nuclei. In any Bol loop, these two nuclei coincide (cf. [8], Proposition 2.1) and form a normal subgroup of the loop (see Lemma 1). Thus, if L denotes a (left) Bol loop, one can speak of the factor loop L/N_λ of L by the left nucleus N_λ.

Lemma 1
Let $(L, \cdot)$ be a (left) Bol loop. Then its left nucleus N_λ is a normal subgroup of L.

Proof. Let $(L, \cdot)$ be a left Bol loop and consider the groups $G_{\text{left}}(L)$ and $G_{\text{right}}(L)$. Let $M(L)$ denote the group generated by $G_{\text{left}}(L)$ and $G_{\text{right}}(L)$. The Bol identity $x \cdot (y \cdot xz) = (x \cdot yx) \cdot z$ can also be expressed by $\rho_{xz}\lambda_x = \rho_x\lambda_x\rho_z$, or equivalently, $\lambda_x\rho_z\lambda_x^{-1} = \rho_{xz}\rho_z^{-1} \in G_{\text{right}}(L)$. This means that $G_{\text{right}}(L)$ is a normal subgroup of $M(L)$.

Let now U be a permutation of L with $1^U = u$ and let us suppose that U centralizes the group $G_{\text{right}}(L)$. Then we have for any $x \in L$

$$x^U = 1^{\rho_x U} = 1^{U\rho_x} = ux,$$

that is, $U = \lambda_u$. Moreover, $\lambda_u\rho_x = \rho_x\lambda_u$ for all $x \in L$ means exactly that u is an element of the left nucleus $N_\lambda(L)$ of L. Hence, $T = \{\lambda_u : u \in N_\lambda(L)\}$ is the centralizer of the normal subgroup $G_{\text{right}}(L)$ in $M(L)$, it is normal also. This implies that $N_\lambda(L) = 1^T$ is a normal subgroup of L, see [1], Theorem 3. ∎

Remark. Clearly, if L is a Burn loop, the factor loop L/N_λ is Burn as well. This means that in the quotient loop L/N_λ of a Burn loop L every element has order 2.

3 The kernel of the map Φ in Burn loops

In this chapter, the kernel of the map Φ will be determined, for the case that the loop is of Burn type. The elements of $\ker\Phi$ are of the form (λ, id), with $\lambda \in G(L)$; thus $\ker\Phi$ is isomorphic to a subgroup of $G(L)$, say K. (By Theorem 3.1 of [7], even $K \leq S(N_\lambda)$ holds.)

If $a_1, \ldots, a_k$ are elements of a group, then $[a_1, \ldots, a_k]$ denotes the commutator $a_1^{-1} \cdots a_k^{-1} a_1 \cdots a_k$. Let L be a Burn loop. For $k \geq 2$, we define the following subgroup H_k of $G(L)$:

$$H_k = \langle [\lambda_{x_1}, \ldots, \lambda_{x_k}] | x_1, \ldots, x_k \in L, \lambda_{x_1} \cdots \lambda_{x_k} \in S(L) \rangle.$$

Lemma 2
In any Bol loop, $K = \cup_k H_k$. If the loop is of Burn type, we have $\ker\Phi \lhd G(L)$.

Proof. An element of $\ker\Phi$ is of the form $(\rho_{x_0}\lambda_{x_0} \cdots \rho_{x_k}\lambda_{x_k}, \lambda_{x_0}^{-1} \cdots \lambda_{x_k}^{-1})$, where $\lambda_{x_0}^{-1} \cdots \lambda_{x_k}^{-1} = id$, $\lambda_{x_0} = \lambda_{x_1}^{-1} \cdots \lambda_{x_k}^{-1}$. Thus

$$x_0 \cdot (\ldots \cdot (x_{k-2} \cdot x_{k-1}x_k) \ldots) = 1.$$

The Bol property immediately implies that $\rho_x\lambda_x\rho_y = \rho_{xy}\lambda_x$ for all $x, y \in L$. Then

$$\begin{aligned} \rho_{x_0}\lambda_{x_0} \cdots \rho_{x_k}\lambda_{x_k} &= \rho_{x_0\cdot(\ldots\cdot(x_{k-2}\cdot x_{k-1}x_k)\ldots)}\lambda_{x_0} \cdots \lambda_{x_k} \\ &= \lambda_{x_0} \cdots \lambda_{x_k} \\ &= \lambda_{x_1}^{-1} \cdots \lambda_{x_k}^{-1}\lambda_{x_0} \cdots \lambda_{x_k} \\ &= [\lambda_{x_1}, \ldots, \lambda_{x_k}]. \end{aligned}$$

By the left inverse property, there exists an $x_0 \in L$ such that $\lambda_{x_0} \cdots \lambda_{x_k} = id$ if and only if $\lambda_{x_1} \cdots \lambda_{x_k} \in S(L)$. So we have

$$\ker\Phi = \langle [\lambda_{x_0}, \ldots, \lambda_{x_k}] | x_0, \ldots, x_k \in L, \lambda_{x_0} \cdots \lambda_{x_k} \in S(L) \rangle = \bigcup_k H_k.$$

Since in a Burn loop, the set $S(L)$ is invariant under the conjugation with elements λ_y, we have $\ker\Phi \lhd G(L)$. ■

As the square of any element of the Burn loop L is in N_λ, for all $n \in N_\lambda$, $x, y \in L$, the commutators $[\lambda_n, \lambda_x]$ and $[\lambda_x^2, \lambda_y]$ belong to H_2. Using this we prove the following lemma.

Lemma 3
Let $\alpha_1, \ldots, \alpha_k \in S(L)$ and $\bar{\alpha}_i \in S(N_\lambda)$.

(i) $[\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_k] \equiv [\alpha_1, \ldots, \alpha_{i+1}, \alpha_i^{\alpha_{i+1}}, \ldots, \alpha_k] \pmod{H_2}$;

(ii) $[\alpha_1, \ldots, (\alpha_i\bar{\alpha}_i), \ldots, \alpha_k] \equiv [\alpha_1, \ldots, \bar{\alpha}_i, \alpha_i, \ldots, \alpha_k] \pmod{H_2}$;

(iii) $[\alpha_1 \cdots \alpha_k, \bar{\alpha}_i] \in H_2$;

(iv) $[\alpha_1, \ldots, \alpha_i, \bar{\alpha}_i, \ldots, \alpha_k] \equiv [\alpha_1, \ldots, \alpha_k] \pmod{H_2}$.

(v) If the element on the right side of the equivalence (i), (ii) or (iv) is in H_k, then the element on the left side is in H_k, as well.

Proof. (i) We have $\alpha_1 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_k = \alpha_1 \cdots \alpha_{i+1} \alpha_i^{\alpha_{i+1}} \cdots \alpha_k$. On the other hand,

$$\begin{aligned} \alpha_1^{-1} \cdots \alpha_i^{-1} \alpha_{i+1}^{-1} \cdots \alpha_k^{-1} &= \alpha_1^{-1} \cdots \alpha_{i+1}^{-1} (\alpha_i^{-1})^{\alpha_{i+1}} [\alpha_i^{\alpha_{i+1}} (\alpha_i^{-1})^{\alpha_{i+1}^{-1}}] \alpha_{i+2}^{-1} \cdots \alpha_k^{-1} \\ &= \alpha_1^{-1} \cdots \alpha_{i+1}^{-1} (\alpha_i^{-1})^{\alpha_{i+1}} \cdots \alpha_k^{-1} [\alpha_i^{\alpha_{i+1}} (\alpha_i^{-1})^{\alpha_{i+1}^{-1}}]^{\beta}, \end{aligned}$$

where $\beta = \alpha_{i+2}^{-1} \cdots \alpha_k^{-1} \in S(L)$. Now, it is sufficient to show that the expression in the square bracket is an element of H_2: $\alpha_i^{\alpha_{i+1}} (\alpha_i^{-1})^{\alpha_{i+1}^{-1}} = [\alpha_{i+1}^2, \alpha_i^{-1}]^{\alpha_{i+1}^{-1}} \in H_2$.

(ii) By some similar calculation one can show that

$$[\alpha_1, \ldots, (\alpha_i \bar{\alpha}_i), \ldots, \alpha_k] = [\alpha_1, \ldots, \bar{\alpha}_i, \alpha_i, \ldots, \alpha_k][\alpha_i, \bar{\alpha}_i]^{\alpha_{i+1} \cdots \alpha_k},$$

and because of $\bar{\alpha}_i \in S(N_\lambda)$, the last factor is an element of H_2.

$$\begin{aligned} \text{(iii) } [\alpha_1 \cdots \alpha_k, \bar{\alpha}_i] &= [\alpha_2 \cdots \alpha_k, \bar{\alpha}_i^{\alpha_1}][\alpha_1, \bar{\alpha}_i] \\ &\equiv [\alpha_2 \cdots \alpha_k, \bar{\alpha}_i^{\alpha_1}] \equiv \cdots \equiv [\alpha_k, \bar{\alpha}_i^{\alpha_1 \cdots \alpha_k}] \equiv id \pmod{H_2}. \end{aligned}$$

$$\begin{aligned} \text{(iv) } [\alpha_1, \ldots, \alpha_i, \bar{\alpha}_i, \ldots, \alpha_k] &= [\alpha_1, \ldots, \alpha_k, \bar{\alpha}_i^{\alpha_{i+1} \cdots \alpha_k}] \\ &= [\alpha_1, \ldots, \alpha_k][\alpha_1 \cdots \alpha_k, \bar{\alpha}_i^{\alpha_{i+1} \cdots \alpha_k}] \\ &\overset{(iii)}{\equiv} [\alpha_1, \ldots, \alpha_k] \pmod{H_2}. \end{aligned}$$

(v) This follows from $H_2 \lhd H_k \lhd G(L)$. ∎

Proposition 1
Let L be a Burn loop and Φ and H_k $(k \geq 2)$ be defined as in the beginning of this section and let $s = |L : N_\lambda|$. Then $\ker \Phi = H_{s-1}$ if $s \geq 3$, and $\ker \Phi = H_2$ if $s = 1$ or 2.

Proof. Let B be a set of representatives from the cosets of N_λ in L such that $1 \in B$. Then any element of L can be written in a unique way as the product nb, with $n \in N_\lambda$, $b \in B$. Let us choose elements $x_1, \ldots, x_k$, $x_i = n_i b_i$, from L such that $\lambda_{x_1} \cdots \lambda_{x_k} \in S(L)$. By Lemma 3 (ii) and (iv), $[\lambda_{x_1}, \ldots, \lambda_{x_k}] \equiv [\lambda_{b_1}, \ldots, \lambda_{b_k}] \pmod{H_2}$. Applying Lemma 3 and $b_i^2 \in N_\lambda$ several times, one gets $[\lambda_{x_1}, \ldots, \lambda_{x_k}] \equiv [\lambda_{b'_1}, \ldots, \lambda_{b'_m}] \pmod{H_2}$, where $b'_1, \ldots, b'_m$ are different elements of $B \backslash \{1\}$. Moreover, $\lambda_{x_1} \cdots \lambda_{x_k} \equiv \lambda_{b'_1} \cdots \lambda_{b'_m} \pmod{S(N_\lambda)}$, hence $[\lambda_{x_1}, \ldots, \lambda_{x_k}] \in H_m$, with $m \leq |B| - 1$. ∎

Corollary

If the loop L is a group, then $\ker \Phi \cong H_2 = L'$.

Lemma 4
Let the subset B of L be defined as before and let us choose elements $b_1, b_2, b_3 \in B$ such that $b_3N_\lambda \cdot (b_2N_\lambda \cdot b_3N_\lambda) = N_\lambda$ holds in the quotient loop L/N_λ. Then the following conditions are equivalent.

(i) $\lambda_{b_1}\lambda_{b_2}\lambda_{b_3} \in S(L)$.

(ii) $\lambda_{b_i}\lambda_{b_j}\lambda_{b_k} \in S(L)$ *with* $\{i,j,k\} = \{1,2,3\}$.

(iii) $\lambda_{b_1}\lambda_{b_2} \in S(L)$.

(iv) $\lambda_{b_i}\lambda_{b_j} \in S(L)$ *for all* $i,j \in \{1,2,3\}$.

Proof. (i) $\Rightarrow$ (iii). From $b_3N_\lambda \cdot (b_2N_\lambda \cdot b_3N_\lambda) = N_\lambda$ we get $\lambda_{b_1}\lambda_{b_2}\lambda_{b_3} = \lambda_n$, $n \in N_\lambda$. Hence $\lambda_{b_1}\lambda_{b_2} = \lambda_{b_3^{-1}n} \in S(L)$.

(iii) $\Rightarrow$ (i). The quotient is a Burn loop, thus $b_3N_\lambda = b_2N_\lambda \cdot b_1N_\lambda$, $b_2b_1 = b_3n$, $\lambda_{b_1}\lambda_{b_2} = \lambda_n\lambda_{b_3}$, and so $\lambda_{b_1}\lambda_{b_2}\lambda_{b_3} = \lambda_{b_3^2n} \in S(L)$.

The equivalence (ii) $\Leftrightarrow$ (iv) can be shown in the same manner. (iv) $\Rightarrow$ (iii) being trivial, we still have to show (i) $\Rightarrow$ (ii). Supposing (i), we have

$$\lambda_{b_2}\lambda_{b_3}\lambda_{b_1} = \lambda_{b_1}^{-1}\lambda_{b_1}\lambda_{b_2}\lambda_{b_3}\lambda_{b_1} \in S(L)$$

and

$$S(L) \ni \lambda_{b_3}^{-1}\lambda_{b_2}^{-1}\lambda_{b_1}^{-1} = \lambda_{b_3}\lambda_{n_3}\lambda_{b_2}\lambda_{n_2}\lambda_{b_1}\lambda_{n_1} = \lambda_{b_3}\lambda_{b_2}\lambda_{b_1}\lambda_n,$$

with $n_1, n_2, n_3, n \in N_\lambda$, and so $\lambda_{b_3}\lambda_{b_2}\lambda_{b_1} \in S(L)$. This is sufficient to imply (ii). ∎

Proposition 2
If $s = |L : N_\lambda| \leq 7$, then $s \in \{1,2,4\}$ and

$$\ker\Phi = [S(N_\lambda), G(L)] = \langle[\lambda_n, \lambda_x] | n \in N_\lambda, x \in L\rangle.$$

Proof. The quotient L/N_λ is a Bol loop of order $s \leq 7$, and so a group (cf. [5]). In L, the square of any element is in N_λ, since L/N_λ is an elementary abelian 2-group, $s \in \{1,2,4\}$. For $s = 1$ or 2 the statement follows directly from Proposition 1. Let us suppose that $s = 4$. If $b_1N_\lambda, b_2N_\lambda, b_3N_\lambda$ are different nontrivial elements of L/N_λ, then $b_3N_\lambda \cdot b_2N_\lambda \cdot b_1N_\lambda = N_\lambda$. Suppose that $\lambda_{b_1}\lambda_{b_2}$ or $\lambda_{b_1}\lambda_{b_2}\lambda_{b_3}$ is an element of $S(L)$. Then, by Lemma 4, for all $i,j \in \{1,2,3\}$, one has $\lambda_{b_i}\lambda_{b_j} \in S(L)$. This means that for any $x_i, x_j \in L$, $x_{i,j} = b_{i,j}n_{i,j}$ with $n_{i,j} \in N_\lambda$,

$$\lambda_{x_i}\lambda_{x_j} = \lambda_{n_i}\lambda_{b_i}\lambda_{n_j}\lambda_{b_j} = \lambda_{n_j'n_i}\lambda_{b_jb_i} \in S(L),$$

thus L is a group, which contradicts $s = 4$.

This shows that $\ker\Phi = H_3 = [S(N_\lambda), G(L)]$. ∎

4 The groups generated by the Bol reflections and the cores of the loops B_{4n} and C_{4n}

Let us denote by σ_m the Bol reflection with axis $x = m$ (see [7]), by N^+ the collineation group generated by these reflections and by N the subgroup generated by products of even length of reflections. Since a Bol reflection interchanges the horizontal and transversal directions, N^+ does not preserve the directions, but the group N does.

Clearly, N is a normal subgroup of index 2 of N^+ and by the geometric properties of Bol reflections, the set $\Sigma = \{\sigma_x | x \in L\}$ is invariant in N^+. Thus, the elements $\sigma_x\sigma_1$ generate N. Using coordinates, we get the form $\sigma_x\sigma_1 = (p_x, \lambda_x)$ for these generators, where $p_x = \lambda_x^{-1}\rho_x^{-1}$, see [8].

The following lemma will help us to determine the orbit of the y-axis under N.

Lemma 5
Let $(L, \cdot)$ be a Burn loop and let us define the groups

$$F = \langle p_x | x \in L \rangle, \qquad U = \langle \lambda_x^2 | x \in L \rangle.$$

Then, the orbits 1^F and 1^U coincide.

Proof. Using the fact that L is left conjugacy closed, we have

$$1^{p_{y_1} \cdots p_{y_k}} = 1^{\lambda_{y_k}^{-1} \cdots \lambda_{y_1}^{-2} \cdots \lambda_{y_k}^{-1}} = 1^{\lambda_{y_1'}^{-2} \cdots \lambda_{y_k'}^{-2}} \in 1^U,$$

which means $1^F \subseteq 1^U$. On the other hand,

$$1^{p_{y_1} \cdots p_{y_k} \lambda_z^2} = 1^{\lambda_z \lambda_{y_k'}^{-1} \cdots \lambda_{y_1'}^{-2} \cdots \lambda_{y_k'}^{-1} \lambda_z} = 1^{p_{y_1'} \cdots p_{y_k'} p_z^{-1}} \in 1^F$$

shows that 1^F is invariant under U. Thus, $1^F = 1^U$. ∎

Lemma 6
Let $(L, \cdot)$ be a Burn loop and $U \subseteq G(L)$ be an abelian group containing the left translations $\{\lambda_m : m \in N_\lambda\}$. Then the group $\Phi^{-1}(U)$ of collineations is abelian, too.

Proof. The action of an arbitrary collineation (u, v) on the set of transversal lines is $v\lambda_a$, where $a = 1^u$, see [2]. If $(u, v) \in \Phi^{-1}(U)$, then by Lemma 5 $a \in N_\lambda$, hence $\lambda_a \in U$ and $v\lambda_a \in U$. And since U is abelian, this means that the commutator elements of $\Phi^{-1}(U)$ act trivially on the set of horizontal and vertical lines, thus on the whole point set. ∎

In the remaining part of this chapter, we describe the structure of the group invariants of the loops B_{4n} and C_{4n}.

Theorem 3
Let $(L, \cdot)$ be one of the loops B_{4n} or C_{4n}, $n \geq 2$. Then, $N = \ker\Phi \rtimes \bar{G}$, where Φ induces an isomorphism from the subgroup $\bar{G}$ to $G(L)$. Denoting the respective generators of $\bar{G}$ by $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$, and by δ the generator of the cyclic group $\ker\Phi$, $\bar{\alpha}$ and $\bar{\gamma}$ act trivially on $\ker\Phi$, and $\bar{\beta}\delta\bar{\beta} = \delta^{-1}$.

	B_{4n}, n odd	B_{4n}, n even	C_{4n}, $n \equiv 2 \pmod 4$	C_{4n}, $n \equiv 0 \pmod 4$
$\ker\Phi$	Z_n	$Z_{\frac{n}{2}}$	$Z_{\frac{n}{2}}$	$Z_{\frac{n}{2}}$
$\lvert (y\text{-axis})^N \rvert$	n	$\frac{n}{2}$	n	$\frac{n}{2}$

Table 1: The kernel of Φ and the orbit of the y-axis under N

$(L,\cdot)$	λ_x	(p_x,λ_x)
(a) $B_{4n},\ C_{4n},\ n \geq 2$	α^{2i}	$\bar\alpha^{2i}\delta^i$
	$\alpha^{2j+1}\beta$	$\bar\alpha^{2j+1}\bar\beta$
(b) $B_{4n},\ n \geq 2$	$\alpha^k\beta\gamma$	$\bar\alpha^k\bar\beta\bar\gamma$
(c) $C_{4n},\ n \equiv 0 \pmod 4$	$\alpha^k\beta\gamma$	$\bar\alpha^k\bar\beta\bar\gamma\delta^{\frac{n}{4}}$
(d) $C_{4n},\ n \equiv 2 \pmod 4$	$\alpha^k\beta\gamma$	$\bar\alpha^k\bar\beta\bar\gamma$

Table 2: Generating elements for $G(L)$ and N

Proof. In each case of L, $\ker\Phi$ is isomorphic to a subgroup of the cyclic group N_λ of order n. Moreover, Proposition 2 implies the results of Table 1.

If L is either B_{4n}, $n \geq 2$ or C_{4n}, $n \equiv 0 \pmod 4$, then by Table 1, $\ker\Phi$ acts regularly on the orbit $(y\text{-axis})^N$. Hence, in these cases, $\bar G = N_{y\text{-axis}}$ is a good choice.

Let us suppose $L = C_{4n}$, $n \equiv 2 \pmod 4$. Let m be 1^{α^2}. Then m has order n in L, it is a generator of the cyclic group N_λ, and the generating element δ of $\ker\Phi$ can be assumed to be of the form (λ_m^{-2}, id). Let X be the set of vertical lines of equation $x = 1$ or $x = m^{\frac{n}{2}}$. Let us define the subgroup $\bar G$ as the setwise stabilizer of X in N. We associate the N-generator (p_x,λ_x) with the left translation $\lambda_x = \beta\gamma$. Since $1^{p_x} = 1^{(\beta\gamma)^2} = 1^{\alpha^n} = m^{\frac{n}{2}}$, this generator interchanges the lines in X. Therefore $|\bar G : N_{y\text{-axis}}| = 2$ and $|N : \bar G| = n/2$. Clearly, $\bar G \cap \ker\Phi = \{id\}$, and so, $\bar G$ is a transversal to $\ker\Phi$.

To complete the proof, we consider the action of $\bar G$ on $\ker\Phi$. Applying Lemma 6 to $U = \langle\alpha,\gamma\rangle$ we see that $\bar\alpha$ and $\bar\gamma$ commute with $\ker\Phi$. Furthermore, since in each cases of L, $\bar\beta \in N_{y\text{-axis}}$, hence $\bar\beta = (\beta,\beta) \in N_{(1,1)}$ and $\delta^{\bar\beta} = \delta^{-1}$. ■

Lemma 7
The reflection σ_1 is an automorphism of N, which inverts the generators (p_x,λ_x). It always leaves $\bar\alpha$ and $\bar\beta$ invariant and acts on $\bar\gamma$ and δ in the following way.

$$\sigma_1 : \begin{cases} \bar\gamma \mapsto \bar\gamma, & \delta \mapsto \bar\alpha^{-4}\delta^{-1} \quad \text{if } L = B_{4n}, n \geq 2 \text{ or } C_{4n}, n \equiv 0 \pmod 4. \\ \bar\gamma \mapsto \bar\alpha^n\bar\gamma, & \delta \mapsto \bar\alpha^{-4}\delta^{-1} \quad \text{if } L = C_{4n}, n \equiv 2 \pmod 4; \end{cases}$$

Proof. Since $(p_x,\lambda_x) = \sigma_x\sigma_1$, the first statement is immediate. To determine the action of σ_1 on the elements $\bar\alpha$, $\bar\beta$, $\bar\gamma$ and δ, we have to express the generators (p_x,λ_x) of N by these elements. We claim that this is done in Table 2. We therefore use the fact that two collineations (u, v) and (u', v') coincide if $v = v'$ and $1^u = 1^{u'}$, see [2]. Moreover, if (u, v) is a generator element for N, then we have $1^u = 1^{v^{-2}}$.

Again, the cases $L = B_{4n}$, $n \geq 2$ or C_{4n}, $n \equiv 0 \pmod 4$ are trivial, since then $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ stabilize the y-axis and δ acts on it in a well known way. Let us suppose $L = C_{4n}$, $n \equiv 2 \pmod 4$ and denote the N-generator associated to $\alpha^k\beta\gamma$ by $(u, \alpha^k\beta\gamma)$. Then one has $1^u = 1^{(\alpha^k\beta\gamma)^2} = 1^{\alpha^n} = m^{\frac{n}{2}}$, and so, $(u, \alpha^k\beta\gamma) \in \bar{G}$. This gives $(u, \alpha^k\beta\gamma) = \bar{\alpha}^k\bar{\beta}\bar{\gamma}$. The results of Table 2 and the lemma follow. ■

The core of an arbitrary Bol loop $(L, \cdot)$ is the groupoid $(L, +)$ with $x + y = x \cdot y^{-1} x$. Isomorphic versions of the groupoid can be defined in the following ways.

$$(S(L), \oplus), \qquad \lambda_x \oplus \lambda_y = \lambda_x \lambda_y^{-1} \lambda_x;$$
$$(\Sigma, \otimes), \quad \Sigma = \{\sigma_x : x \in L\}, \qquad \sigma_x \otimes \sigma_y = \sigma_x \sigma_y \sigma_x.$$

The isomorphism $(L, +) \cong (S(L), \oplus)$ is trivial, and $(S(L), \oplus) \cong (\Sigma, \otimes)$ can be shown using $\sigma_x\sigma_1 = (p_x, \lambda_x)$. Hence, the permutation group generated by the core acts on L like N^+ acts on Σ by conjugation and this action equals to the action of N^+ on the set of vertical lines. And since Σ generates N^+, the group G_{core} generated by the core is isomorphic to $N^+/Z(N^+)$.

These general properties of the core imply the following result for our special loops B_{4n} and C_{4n}.

Theorem 4
Let L be equal to B_{4n} or C_{4n}. Then the group G_{core} generated by the core is isomorphic to $N^+/Z(N^+)$ where

$$Z(N^+) = \begin{cases} \langle \bar{\alpha}^n, \bar{\gamma}, \sigma_1 \rangle & \text{if } L = B_8; \\ \langle \bar{\alpha}^n, \bar{\gamma} \rangle & \text{if } L = B_{4n}, n \not\equiv 0 \pmod 4, n > 2; \\ \langle \bar{\alpha}^n, \bar{\gamma}, \delta^{\frac{n}{4}} \rangle & \text{if } L = B_{4n}, n \equiv 0 \pmod 4; \\ \langle \bar{\gamma}\bar{\alpha}^{\frac{n}{2}}, \delta^{\frac{n}{4}} \rangle & \text{if } L = C_{4n}, n \equiv 0 \pmod 4; \\ \langle \bar{\alpha}^n \rangle & \text{if } L = C_{4n}, n \equiv 2 \pmod 4. \end{cases}$$

Proof. One only has to compute the centre $Z(N^+)$. If $L = B_8$, then σ_1 acts trivially on N. In any other case, σ_1 is a non-trivial outer automorphism and we have $Z(N^+) = C_{Z(N)}(\sigma_1)$, which is very easy to calculate. ■

5 Automorphisms of Burn loops of type B_{4n} and C_{4n}

Let $(L, \cdot)$ be a loop and let u denote an automorphism of L. Then, by conjugation, u induces an automorphism of the group $G(L)$. Moreover u leaves the section $S(L)$ and the stabilizer $G(L)_1$ invariant. Conversely, let u be an automorphism of $G(L)$, normalizing the subgroup $G(L)_1$ and the set $S(L)$. Then u induces a permutation on the cosets of $G(L)_1$, hence on L. The induced permuation will fix 1 and normalize $S(L)$, thus $u^{-1}\lambda_x u = \lambda_y$ for all $x \in L$. Applying this to 1, one gets $y = x^u$, hence $\lambda_x^u = \lambda_{x^u}$ for all $x \in L$. This means $u \in \text{Aut}(L)$.

In the case of the given loops the stabilizer of 1 consists of $\{id, \beta\}$. First we calculate its normalizer in the automorphism groups of the left translation groups, that is, the groups $C_{\text{Aut}(G)}(\beta)$, where G is G_{8n} or H_{8n}.

Lemma 8
Let G denote the group G_{8n}, n odd. Then $C_{\mathrm{Aut}(G)}(\beta) \cong Z_n^* \times S_3$, and the elements of $C_{\mathrm{Aut}(G)}(\beta)$ normalize $S(B_{4n})$.

Proof. Let us define the subgroups $A = \langle \alpha^2 \rangle$ and $B = \langle \alpha^n, \beta, \gamma \rangle$ of G. As $|A| = n$ is odd, A is a characteristic subgroup of $G = A \times B$. Moreover, $B = Z(G)\langle \beta \rangle$ is invariant in $C_{\mathrm{Aut}(G)}(\beta)$, as well. Hence, $C_{\mathrm{Aut}(G)}(\beta) = \mathrm{Aut}(A) \times C_{\mathrm{Aut}(B)}(\beta) \cong Z_n^* \times S_3$.

On the other hand, $S(L) = A\{id, \alpha^n\beta, \beta\gamma, \alpha^n\beta\gamma\}$. Since the set

$$\{id, \alpha^n\beta, \beta\gamma, \alpha^n\beta\gamma\}$$

is invariant under $C_{\mathrm{Aut}(B)}(\beta)$, the statement follows. ∎

Lemma 9
Let G denote the group G_{8n}, n even. Then $C_{\mathrm{Aut}(G)}(\beta) \cong Z_n^* \times D_8$, and the elements of $C_{\mathrm{Aut}(G)}(\beta)$ normalize $S(B_{4n})$.

Proof. It is enough to consider the possible images of α and γ, let us write them as $\hat{\alpha} = \alpha^i\gamma^k\beta^j$ and $\hat{\gamma} = \alpha^p\gamma^q\beta^s$, respectively. Clearly, $\hat{\beta} = \beta$.

If $j = 1$ then $\hat{\alpha}^2 = id$, which is impossible. The order of $\hat{\alpha}$ must be $2n$, thus $i \in Z_{2n}^*$. The elements $\hat{\alpha}$ and $\hat{\gamma}$ must commute, s cannot be 1. Also the elements $\hat{\beta}$ and $\hat{\gamma}$ commute, we must have $p = ln$ with $l \in Z_2$.

Let us now suppose that $q = 0$. Then $l = 0$ implies $\hat{\gamma} = id$ and $k = 0$ implies $\gamma \notin \langle \hat{\alpha}, \hat{\beta}, \hat{\gamma} \rangle$, hence we have $l = k = 1$. This means $\hat{\alpha}^n = \alpha^{ni} = \alpha^n = \hat{\gamma}$, a contradiction.

Let us denote by $u(i, k, l)$ the automorphism induced by

$$\alpha \mapsto \alpha^i\gamma^k, \qquad \beta \mapsto \beta, \qquad \gamma \mapsto \alpha^{ln}\gamma,$$

with $i \in Z_{2n}^*$, $k, l \in Z_2$. It is easy to check that this is really an element of $C_{\mathrm{Aut}(G)}(\beta)$. Moreover,

$$u(i, j, k)u(i', j', k') = u(ii' + lk'n, k + k', l + l'),$$

where one calculates modulo $2n$ in the first and modulo 2 in the second and third position.

Let us decompose Z_{2n}^* into $Z_n^* \times Z_2$ by $i = i_0 + i_1 n$, $i_0 \in Z_n^*$, $i_1 \in Z_2$. Then the group $C_{\mathrm{Aut}(G)}(\beta)$ decomposes into the direct factors

$$\{u(i_0, 0, 0) : i_0 \in Z_n^*\} \text{ and } \{u(i_1 n, k, l) : i_1, k, l \in Z_2\}.$$

An easy calculation shows that the second factor is isomorphic to the dihedral group D_8 of 8 elements.

Since we explicitely gave the elements of $C_{\mathrm{Aut}(G)}(\beta)$, it can be checked directly that they leave $S(L)$ invariant. ∎

Lemma 10
Let G denote the group H_{8n}, $n > 2$ even. Then $C_{\mathrm{Aut}(G)}(\beta) \cong Z_{2n}^* \times Z_2$, and the elements of $C_{\mathrm{Aut}(G)}(\beta)$ normalize $S(C_{4n})$.

Proof. As in the preceding proof, we consider the images $\hat{\alpha} = \alpha^i\gamma^k\beta^j$, $\hat{\gamma} = \alpha^p\gamma^q\beta^s$ of α and γ.

If $j = 1$, then $\hat{\alpha}^2 = \alpha^i\gamma^k\beta\alpha^i\gamma^k\beta = (\gamma^k\beta)^2 = \alpha^{kn}$, $\hat{\alpha}^4 = id$, which is not possible because of $n > 2$. If $k = 1$, then $(\hat{\alpha}\hat{\beta})^2 = (\gamma\beta)^2 = \alpha^n \neq id$, hence $k = 0$ and $\hat{\alpha} = \alpha^i$, with $i \in Z_{2n}^*$.

As before, $\hat{\alpha}\hat{\gamma} = \hat{\gamma}\hat{\alpha}$ implies $s = 0$ and $\gamma \in \langle\hat{\alpha}, \hat{\beta}, \hat{\gamma}\rangle$ implies $q \neq 0$. Finally, $p \in \{0, n\}$, since $\hat{\gamma} = (\alpha^p\gamma)^2 = \alpha^{2p} = id$.

Thus, any element of $C_{\mathrm{Aut}(G)}(\beta)$ is induced by

$$\alpha \mapsto \alpha^i, \qquad \beta \mapsto \beta, \qquad \gamma \mapsto \alpha^{ln}\gamma,$$

and it leaves $S(L)$ invariant. ∎

Theorem 5
Let $(L, \cdot)$ be one of the loops B_{4n} or C_{4n} defined at the beginning of this section. Then

$$\mathrm{Aut}(L) \cong \begin{cases} Z_n^* \times S_3 & \text{if } L = B_{4n},\ n \text{ odd} \\ Z_n^* \times D_8 & \text{if } L = B_{4n},\ n \text{ even} \\ Z_{2n}^* \times Z_2 & \text{if } L = C_{4n},\ n > 2,\ n \text{ even} \\ D_8 & \text{if } L = C_8 \end{cases}$$

Moreover, in any of these loops, each left pseudo-automorphism is an automorphism.

Proof. The case $L = C_8$ is handled in [8], the others in Lemmas 8, 9 and 10. We only have to prove the second statement. Therefore, let us suppose that u is a left pseudo-automorphism of L with companion c, that is, for all $x, y \in L$,

$$(c \cdot x^u) \cdot y^u = c \cdot (xy)^u.$$

This can be expressed by $u\lambda cx^u = \lambda_x u\lambda_c$, which implies $S(L)^u = S(L)\lambda_c^{-1}$.

The following results can be found in [6]. If $L = B_{4n}$, then the principal isotopes of L have the four representations $S(L)$, $\alpha\beta S(L)$, $\alpha\beta\gamma S(L)$, and $\beta\gamma S(L)$. If n is even, then these sections contain $3n + 1$, $n + 3$, $n + 3$ and $n + 1$ elements of order 2. If n is odd, $S(L)$ contains $3n$ elements of order 2 and the others contain $n + 2$ elements of order 2, $n > 2$. That means that c is a left companion element of L if and only if $S(L)\lambda_c = S(L)$, that is, $c \in N_\lambda$ and u is an automorphism.

Let now L be equal to C_{4n}. Again the principal isotopes are $S(C_{4n})$, $\alpha\beta S(C_{4n})$, $\alpha\beta\gamma S(C_{4n})$, and $\beta\gamma S(C_{4n})$, they contain $n + 1$, $n + 3$, 3 and 1 involutions, respectively. If $n > 2$, then one sees with the above argument that $c \in N_\lambda$ and u is an automorphism. ∎

6 Collineation groups of the given 3-nets

In this chapter, we determine the full collineation group Γ of the 3-nets belonging to B_{4n}, $n \geq 3$, and C_{4n}, $n \geq 4$, n even. The cases B_8 and C_8 are completely descripted in [8].

Denote by P the orbit $(1, 1)^\Gamma$ of the origin under Γ. As we know by Corollary 2.8 of [8], for any Burn loop, P is a union of vertical lines and its intersection with the x-axis constitute of the points belonging to the left companion elements. In our cases, these are the elements of N_λ, see Theorem 5. Hence $|P| = 4n^2$.

Let Λ_0 be the subgroup $\langle \alpha, \gamma \rangle$ of $G(L)$. The centralizer element $\alpha^i \beta \gamma^j \notin \Lambda_0$ in Λ_0 has order 4, that is, any abelian subgroup not contained in Λ_0 has order at most 8. This means that if $n > 2$ then Λ_0 is the only abelian subgroup of index 2 in $G(L)$, it must therefore be characteristic in $G(L)$.

Now, we define the following subgroups of Γ.

$$T = \{(\lambda_m, id) : m \in N_\lambda\}, \qquad \Lambda = \Phi^{-1}(\Lambda_0),$$
$$A = \{(\sigma, \sigma) : \sigma \in \mathrm{Aut}(L)\}, \qquad M = T\Lambda.$$

Lemma 11
The subgroup M is an abelian normal subgroup of Γ. Moreover, it is isomorphic to the direct product $N_\lambda \times \Lambda_0$ and acts regularly on the orbit P of the origin.

Proof. First we show that M is abelian. By Lemma 6, one sees that the permutation action of the elements of Λ are all in $\langle \alpha, \gamma \rangle$; the same can be said about the elements of T. These actions commute, and so, all the elements must commute.

Clearly, T is normal in Γ. The subgroup Λ is invariant in Γ as well, for it is the homomorphic preimage of a characteristic subgroup.

Suppose that (u, v) is an element of $M_{(1,1)}$. Then $v = id$, since $v = \beta$ is not possible. This implies $u = \lambda_m$, $m \in N_\lambda$; this yields $u = id$. Furthermore, on the one hand, by $\Lambda \cap T = \ker \Phi$, we have $M_{y\text{-axis}} \cong M/T \cong \Lambda_0$. On the other hand, $T \subset M_{x\text{-axis}}$ acts transitively on $P \cap y$-axis. This means that M acts transitively on P, thus, regularly. Finally, $M = T \times M_{y\text{-axis}} \cong N_\lambda \times \Lambda_0$. ∎

Theorem 6
Let Γ be the full collineation group of a 3-net, coordinatized by a loop L, with $L = B_{4n}$ or C_{4n}, $n > 2$. Then, Γ can be written as the semidirect product $M \rtimes \mathrm{Aut}(L)$, where M is defined as above and the action of $\mathrm{Aut}(L)$ on M is defined by $(u, v)^\sigma = (u^\sigma, v^\sigma)$.

Proof. Obviously, A is isomorphic to $\mathrm{Aut}(L)$. By Theorem 10.1 of [2], A is equal to the stabilizer $\Gamma_{(1,1)}$ of the origin $(1, 1)$ in Γ. By Lemma 11, M is a normal subgroup of Γ, acting regularly on the orbit $P = (1, 1)^\Gamma$. Then, Γ can be written as the semidirect product $M \rtimes A \cong M \rtimes \mathrm{Aut}(L)$. ∎

Remark. Note that there is an interesting analogy with the case of group 3-nets: then one has $\Gamma \cong (G \times G) \rtimes \mathrm{Aut}(G)$ (cf. [2], Theorem 10.1).

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