A note on small complete caps in the Klein quadric

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Abstract

We give a lower bound for the size of a cap being contained and complete in the Klein quadric in $\mathrm{PG}(5,q)$, or equivalently, for the size of a set $\mathcal{L}$ of lines in $\mathrm{PG}(3,q)$ such that no three of them are concurrent and coplanar in the same time, and being also maximal for this property.

The *Klein quadric* is a quadric hypersurface in $\mathrm{PG}(5,q)$, the points of which correspond to the lines of $\mathrm{PG}(3,q)$, see e.g. [5]. Three points of the Klein quadric are collinear if and only if their corresponding lines are concurrent and coplanar in $\mathrm{PG}(3,q)$.

A *cap* in $\mathrm{PG}(n,q)$ is a set of points, no three of which are collinear. In this paper we are interested in caps being contained and complete in the Klein quadric, which correspond to maximal sets $\mathcal{L}$ of lines in $\mathrm{PG}(3,q)$ such that no three of them are concurrent and coplanar at the same time.

If you take a point P of $\mathrm{PG}(3,q)$ then the lines of $\mathcal{L}$ through P (more precisely: their residues with respect to P, in design theory terminology) form an arc. So there are at most $q+1$ or $q+2$ lines through any point according as q is odd or even. This implies the upper bound

$$|\mathcal{L}| \le (q+1)(q^2+1) \text{ if } 2 \nmid q \quad \text{or} \quad |\mathcal{L}| \le (q+2)(q^2+1) \text{ if } 2|q.$$

For q odd this bound is sharp, see Glynn [4]. We remark that the extremal example comes from a full Singer line orbit. There is an embedding-type result for q even,

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see Ebert, Metsch and Szőnyi [3]. For q even it is conjectured that the upper bound cannot be reached.

Here we will examine the 'other side', i.e. small complete caps. We give a lower bound for the size of a cap being contained and complete in the Klein quadric in $\mathrm{PG}(5, q)$.

Our motivation is the following: Cossidente, Hirschfeld and Storme in [1, 2] constructed a cap of size $2(q^2+q+1)$ which is complete on the Klein quadric. We want to prove a lower bound for the size of such a cap. The trivial lower bound is $q^{3/2}$, which can be shown by the following easy argument:

Let $\mathcal{L}$ be a set of lines in $\mathrm{PG}(3, q)$ such that no three of them are concurrent and coplanar in the same time, and suppose that $\mathcal{L}$ is complete for this property. It means that the intersecting pairs of lines from $\mathcal{L}$ 'block' all the other lines being concurrent and coplanar with them, i.e. $q-1$ lines, and the intersecting pairs from $\mathcal{L}$ block all the lines not in $\mathcal{L}$. As $\mathrm{PG}(3, q)$ has roughly q^4 lines, and the number of intersecting pairs of lines is at most $\binom{|\mathcal{L}|}{2}$, the bound of magnitude $q^{3/2}$ follows.

Theorem 1 *Let $\mathcal{L}$ be a set of lines in* $\mathrm{PG}(3, q)$ *such that no three of them are concurrent and coplanar at the same time, and let $\mathcal{L}$ be also maximal for this property. Then* $|\mathcal{L}| \geq \mathrm{const} \cdot q^{12/7}$.

This is equivalent to the following.

Theorem 2 *Any complete cap on the Klein quadric has size at least* $\mathrm{const} \cdot q^{12/7}$.

Proof. Let $G(V, E)$ be the graph with the lines of $\mathcal{L}$ as vertices, and $(l_i, l_j) \in E$ if and only if the lines are intersecting. Let $n = |V| = |\mathcal{L}|, e = |E|$ and let d_i the degree of the vertex (line) l_i.

1. Claim: $e \geq q^3$. Proof: the intersecting pairs block the lines not in $\mathcal{L}$, their number is greater than q^4, one pair blocks $q-1$ other lines.

2. We estimate the number Q of quintuples $(l_0; l_1, l_2, l_3, l_4)$, where

- $l_i \in \mathcal{L}$;
- l_0 meets l_1, l_2, l_3 and l_4;
- l_1, l_2, l_3, l_4 are pairwise skew and they are not contained in a regulus.

We get a lower bound for this number in the following way: after choosing l_0 with degree d, one may choose l_1 in d ways, then l_2 in at least $d-2q$ ways (as any two intersecting lines can have at most $2q$ common neighbours), then l_3 in at least $d-4q$ ways, finally l_4 in at least $d-7q+2$ ways (because you have to exclude the $q-2$ remaining lines of the regulus determined by l_1, l_2 and l_3). So

$$Q \geq \sum_i d_i(d_i-2q)(d_i-4q)(d_i-7q+2) \geq \sum_i (d_i-7q)^4 \geq$$

$$\frac{(\sum(d_i-7q))^4}{n^3} \geq \frac{(2e-7qn)^4}{n^3} \geq \frac{(2q^3-7qn)^4}{n^3}.$$

On the other hand, we get an upper bound in the following way: one may choose l_1, l_2, l_3 and l_4 in less than n^4 ways. Finally at most two lines can be chosen as l_0, because if there were three lines meeting l_1, l_2 and l_3, then all the other lines (so l_4 as well) meeting these three lines would be in the regulus determined by l_1, l_2 and l_3. This is less than $2n^4$ in total.

So we have

$$2n^4 \geq Q \geq \frac{(2q^3 - 7qn)^4}{n^3}$$

and the desired inequality follows. ∎

To end this note we remark that if such a configuration of lines exists, with significantly less than q^2 lines, then it has some interesting properties; for example it contains "large" parts from "many" reguli.

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References

[1] A. Cossidente, J. W. P. Hirschfeld and L. Storme. Applications of line geometry, III: The quadric Veronesean and the chords of a twisted cubic. *Australas. J. Combin.* **16** (1997), 99-111.

[2] A. Cossidente. Private communication.

[3] G. L. Ebert, K. Metsch and T. Szőnyi. Caps embedded in Grassmanian surfaces. *Manuscript*.

[4] D. G. Glynn. On a set of lines of $\mathrm{PG}(3, q)$ corresponding to maximal cap contained in the Klein quadric of $\mathrm{PG}(5, q)$. *Geom. Dedicata* **26** (1988), 273-280.

[5] J. W. P. Hirschfeld. *Finite projective spaces of three dimensions*. Oxford University Press, Oxford (1985).

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