Bilinear mappings on topological modules

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Abstract

We study boundedness, equicontinuity and sequential equicontinuity for sets of separately continuous bilinear mappings between topological modules.

In this paper, we consider various types of boundedness, equicontinuity and sequential equicontinuity for sets of separately continuous bilinear mappings between topological modules. Our purpose here is to establish relations among the various notions of boundedness (resp. equicontinuity, sequential equicontinuity) under consideration, as well as to establish relations among notions of a different nature; for example, to obtain conditions under which pointwise boundedness implies separate equicontinuity. The key result for our study is a version of the Banach-Steinhaus theorem in the context of topological modules, proved in [7].

It should be mentioned that this paper was written under the influence of [2] (see also [1] and [8]), where the same notions for sets of separately continuous bilinear mappings between (real or complex) topological vector spaces have been studied.

Throughout this work, A denotes a commutative topological ring with an identity element and A^* denotes the multiplicative group of all invertible elements of A. All modules under consideration are unitary A-modules. E, F and G represent topological A-modules, $\mathcal{M}$ (resp. $\mathcal{N}$) represents a set of bounded subsets of E (resp. F), and $\mathcal{L}_{\text{sep}}(E, F; G)$ represents the A-module of all separately continuous A-bilinear mappings from $E \times F$ into G.

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Definition 1. Let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$.

(B1) $\mathfrak{X}$ is *pointwise bounded* if, for each $(x, y) \in E \times F$, the set

$$\mathfrak{X}(x, y) = \{u(x, y); u \in \mathfrak{X}\}$$

is bounded in G.

(B2)(a) $\mathfrak{X}$ is $\mathcal{M}$-*uniformly bounded* if, for each $y \in F$ and for each $B \in \mathcal{M}$, the set

$$\{u(x, y); u \in \mathfrak{X}, x \in B\}$$

is bounded in G.

(B2)(b) $\mathfrak{X}$ is $\mathcal{N}$-*uniformly bounded* if, for each $x \in E$ and for each $C \in \mathcal{N}$, the set

$$\{u(x, y); u \in \mathfrak{X}, y \in C\}$$

is bounded in G.

(B3) $\mathfrak{X}$ is $(\mathcal{M}, \mathcal{N})$-*uniformly bounded* if $\mathfrak{X}$ is $\mathcal{M}$-uniformly bounded and $\mathcal{N}$-uniformly bounded.

(B4) $\mathfrak{X}$ is $(\mathcal{M} \times \mathcal{N})$-*uniformly bounded* if, for each $B \in \mathcal{M}$ and for each $C \in \mathcal{N}$, the set

$$\mathfrak{X}(B \times C) = \{u(x, y); u \in \mathfrak{X}, x \in B, y \in C\}$$

is bounded in G.

Remark 2. Since every subset of a topological module over a discrete ring is necessarily bounded, there is no interest in the study of the notions just defined when A is a discrete ring.

Remark 3. (a) If $\bigcup_{B \in \mathcal{M}} B = E$ (resp. $\bigcup_{C \in \mathcal{N}} C = F$), then (B2)(a) implies (B1) (resp. (B2)(b) implies (B1)).
(b) If $\bigcup_{C \in \mathcal{N}} C = F$ (resp. $\bigcup_{B \in \mathcal{M}} B = E$), then (B4) implies (B2)(a) (resp. (B4) implies (B2)(b)). In particular, if $\bigcup_{B \in \mathcal{M}} B = E$ and $\bigcup_{C \in \mathcal{N}} C = F$, then (B4) implies (B3).

There are examples showing that the reverse implications in Remark 3 are not valid in general; see [2]. In the sequel we shall see conditions under which such reverse implications hold.

Definition 4. Let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$.

(E1)(a) $\mathfrak{X}$ is *left equicontinuous* if, for each $y \in F$, the set

$$\{x \in E \mapsto u(x, y) \in G; u \in \mathfrak{X}\}$$

is equicontinuous.

(E1)(b) $\mathfrak{X}$ is *right equicontinuous* if, for each $x \in E$, the set

$$\{y \in F \mapsto u(x, y) \in G; u \in \mathfrak{X}\}$$

is equicontinuous.

(E2)(a) $\mathfrak{X}$ is $\mathcal{N}$-*equihypocontinuous* if, for each $C \in \mathcal{N}$, the set

$$\{x \in E \mapsto u(x,y) \in G; u \in \mathfrak{X}, y \in C\}$$

is equicontinuous.

(E2)(b) $\mathfrak{X}$ is $\mathcal{M}$-*equihypocontinuous* if, for each $B \in \mathcal{M}$, the set

$$\{y \in F \mapsto u(x,y) \in G; u \in \mathfrak{X}, x \in B\}$$

is equicontinuous.

(E3) $\mathfrak{X}$ is $(\mathcal{M},\mathcal{N})$-*equihypocontinuous* if $\mathfrak{X}$ is $\mathcal{M}$-equihypocontinuous and $\mathcal{N}$-equihypocontinuous.

(E4) $\mathfrak{X}$ is *equicontinuous* if, for each $(x,y) \in E \times F$ and for each neighborhood W of zero in G, there exist a neighborhood U of zero in E and a neighborhood V of zero in F such that the relations $u \in \mathfrak{X}$, $x' \in U$, $y' \in V$ imply $u(x'+x, y'+y) - u(x,y) \in W$.

Remark 5. If $\bigcup_{C\in\mathcal{N}} C = F$ (resp. $\bigcup_{B\in\mathcal{M}} B = E$), then (E2)(a) implies (E1)(a) (resp. (E2)(b) implies (E1)(b)). In particular, if $\bigcup_{B\in\mathcal{M}} B = E$ and $\bigcup_{C\in\mathcal{N}} C = F$, then (E3) implies (E1)(a) and (E1)(b).

Remark 6. Suppose that the product of any neighborhood of zero in A by any neighborhood of zero in E (resp. F) is a neighborhood of zero in E (resp. F). Then (E4) implies (E2)(a) (resp. (E4) implies (E2)(b)); in particular, if these two properties hold, then (E4) implies (E3). In fact, let $\mathfrak{X} \subset \mathcal{L}_{\mathrm{sep}}(E,F;G)$ be equicontinuous, let $C \in \mathcal{N}$, and assume that the product of any neighborhood of zero in A by any neighborhood of zero in E is a neighborhood of zero in E. Given an arbitrary neighborhood W of zero in G, there are a neighborhood U of zero in E and a neighborhood V of zero in F such that $\mathfrak{X}(U \times V) \subset W$. By the boundedness of C, there exists a neighborhood L of zero in A such that $LC \subset V$. Thus

$$\mathfrak{X}((LU) \times C) = \mathfrak{X}(U \times (LC)) \subset \mathfrak{X}(U \times V) \subset W.$$

Since, by assumption, LU is a neighborhood of zero in E, we have just verified that the set

$$\{x \in E \mapsto u(x,y) \in G; u \in \mathfrak{X}, y \in C\}$$

is equicontinuous. Therefore $\mathfrak{X}$ is $\mathcal{N}$-equihypocontinuous. By interchanging the roles of E and F, we conclude that the other assertion is also true.

There are examples showing that the reverse implications in Remarks 5 and 6 are not valid in general; see [2]. In the sequel we shall see conditions under which such reverse implications hold.

Theorem 7 and Corollary 8 below have already been established in [7].

Theorem 7. Suppose that E and F are metrizable and E is barrelled [7] (resp. F is barrelled). If $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$ is right equicontinuous (resp. left equicontinuous), then $\mathfrak{X}$ is equicontinuous.

Corollary 8. Suppose that E and F are metrizable and barrelled. If $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$ is pointwise bounded, then $\mathfrak{X}$ is equicontinuous.

Corollary 9. Suppose that there exists a countable subset C of A^* such that $0 \in \overline{C}$. Let E and F be metrizable, and let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$.

(a) If E is complete (resp. F is complete) and $\mathfrak{X}$ is right equicontinuous (resp. left equicontinuous), then $\mathfrak{X}$ is equicontinuous.

(b) If E and F are complete, and $\mathfrak{X}$ is pointwise bounded, then $\mathfrak{X}$ is equicontinuous.

Proof. (a) follows from Proposition 2.4 of [7] and Theorem 7; (b) follows from Proposition 2.4 of [7] and Corollary 8.

Remarks 10 and 11 below are concerned with relations among some of the notions considered in Definitions 1 and 4.

Remark 10. By Theorem 25.5 of [9], (E1)(a) implies (B2)(a), (E1)(b) implies (B2)(b), (E2)(a) implies (B4), and (E2)(b) implies (B4) (in particular, (E3) implies (B4)).

Remark 11. Suppose that the product of any two neighborhoods of zero in A is a neighborhood of zero in A. Then (E4) implies (B4). In fact, let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$ be equicontinuous, and let $B \in \mathcal{M}$, $C \in \mathcal{N}$. Given an arbitrary neighborhood W of zero in G, there are a neighborhood U of zero in E and a neighborhood V of zero in F such that $\mathfrak{X}(U \times V) \subset W$. By the boundedness of B and C, there exists a neighborhood L of zero in A such that $LB \subset U$ and $LC \subset V$. Thus

$$(LL)\,\mathfrak{X}(B \times C) = \mathfrak{X}((LB) \times (LC)) \subset \mathfrak{X}(U \times V) \subset W.$$

Since, by assumption, LL is a neighborhood of zero in A, we have just verified that the set $\mathfrak{X}(B \times C)$ is bounded. Therefore $\mathfrak{X}$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded.

We have seen in Corollaries 8 and 9(b) conditions under which (B1) implies (E4). In Theorem 12 (resp. Proposition 16) below, we shall see conditions under which (B2)(b) implies (E2)(a), (B2)(a) implies (E2)(b), and (B1) implies (E1)(a) or (E1)(b) (resp. (B4) implies (E2)(a) or (E2)(b), (B2)(a) implies (E1)(a), and (B2)(b) implies (E1)(b)). Some consequences of Theorem 12 and Proposition 16 are also derived.

Theorem 12. Suppose that E is barrelled (resp. F is barrelled), and let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$. If $\mathfrak{X}$ is $\mathcal{N}$-uniformly bounded (resp. $\mathcal{M}$-uniformly bounded), then $\mathfrak{X}$ is $\mathcal{N}$-equihypocontinuous (resp. $\mathcal{M}$-equihypocontinuous). In particular, if $\mathfrak{X}$ is pointwise bounded, then $\mathfrak{X}$ is left equicontinuous (resp. right equicontinuous).

Proof. Assume that E is barrelled and $\mathfrak{X}$ is $\mathcal{N}$-uniformly bounded. In order to prove that $\mathfrak{X}$ is $\mathcal{N}$-equihypocontinuous, let $C \in \mathcal{N}$ and consider the set

$$\mathcal{Z} = \{x \in E \mapsto u(x, y) \in G; u \in \mathfrak{X}, y \in C\}$$

of continuous A-linear mappings from E into G. By hypothesis, $\mathcal{Z}(x)$ is bounded in G for each $x \in E$. Thus, by Theorem 3.1 of [7], $\mathcal{Z}$ is equicontinuous. Therefore $\mathfrak{X}$ is $\mathcal{N}$-equihypocontinuous. By interchanging the roles of E and F, we conclude that the other assertion is also true.

Corollary 13. Suppose that E and F are barrelled, and let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$. If $\mathfrak{X}$ is $(\mathcal{M}, \mathcal{N})$-uniformly bounded, then $\mathfrak{X}$ is $(\mathcal{M}, \mathcal{N})$-equihypocontinuous. In particular, if $\mathfrak{X}$ is pointwise bounded, then $\mathfrak{X}$ is left equicontinuous and right equicontinuous.

Proof. Immediate from Theorem 12.

The following corollary was suggested by Proposition 14, p.44 of [4].

Corollary 14. Suppose that F is barrelled (resp. E is barrelled), and let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$. If $\bigcup_{C \in \mathcal{N}} C = F$ (resp. $\bigcup_{B \in \mathcal{M}} B = E$) and $\mathfrak{X}$ is $\mathcal{N}$-equihypocontinuous (resp. $\mathcal{M}$-equihypocontinuous), then $\mathfrak{X}$ is $(\mathcal{M}, \mathcal{N})$-equihypocontinuous. In particular, if $\mathfrak{X}$ is left equicontinuous (resp. right equicontinuous), then $\mathfrak{X}$ is $\mathcal{M}$-equihypocontinuous (resp. $\mathcal{N}$-equihypocontinuous). Thus, if E and F are barrelled and $\mathfrak{X}$ is left equicontinuous and right equicontinuous, then $\mathfrak{X}$ is $(\mathcal{M}, \mathcal{N})$-equihypocontinuous.

Proof. Immediate from Remarks 5 and 10, and Theorem 12.

Corollary 15. Suppose that E is barrelled (resp. F is barrelled), and let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$. If $\mathfrak{X}$ is $\mathcal{N}$-uniformly bounded (resp. $\mathcal{M}$-uniformly bounded), then $\mathfrak{X}$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded. In particular, if $\mathfrak{X}$ is pointwise bounded, then $\mathfrak{X}$ is $\mathcal{M}$-uniformly bounded (resp. $\mathcal{N}$-uniformly bounded). Thus, if E and F are barrelled and $\mathfrak{X}$ is pointwise bounded, then $\mathfrak{X}$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded.

Proof. Immediate from Theorem 12 and Remark 10.

Proposition 16. Suppose that E is bornological [3] (resp. F is bornological), $\mathcal{M}$ is the set of all bounded subsets of E (resp. $\mathcal{N}$ is the set of all bounded subsets of F), and let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$. If $\mathfrak{X}$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded, then $\mathfrak{X}$ is $\mathcal{N}$-equihypocontinuous (resp. $\mathcal{M}$-equihypocontinuous). In particular, if $\mathfrak{X}$ is $\mathcal{M}$-uniformly bounded (resp. $\mathcal{N}$-uniformly bounded), then $\mathfrak{X}$ is left equicontinuous (resp. right equicontinuous).

Proof. The proof is analogous to that of Theorem 12. For example, to prove one of the assertions, assume that E is bornological and $\mathfrak{X}$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded, $\mathcal{M}$ being the set of all bounded subsets of E. For a given $C \in \mathcal{N}$, the set

$$\mathcal{Z} = \{x \in E \mapsto u(x, y) \in G; u \in \mathfrak{X}, y \in C\}$$

of (continuous) A-linear mappings transforms bounded subsets of E into bounded subsets of G. By the theorem proved in [3], $\mathcal{Z}$ is equicontinuous. Therefore $\mathfrak{X}$ is

$\mathcal{N}$-equihypocontinuous.

Corollary 17. Suppose that E and F are bornological, $\mathcal{M}$ is the set of all bounded subsets of E and $\mathcal{N}$ is the set of all bounded subsets of F. If $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded, then $\mathfrak{X}$ is $(\mathcal{M}, \mathcal{N})$-equihypocontinuous.

Proof. Immediate from Proposition 16.

Corollary 18. Suppose that F is bornological (resp. E is bornological), and let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$. If $\mathcal{N}$ is the set of all bounded subsets of F (resp. $\mathcal{M}$ is the set of all bounded subsets of E) and $\mathfrak{X}$ is $\mathcal{N}$-equihypocontinuous (resp. $\mathcal{M}$-equihypocontinuous), then $\mathfrak{X}$ is $(\mathcal{M}, \mathcal{N})$-equihypocontinuous.

Proof. Immediate from Remark 10 and Proposition 16.

Definition 19. Let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$.

(S1)(a) $\mathfrak{X}$ is *sequentially left equicontinuous* if, for each $y \in F$ and for each null sequence $(x_n)_{n\in\mathbf{N}}$ in E, $(u(x_n, y))_{n\in\mathbf{N}}$ converges uniformly to zero for $u \in \mathfrak{X}$.

(S1)(b) $\mathfrak{X}$ is *sequentially right equicontinuous* if, for each $x \in E$ and for each null sequence $(y_n)_{n\in\mathbf{N}}$ in F, $(u(x, y_n))_{n\in\mathbf{N}}$ converges uniformly to zero for $u \in \mathfrak{X}$.

(S2)(a) $\mathfrak{X}$ is *sequentially $\mathcal{N}$-equihypocontinuous* if, for each $C \in \mathcal{N}$ and for each null sequence $(x_n)_{n\in\mathbf{N}}$ in E, $(u(x_n, y))_{n\in\mathbf{N}}$ converges uniformly to zero for $u \in \mathfrak{X}$, $y \in C$.

(S2)(b) $\mathfrak{X}$ is *sequentially $\mathcal{M}$-equihypocontinuous* if, for each $B \in \mathcal{M}$ and for each null sequence $(y_n)_{n\in\mathbf{N}}$ in F, $(u(x, y_n))_{n\in\mathbf{N}}$ converges uniformly to zero for $u \in \mathfrak{X}$, $x \in B$.

(S3) $\mathfrak{X}$ is *sequentially $(\mathcal{M}, \mathcal{N})$-equihypocontinuous* if $\mathfrak{X}$ is sequentially $\mathcal{M}$-equihypocontinuous and sequentially $\mathcal{N}$-equihypocontinuous.

(S4) $\mathfrak{X}$ is *sequentially equicontinuous* if, for each $(x, y) \in E \times F$ and for each sequence $((x_n, y_n))_{n\in\mathbf{N}}$ in $E \times F$ converging to (x, y), $(u(x_n, y_n))_{n\in\mathbf{N}}$ converges uniformly to $u(x, y)$ for $u \in \mathfrak{X}$.

Remark 20. If $\bigcup\limits_{C\in\mathcal{N}} C = F$ (resp. $\bigcup\limits_{B\in\mathcal{M}} B = E$), then (S2)(a) implies (S1)(a) (resp. (S2)(b) implies (S1)(b)). In particular, if $\bigcup\limits_{B\in\mathcal{M}} B = E$ and $\bigcup\limits_{C\in\mathcal{N}} C = F$, then (S3) implies (S1)(a) and (S1)(b).

Remark 21. Obviously, (Ei)(a) implies (Si)(a) and (Ei)(b) implies (Si)(b) for $i = 1, 2$; in particular, (E3) implies (S3). Moreover, (E4) implies (S4).

Proposition 22. (a) Suppose that E is metrizable, and let $\mathcal{Z}$ be a set of A-linear mappings from E into G. In order that $\mathcal{Z}$ be equicontinuous, it is necesssary and sufficient that, for each null sequence $(x_n)_{n\in\mathbf{N}}$ in E, $(u(x_n))_{n\in\mathbf{N}}$ converges uniformly to zero for $u \in \mathcal{Z}$.

(b) Suppose that E and F are metrizable, and let $\mathfrak{X}$ be a set of A-bilinear mappings from $E \times F$ into G. In order that $\mathfrak{X}$ be equicontinuous, it is necessary and sufficient that, for each $(x, y) \in E \times F$ and for each sequence $((x_n, y_n))_{n\in\mathbf{N}}$ in $E \times F$ converging to (x, y), $(u(x_n, y_n))_{n\in\mathbf{N}}$ converges uniformly to $u(x, y)$ for $u \in \mathfrak{X}$.

Proof. We shall prove (a), the proof of (b) being analogous. The condition is obviously necessary for every E. In order to prove the sufficiency of the condition, let us fix a decreasing countable fundamental system $(U_n)_{n\in\mathbf{N}}$ of neighborhoods of zero in E (E is metrizable). If $\mathcal{Z}$ is not equicontinuous, there exist a neighborhood W of zero in G, a sequence $(x_n)_{n\in\mathbf{N}}$ in E and a sequence $(u_n)_{n\in\mathbf{N}}$ in $\mathcal{Z}$ such that $x_n \in U_n$ and $u_n(x_n) \notin W$ for all $n \in \mathbb{N}$. Therefore $(x_n)_{n\in\mathbf{N}}$ is a null sequence in E such that $(u(x_n))_{n\in\mathbf{N}}$ does not converge uniformly to zero for $u \in \mathcal{Z}$. This concludes the proof.

Corollary 23. Suppose that E is metrizable (resp. F is metrizable), and let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$. If $\mathfrak{X}$ is sequentially $\mathcal{N}$-equihypocontinuous (resp. sequentially $\mathcal{M}$-equihypocontinuous), then $\mathfrak{X}$ is $\mathcal{N}$-equihypocontinuous (resp. $\mathcal{M}$-equihypocontinuous). In particular, if $\mathfrak{X}$ is sequentially left equicontinuous (resp. sequentially right equicontinuous), then $\mathfrak{X}$ is left equicontinuous (resp. right equicontinuous).

Proof. Immediate from Proposition 22(a).

The following proposition generalizes Theorems 16 and 21 of [2], and improves Remark 10 (recall Remark 21) when A is metrizable.

Proposition 24. Suppose that A is metrizable, and let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$. If $\mathfrak{X}$ is sequentially $\mathcal{N}$-equihypocontinuous (resp. sequentially $\mathcal{M}$-equihypocontinuous), then $\mathfrak{X}$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded. In particular, if $\mathfrak{X}$ is sequentially left equicontinuous (resp. sequentially right equicontinuous), then $\mathfrak{X}$ is $\mathcal{M}$-uniformly bounded (resp. $\mathcal{N}$-uniformly bounded).

Proof. Assume that $\mathfrak{X}$ is sequentially $\mathcal{N}$-equihypocontinuous, and let $B \in \mathcal{M}$, $C \in \mathcal{N}$ be given. We shall prove that the set $\mathfrak{X}(B \times C)$ is bounded. For this purpose, let $(a_n)_{n\in\mathbf{N}}$ be a null sequence in A, $(u_n)_{n\in\mathbf{N}}$ a sequence in $\mathfrak{X}$, $(x_n)_{n\in\mathbf{N}}$ a sequence in B, and $(y_n)_{n\in\mathbf{N}}$ a sequence in C. Since $(a_n x_n)_{n\in\mathbf{N}}$ converges to zero in E, it follows from the hypothesis that the sequence $(a_n u_n(x_n, y_n))_{n\in\mathbf{N}}$ converges to zero in G. By Theorem 15.3 of [9], $\mathfrak{X}(B \times C)$ is bounded. Therefore $\mathfrak{X}$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded. By interchanging the roles of E and F, we conclude that the other assertion is also true.

There are examples showing that the reverse implications in Proposition 24 are not valid in general; see [2]. In Theorem 28 below, we shall see conditions under which such reverse implications hold. In order to reach our purpose, we shall need the following

Definition 25. Let E be a topological A-module and let $(x_n)_{n\in\mathbf{N}}$ be a sequence in E; $(x_n)_{n\in\mathbf{N}}$ *converges to zero in the Mackey sense* if there exists a null sequence $(a_n)_{n\in\mathbf{N}}$ in A consisting of elements of A^* such that $(a_n^{-1} x_n)_{n\in\mathbf{N}}$ is a null sequence in E. In this case, we shall write $(x_n)_{n\in\mathbf{N}} \xrightarrow{M} 0$.

Clearly, if $(x_n)_{n\in\mathbf{N}} \xrightarrow{M} 0$, then $(x_n)_{n\in\mathbf{N}}$ is a null sequence.

E is said to be an *M-topological A-module* if every null sequence in E converges to zero in the Mackey sense.

The following result, whose proof is due to N.C. Bernardes Jr., contains a well known result ([5], p. 149) as a special case.

Proposition 26. Suppose that A contains a null sequence of elements of A^*, and let E be a metrizable topological A-module. Then E is an M-topological A-module.

Proof. Let $(x_n)_{n\in\mathbf{N}}$ be an arbitrary null sequence in E. Let $(U_n)_{n\in\mathbf{N}}$ be a decreasing countable fundamental system of neighborhoods of zero in E (E is metrizable), and let $(a_n)_{n\in\mathbf{N}}$ be a null sequence in A consisting of elements of A^*.

We claim that there exists a strictly increasing sequence $(m_i)_{i\in\mathbf{N}}$ in $\mathbb{N}$ such that, for each $i \in \mathbb{N}$, the relation $n \geq m_i$ implies $x_n \in a_i U_i$. We argue by induction on i. Since $a_0 U_0$ is a neighborhood of zero in E by Theorem 12.4(1) of [9], there is an $m_0 \in \mathbb{N}$ such that the relation $n \geq m_0$ implies $x_n \in a_0 U_0$. Assume that, for a certain $i \in \mathbb{N}^*$, $m_0 < \cdots < m_i$ have been constructed in such a way that the relation $n \geq m_\ell$ implies $x_n \in a_\ell U_\ell$ for $\ell = 0, \ldots, i$. Since $a_{i+1} U_{i+1}$ is a neighborhood of zero in E, there exists an $m_{i+1} \in \mathbb{N}$, $m_{i+1} > m_i$, such that the relation $n \geq m_{i+1}$ implies $x_n \in a_{i+1} U_{i+1}$. Therefore our claim is verified.

By what we have just proved, for each $i \in \mathbb{N}$ and for each $m_i \leq n < m_{i+1}$, there exists a $y_n \in U_i$ so that $x_n = b_n y_n$, where $b_n = a_i$ for $m_i \leq n < m_{i+1}$. Since $(y_n)_{n\geq m_0}$ converges to zero in E, it follows that $(x_n)_{n\in\mathbf{N}} \xrightarrow{M} 0$, as we wished to prove.

Remark 27. In Proposition 26, A may be taken as being metrizable and such that $0 \in \overline{A^*}$.

The following theorem generalizes Theorems 19 and 23 of [2].

Theorem 28. Suppose that E is an M-topological A-module (resp. F is an M-topological A-module), $\mathcal{M}$ contains all sets consisting of points of null sequences in E (resp. $\mathcal{N}$ contains all sets consisting of points of null sequences in F), and let $\mathfrak{X} \subset \mathcal{L}_{\text{sep}}(E, F; G)$. If $\mathfrak{X}$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded, then $\mathfrak{X}$ is sequentially $\mathcal{N}$-equihypocontinuous (resp. sequentially $\mathcal{M}$-equihypocontinuous). In particular, if $\mathfrak{X}$ is $\mathcal{M}$-uniformly bounded (resp. $\mathcal{N}$-uniformly bounded), then $\mathfrak{X}$ is sequentially left equicontinuous (resp. sequentially right equicontinuous).

Proof. Let E, F and $\mathcal{N}$ be arbitrary. We claim that $\mathfrak{X}$ is sequentially $\mathcal{N}$-equihypocontinuous if and only if for each sequence $(u_n)_{n\in\mathbf{N}}$ in $\mathfrak{X}$, for each null sequence $(x_n)_{n\in\mathbf{N}}$ in E and for each sequence $(y_n)_{n\in\mathbf{N}}$ in F such that the set $\{y_n; n \in \mathbb{N}\}$ is contained in an element of $\mathcal{N}$, the sequence $(u_n(x_n, y_n))_{n\in\mathbf{N}}$ converges to zero in G. Indeed, it is obvious that the above-mentioned property is valid if $\mathfrak{X}$ is sequentially $\mathcal{N}$-equihypocontinuous. On the other hand, if $\mathfrak{X}$ is not sequentially $\mathcal{N}$-equihypocontinuous, there are a $C \in \mathcal{N}$ and a null sequence $(x_n)_{n\in\mathbf{N}}$ in E in such a way that $(u(x_n, y))_{n\in\mathbf{N}}$ does not converge uniformly to zero for $u \in \mathfrak{X}$, $y \in C$. Thus there is a neighborhood W of zero in G such that, for every $n \in \mathbb{N}$, there are an

$m \in \mathbb{N}$, $m > n$, a $u \in \mathfrak{X}$ and a $y \in C$ so that $u(x_m, y) \notin W$. It then follows that there exist a strictly increasing sequence $(n_k)_{k\in\mathbf{N}}$ in $\mathbb{N}$, a sequence $(y_{n_k})_{k\in\mathbf{N}}$ in C and a sequence $(u_{n_k})_{k\in\mathbf{N}}$ in $\mathfrak{X}$ such that $u_{n_k}(x_{n_k}, y_{n_k}) \notin W$ for all $k \in \mathbb{N}$. Therefore $(u_{n_k}(x_{n_k}, y_{n_k}))_{k\in\mathbf{N}}$ does not converge to zero, $(x_{n_k})_{k\in\mathbf{N}}$ being a null sequence in E. This proves our claim.

Now, assume that E is an M-topological A-module and that $\mathfrak{X}$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded, $\mathcal{M}$ being as in the statement of the theorem. Let $(u_n)_{n\in\mathbf{N}}$, $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ be as indicated before, and let $(a_n)_{n\in\mathbf{N}}$ be a null sequence in A consisting of elements of A^* such that $(a_n^{-1} x_n)_{n\in\mathbf{N}}$ converges to zero in E. Since the set $\{a_n^{-1} x_n; \quad n \in \mathbb{N}\}$ belongs to $\mathcal{M}$ and the set $\{y_n; n \in \mathbb{N}\}$ is contained in an element of $\mathcal{N}$, it follows that the sequence $(u_n(a_n^{-1} x_n, y_n))_{n\in\mathbf{N}}$ is bounded. Hence $(u_n(x_n, y_n))_{n\in\mathbf{N}}$ converges to zero in G because $u_n(x_n, y_n) = a_n u_n(a_n^{-1} x_n, y_n)$ for all $n \in \mathbb{N}$. Therefore $\mathfrak{X}$ is sequentially $\mathcal{N}$-equihypocontinuous. By interchanging the roles of E and F, we conclude that the other assertion is also true.

Our next result is a bilinear version of the Banach-Steinhaus theorem, and was suggested by Theorem 25 of [2].

Theorem 29. Suppose that E and F are barrelled, E is an M-topological A-module, and G is separated. Let $(u_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{L}_{\text{sep}}(E, F; G)$ which is pointwise convergent to a mapping $u\colon E \times F \to G$. Then $(u_n)_{n\in\mathbf{N}}$ is sequentially equicontinuous, $u \in \mathcal{L}_{\text{sep}}(E, F; G)$, and u is sequentially continuous.

Proof. It is easily seen that u is a bilinear mapping; and, by Corollary 3.1 of [7], $u \in \mathcal{L}_{\text{sep}}(E, F; G)$.

Now, let us prove that $(u_n)_{n\in\mathbf{N}}$ is sequentially equicontinuous. For this purpose, let $\mathcal{M}$ (resp. $\mathcal{N}$) be the set of all bounded subsets of E (resp. F). Since $(u_n)_{n\in\mathbf{N}}$ is pointwise bounded, Corollary 15 ensures that $(u_n)_{n\in\mathbf{N}}$ is $(\mathcal{M} \times \mathcal{N})$-uniformly bounded. Therefore, by Theorem 28, $(u_n)_{n\in\mathbf{N}}$ is sequentially $\mathcal{N}$-equihypocontinuous. Let $(x, y) \in E \times F$, and let $((x_k, y_k))_{k\in\mathbf{N}}$ be a sequence in $E \times F$ converging to (x, y). Since

$$u_n(x_k, y_k) - u_n(x, y) = u_n(x, y_k - y) + u_n(x_k - x, y) + u_n(x_k - x, y_k - y)$$

for all $n, k \in \mathbb{N}$, and since the set

$$\{z \in F \mapsto u_n(x, z) \in G; n \in \mathbb{N}\}$$

is equicontinuous by Theorem 3.1 of [7], it follows that $(u_n(x_k, y_k))_{k\in\mathbf{N}}$ converges uniformly to $u_n(x, y)$ for $n \in \mathbb{N}$. Thus $(u_n)_{n\in\mathbf{N}}$ is sequentially equicontinuous.

Finally, u is sequentially continuous. Indeed, let $((x_k, y_k))_{k\in\mathbf{N}}$ be a sequence in $E \times F$ converging to an element (x, y) in $E \times F$, and let W be a closed neighborhood of zero in G. Then there exists a $k_0 \in \mathbb{N}$ such that $u_n(x_k, y_k) - u_n(x, y) \in W$ for all $n \in \mathbb{N}$ and for all $k \geq k_0$. Consequently, $u(x_k, y_k) - u(x, y) \in W$ for all $k \geq k_0$. Hence u is sequentially continuous, thereby concluding the proof of the theorem.

Corollary 30. Suppose that A contains a null sequence of elements of A^*. Suppose that E and F are metrizable and complete, and that G is separated. If $(u_n)_{n\in\mathbf{N}}$ is a sequence in $\mathcal{L}_{\mathrm{sep}}(E, F; G)$ which is pointwise convergent to a mapping $u\colon E \times F \to G$, then $(u_n)_{n\in\mathbf{N}}$ is equicontinuous and u is continuous.

Proof. Immediate from Propositon 2.4 of [7], Proposition 26, Theorem 29 and Proposition 22(b).

Remark 31. In the particular case when $A = \mathbb{R}$ or $\mathbb{C}$, E and F are metrizable (one or both being complete) and $\mathfrak{X}$ is a countable set, some of our results have already been obtained, by different methods; see Corollaries 6, 10, 13, 16 and 17 of [8]. Also, our Corollary 30 has already been obtained when E and F are metrizable and complete topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$; see Corollary 14 of [8].

We conclude the paper with an application (Proposition 33) of one of our results. In order to prove it, we shall need the following

Proposition 32. Suppose that the product of any two neighborhoods of zero in A is a neighborhood of zero in A. Suppose that $\bigcup\limits_{B\in\mathcal{M}} B = E$ and $\bigcup\limits_{C\in\mathcal{N}} C = F$, and consider the A-module $\mathcal{L}(E, F; G)$ of all continuous A-bilinear mappings from $E \times F$ into G endowed with the topology $\tau_{\mathcal{M}\times\mathcal{N}}$ of uniform convergence on all subsets of $E \times F$ of the form $B \times C$, where $B \in \mathcal{M}$ and $C \in \mathcal{N}$. Then $(\mathcal{L}(E, F; G), \tau_{\mathcal{M}\times\mathcal{N}})$ is a topological A-module, which is separated if G is separated.

Proof. Without loss of generality we may assume that for every $B_1, B_2 \in \mathcal{M}$ and for every $C_1, C_2 \in \mathcal{N}$ there are a $B_3 \in \mathcal{M}$ and a $C_3 \in \mathcal{N}$ such that $(B_1 \times C_1) \cup (B_2 \times C_2) \subset B_3 \times C_3$. We claim that if $u \in \mathcal{L}(E, F; G)$, $B \in \mathcal{M}$ and $C \in \mathcal{N}$, then $u(B \times C)$ is bounded. Indeed, let W be a neighborhood of zero in G. Then there exist a neighborhood U of zero in E and a neighborhood V of zero in F such that $u(U \times V) \subset W$, and there exists a neighborhood L of zero in A such that $LB \subset U$ and $LC \subset V$. Consequently, $(LL)u(B \times C) \subset W$, LL being a neighborhood of zero in A by hypothesis. Therefore $u(B \times C)$ is bounded. By Proposition (a) of [6], $(\mathcal{L}(E, F; G), \tau_{\mathcal{M}\times\mathcal{N}})$ is a topological A-module, which is clearly separated if G is separated.

Proposition 33. Suppose that A contains a null sequence of elements of A^*. Suppose that E and F are metrizable and complete, and that G is separated and sequentially complete. If $\bigcup\limits_{B\in\mathcal{M}} B = E$ and $\bigcup\limits_{C\in\mathcal{N}} C = F$, then $(\mathcal{L}(E, F; G), \tau_{\mathcal{M}\times\mathcal{N}})$ is a separated sequentially complete topological A-module.

Proof. It is easily verified that the product of any two neighborhoods of zero in A is a neighborhood of zero in A. Therefore, by Proposition 32, $(\mathcal{L}(E, F; G), \tau_{\mathcal{M}\times\mathcal{N}})$ is a separated topological A-module. Now, let $(u_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in $(\mathcal{L}(E, F; G), \tau_{\mathcal{M}\times\mathcal{N}})$. Since $\bigcup\limits_{B\in\mathcal{M}} B = E$ and $\bigcup\limits_{C\in\mathcal{N}} C = F$, $(u_n(x, y))_{n\in\mathbf{N}}$ is a Cauchy sequence in G for each $(x, y) \in E \times F$. Thus $(u_n(x, y))_{n\in\mathbf{N}}$ converges in G for each $(x, y) \in E \times F$, because G is sequentially complete. By Corollary 30, the mapping $u\colon E \times F \to G$ defined by $u(x, y) = \lim\limits_{n\to\infty} u_n(x, y)$ belongs to $\mathcal{L}(E, F; G)$. Finally,

$(u_n)_{n\in\mathbf{N}}$ converges to u for $\tau_{\mathcal{M}\times\mathcal{N}}$, thereby concluding the proof.

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