A nontrivial model of Weydert's SF_3 minus the Leibniz rules

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Abstract

We consider the positive set theory "Strong Frege 3" (SF_3) proposed by E. Weydert and discussed in [5]. SF_3 is a "three valued" set theory where two binary predicates appear, $\in$ and $\overline{\in}$, mutually exclusive, but none of whom is the negation of the other (so given the sets x and y there are three possibilities: either $x \in y$ holds, or $x \overline{\in} y$ holds, or both fail). SF_3 gives the axiom of extensionality with respect to $\in$ and $\overline{\in}$, and a comprehension schema for those first order formulas which are built positively from $x \in y$, $x \overline{\in} y$, $x = y$ and $\neg(x = y)$.

In this paper we build a model $\mathcal{M}$, which we conjecture to satisfy SF_3, and we prove that $\mathcal{M}$ does satisfy SF_3 but in the logic without Leibniz rules for equality; $\mathcal{M}$ is nontrivial in the sense that its equality relation is not the trivial relation which identifies everything. The construction uses an ad hoc calculus $C\Delta_3$, which is a typed, three-valued variant of the Fitch combinatory logic $C\Delta$ (see [1]).

1 Introduction: the theory SF_3

In this paper we are concerned with the "positive" set theory Strong Frege 3 (SF_3), which can be considered as a sort of "three-valued" set theory where equality is treated classically. The author of SF_3 is E. Weydert. Some theories inspired by the same ideas as SF_3 can be found in the works of Gilmore [2] and Hinnion [3], [4]. As far as I know, the only published paper about SF_3 is my [5]. Here we just define the theory SF_3 (following closely the first section of [5]), and we refer to [5] for more details.

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The name of the theory Strong Frege 3 dates to 1989 and is due to R. Hinnion; its explanation is the following:

- Strong: in contraposition with another theory, called simply Frege 3 (F_3). Shortly, F_3 can be viewed as SF_3 where the equality predicate $x = y$ and its negation $\neg(x = y)$ are replaced by an equivalence $\equiv$ and another predicate $\not\equiv$, and the axiom $x \equiv y \Rightarrow \neg(x \not\equiv y)$ is given, but the axiom $x \equiv y \vee x \not\equiv y$ is not;

- Frege: in honour of G. Frege, the author of the first (although inconsistent) comprehension principle for sets;

- 3: because, given the sets x and y in SF_3, we can have three situations: either $x \in y$ holds, or $x \overline{\in} y$ holds, or both fail; in other words the membership relation between sets may be undetermined.

We call *sets* the inner objects of SF_3. The formal language of SF_3 is the first order language consisting of two binary predicates, $\in$ (membership) and $\overline{\in}$ (bar-membership), and including the equality predicate $=$.

First of all we have the following axiom:

Axiom 1. *(mutual exclusion)* $x \in y \Rightarrow \neg(x \overline{\in} y)$.

This axiom means that each of $\in$ and $\overline{\in}$ is a kind of "weak negation" of the other. However since we do not state the natural counterpart of the axiom (i.e. something like $x \in y \vee x \overline{\in} y$), we do not impose a priori that each of $\in$ and $\overline{\in}$ is the negation of the other; actually this is provably false in SF_3.

A set in SF_3 is a kind of "two-face medal", for it can have (zero or more) members and (zero or more) bar-members. Anyway, a set is determined by its members and its barmembers, as the following axiom states:

Axiom 2. *(extensionality)* $(\forall t((t \in x \Leftrightarrow t \in y) \wedge (t \overline{\in} x \Leftrightarrow t \overline{\in} y))) \Rightarrow x = y$.

Finally we give the very core of SF_3, namely its comprehension schema. The idea is to repeat Frege's comprehension schema for set-theoretic formulas (this time in the $\in, \overline{\in}$-language), but with two changes:

- considering only those formulas which are "positive" in the definition below;

- while defining a set, specifying both its members and its barmembers (so this set will be uniquely determined, by extensionality).

So we first define the positive formulas:

Definition 3. *(positive formulas) The set Pf of the positive formulas is the smallest set of $\in, \overline{\in}$-formulas such that:*

- *$x \in y$, $x \overline{\in} y$, $x = y$, $\neg(x = y)$ are in Pf (these are the basic positive formulas);*

- *if ϕ, ψ are in Pf then $\phi \vee \psi$, $\phi \wedge \psi$, $\exists x\phi$, $\forall x\phi$ are in Pf.*

Let us consider the four kinds of basic positive formulas (and fix two variables x and y): by mutual exclusion, $x \in y$ and $x \overline{\in} y$ are in a status of mutual weak negation or, since they are both positive, of *positive negation*; moreover each of $x = y$ and $\neg(x = y)$ is logically the negation of the other, and we can consider also their correspondance as a positive negation, since they are both positive by definition.

So we have a bijection between basic positive formulas, called positive negation; there is a natural extension of this bijection to all positive formulas by induction, and we give it in the following definition:

Definition 4. *(positive negation of positive formulas) Given the positive formula ϕ, we call positive negation of ϕ the formula $Pn(\phi)$, where:*

- *$Pn(x \in y)$ is $x \overline{\in} y$, $Pn(x \overline{\in} y)$ is $x \in y$, $Pn(x = y)$ is $\neg(x = y)$ and $Pn(\neg(x = y))$ is $x = y$;*
- *$Pn(\phi \vee \psi)$ is $Pn(\phi) \wedge Pn(\psi)$, and $Pn(\phi \wedge \psi)$ is $Pn(\phi) \vee Pn(\psi)$;*
- *$Pn(\exists x \phi)$ is $\forall x Pn(\phi)$, and $Pn(\forall x \phi)$ is $\exists x Pn(\phi)$.*

We write $\overline{\phi}$ for $Pn(\phi)$; we note that, for any positive ϕ, $\overline{\phi}$ is positive and $\overline{\overline{\phi}}$ is ϕ.

Now, because of the presence of $\in$ and $\overline{\in}$ in SF_3, it is natural to associate (certain) sets with (certain) pairs of formulas. For example, given two formulas $\phi(x)$ and $\psi(x)$ with at most x free, by extensionality there is at most one set whose members are those enjoying ϕ and whose barmembers are those enjoying ψ; when ϕ is a positive formula and ψ is $\overline{\phi}$, this set exists by the following axiom schema, where b is a variable which is not free in ϕ:

Axiom 5. ϕ *(positive comprehension schema)* $\exists b \forall x((x \in b \Leftrightarrow \phi) \wedge (x \overline{\in} b \Leftrightarrow \overline{\phi}))$.

We denote $\{x|\phi\}$ the set b of the previous axiom.

We call SF_3 the first order theory whose nonlogical axioms are the axioms of Mutual Exclusion, Extensionality and of the Positive Comprehension schema.

An important detail is still missing. In order to formalize our theory we have to express it within first order logic *with equality*. Usually the logical axioms of this logic include the so-called Leibniz rules, which say that equality is a congruence with respect to every predicate symbol in consideration. In our case, where we have two predicate symbols $\in$ and $\overline{\in}$, the Leibniz rules would be: $x = x' \wedge y = y' \wedge x \in y \Rightarrow x' \in y'$, and similarly for $\overline{\in}$. However for simplicity we do *not* include the Leibniz rules among our axioms. This simplifies a great deal the problem of finding the models of SF_3.

2 The consistency problem for SF_3

As usual, when one proposes a new axiomatic system, the first question is the consistency of the system. The consistency of SF_3 is a longstanding open problem. Many models are known of the theory SF_3 without extensionality; for instance, the Π^1_1 sets are a natural model of this theory once a Π^1_1 enumeration of them is given; unfortunately there is no such *injective* enumeration (see [6]), so that the Π^1_1 sets do not give us a model of the full SF_3 with extensionality. Also, one can find various "term models" of SF_3 minus extensionality. So the main trouble is extensionality.

In this paper we do obtain a model $\mathcal{M}$ of the full SF_3 with extensionality, but we are not able to prove that $\mathcal{M}$ satisfies the Leibniz rules, though we conjecture that it is so. If we call "trivial" a model of SF_3 minus the Leibniz rule where the equality identifies everything, then our model $\mathcal{M}$ is nontrivial. Note that in trivial models, extensionality is trivially satisfied. This is why we call such models "trivial".

So our approach to the consistency problem of SF_3 consists in proving the following theorem:

Theorem 6. *Assume the consistency of set theory ZFC. There exists a nontrivial model $\mathcal{M}$ of SF_3 without Leibniz rules ($\mathcal{M}$ is defined in section 8). Hence if $\mathcal{M}$ verifies the Leibniz rules, then the theory SF_3 is consistent.*

Note that the theorem delivers as a corollary that SF_3 without Leibniz rules is consistent, but this can be seen directly by constructing trivial models, which however is an easy but tedious task which is left to the interested reader. Instead, it seems nonobvious to construct nontrivial models of this theory.

We will prove the theorem above in the next sections. Essentially the idea of the proof is to use Combinatory Logic, and in particular Fitch's combinatory logic $C\Delta$ (see [1]). This system has the right expressiveness, and it has also good extensionality properties, but not exactly the right ones for SF_3; to obtain the "right" extensionality we use a typed, three-valued version of $C\Delta$, called $C\Delta_3$, which will be exposed in the next sections. We perform our construction inside ZFC, but of course some weaker metatheories may go as well.

3 The system $C\Delta_3$: types and terms

In this section we begin the exposition of the Fitch-like calculus $C\Delta_3$. This calculus, unlike that of [1], is typed, so we begin by introducing the types of $C\Delta_3$.

Definition 7. *(types of $C\Delta_3$) The class Type of the types is the least class such that:*

- *B is in Type, and is called the boolean type;*
- *D is in Type, called the main type;*
- *if T and U are in Type, then $T \to U$ is a type, and is called the type of the functions from T to U.*

As usual we abbreviate $T \to (U \to V)$ as $T \to U \to V$.

Our types look similar to the usual type theories, such as those of Martin-Löf. However only some types are important for us. In particular, among the types, we select some types called the predicative types, which will be useful for encoding predicate logic in our system:

Definition 8. *(predicative types) The set of the predicative types is the set $Pt = \{T_n | n \in \mathbf{N}\}$, where:*

- *T_0 is B;*
- *T_{n+1} is $D \to T_n$.*

Intuitively T_n is the type of the n-ary predicates. In particular T_0, namely B, is intended to be the type of the propositions.

Now we define the basic terms of $C\Delta_3$ and their types:

Definition 9. *(basic terms) The set Bt of the basic terms and their types are the following.*

- *the bottom combinator* $bottom_n$ $(n \geq 1)$ *has type* T_n;
- *the B-equality,* $=_B$, *has type* $B \to B \to B$;
- *the D-equality,* $=_D$, *has type* $D \to D \to B$;
- *the membership* M_{nij} $(n \geq 2,\ 1 \leq i \leq n,\ 1 \leq j \leq n,\ i < j)$ *has type* T_n;
- *the identity* I_{nij} $(n \geq 2,\ 1 \leq i \leq n,\ 1 \leq j \leq n,\ i < j)$ *has type* T_n;
- *the permutation* $P_{n\sigma}$ $(n \geq 2,\ \sigma$ *permutation on* $\{1, \ldots, n\})$ *has type* $T_n \to T_n$;
- *the negation* $\neg_n$ $(n \geq 1)$ *has type* $T_n \to T_n$;
- *the disjunction* $\vee_n$ $(n \geq 1)$ *has type* $T_n \to T_n \to T_n$;
- *the existential quantification* E_n $(n \geq 1)$ *has type* $T_{n+1} \to T_n$.

We note that the combinators introduced here are not complete with respect to combinatory logic, but they are just what is needed to obtain a system at least as powerful as SF_3, as we will see. In particular we do not need anything like the combinators K and S of Curry's Combinatory Logic. Now we proceed to defining all the terms of $C\Delta_3$; we mostly follow the usual type theory, and the only new thing here is that we identify the types D and $D \to B$; in this way the type D is endowed with an applicative structure which will be useful in the sequel.

Definition 10. *(terms) The set Term of the terms and the relation "is a term of type" are the least objects such that:*

- *if* t *is a basic term of type* T, *then* t *is a term of type* T;
- *if* a *is a term of type* $T \to U$ *and* b *is a term of type* T, *then* ab *is a term of type* U;
- *if* d *is a term of type* D, *then* d *is a term of type* $D \to B$;
- *if* d *is a term of type* $D \to B$, *then* d *is a term of type* D.

We adopt the usual way of associating terms: we write abc for $(ab)c$.

Among the terms, those beginning with $=_D$ or $=_B$ have a special importance. We call D-equation a term beginning with $=_D$, which has necessarily the form $=_D ab$, where a and b are terms of type D; such a term will be also denoted $a =_D b$. Likewise we call B-equation a term beginning with $=_B$, which has necessarily the form $=_B ab$, where a and b are terms of type B; such a term will be also denoted $a =_B b$.

We note that we can apply any term d of type D to any term d' of type D, modulo viewing d as a function from D to B; the result dd' will be a term of type B.

4 The calculus of a set of D-equations

Let A be a set of D-equations. The calculus CA associated with A is a pair of sets of booleans (PA, RA). Intuitively PA is the set of the propositions which are provable in A, and RA is the set of the propositions which are refutable in A.

The set PA will be constructed in a denumerable sequence of steps $P_n A$. We begin by defining $P_0 A$ and RA:

Definition 11. *($P_O A$ and RA) The pair $(P_0 A, RA)$ is the least pair of sets such that:*

1. *$a =_D b$ is in $P_0 A$ iff $a =_D b$ is in A;*
2. *$a =_D b$ is in RA iff $a =_D b$ is not in A;*
3. *$M_{nij} d_1 \dots d_n$ is in $P_0 A$ (resp. RA) iff $d_i d_j$ is in $P_0 A$ (resp. RA);*
4. *$I_{nij} d_1 \dots d_n$ is in $P_0 A$ (resp. RA) iff $d_i =_D d_j$ is in $P_0 A$ (resp. RA);*
5. *$P_{n\sigma} x d_1 \dots d_n$ is in $P_0 A$ (resp. RA) iff $x d_{\sigma 1} \dots d_{\sigma n}$ is in $P_0 A$ (resp. RA);*
6. *$\neg_n x d_1 \dots d_n$ is in $P_0 A$ (resp. RA) iff $x d_1 \dots d_n$ is in RA (resp. $P_0 A$);*
7. *$\vee_n x y d_1 \dots d_n$ is in $P_0 A$ (RA) iff $x d_1 \dots d_n$ is in $P_0 A$ (RA) or (and) $y d_1 \dots d_n$ is in $P_0 A$ (RA);*
8. *$E_n x d_1 \dots d_n$ is in $P_0 A$ (RA) iff for some (all) d of type D, $x d d_1 \dots d_n$ is in $P_0 A$ (RA).*

Before completing the definition of PA we define the depth of a term b of type B as the maximum number of nested $=_B$'s occurring in b. More formally:

Definition 12. *(depth of a term of type B) We define the depth of a term of type B as follows (by induction on the length of b):*

- *$depth(b)$ is 0 if b is not a B-equation;*
- *$depth(a =_B b)$ is $1 + max(depth(a), depth(b))$.*

Remark 13. *We note that a term of type B and of depth 0 cannot contain any occurrence of the term $=_B$. In fact, reasoning by exclusion, such a term must be an application of two terms of type D, and by definition no term of type D can contain the term $=_B$.*

Now we can complete the definition of PA. The idea is of having only three booleans, up to equality provable in PA: the true, the false and the undefined. In this sense our calculus is three-valued.

Definition 14. *(PA) We define the sets $P_n A$ and PA as follows:*

- *$P_0 A$ is given above;*

- *$P_{n+1}A$ is P_nA plus all the equalities $a =_B b$, where either a and b lie in P_nA, or a and b lie in RA, or a and b have depth $\leq n$ and do not lie neither in P_nA nor in RA;*

- *PA is $\bigcup_{n\in\mathbf{N}} P_nA$.*

We remark that, in the construction of the P_nA and RA, there is no rule for introducing terms beginning with $bottom_n$, hence for any tuple $d_1 \ldots d_n$ of type D, $bottom_n d_1 \ldots d_n$ lies neither in PA nor in RA. So the $bottom_n$ are "empty" combinators, which will be useful in constructing our model of SF_3.

The calculus $CA = (PA, RA)$ has some nice properties; for instance:

Lemma 15. *1. PA and RA have the same closure properties listed in def. 11 for P_0A and RA;*

2. $a =_B b$ is in PA iff either a and b are in PA, or a and b are in RA, or a and b lie neither in PA nor in RA.

3. PA and RA are disjoint.

Proof: 1 follows because P_0A and PA are the same up to some B-equations. 2 follows by induction on the depth of the equation $a =_B b$, taking into account that each term in $P_{k+1}A \backslash P_kA$ has depth $k+1$. 3 can be proved first for P_0A and RA by induction on the structure of P_0A and RA, and extends to PA and RA because the difference $PA - P_0A$ contains only B-equations and RA contains no B-equation. ■

5 Normality

In this section we follow closely the section 4 of [1] and we give some technical notions which will be useful in the sequel. Let us start with the notions of resultant, preresultant and medioresultant.

Definition 16. *(resultant, preresultant and medioresultant) Let a, b be terms of the same type T and let c be a term. By an ab-resultant of c we will mean any result of replacing exactly one occurrence of a by b in c, or of b by a in c. If no terms occur to the left of the replaced occurrence, then the ab-resultant of c will be called an ab-preresultant of c (Thus $bdef$ is an ab-preresultant of $adef$, and viceversa). Otherwise it will be called an ab-medioresultant of c. (Thus $dbefg$ is an ab-medioresultant of $daefg$, and viceversa). Hence every ab-resultant of a term is either an ab-preresultant of that term or else is an ab-medioresultant of it.*

Now let us proceed with the definition of normality, prenormality and equinormality (our definition of equinormality is weaker than that of [1] because we do not impose reflexivity):

Definition 17. *(normality, prenormality and equinormality) Let X be a set of terms. Let $a =_T b$ be an equation (where T is D or B).*

- *The set X is said to be normal in $a =_T b$ if all ab-resultants of members of X are members of X. If E is a set of equations, we say that X is normal in E if it is normal in all members of E.*

- *The set X is said to be prenormal in $a =_T b$ if all ab-preresultants of members of X are members of X. If E is a set of equations, we say that X is prenormal in E if it is prenormal in all members of E.*
- *Assume that X is a set of equations. The set X will be said to be equinormal in $a =_T b$ if $a =_T b$ is in X, and moreover all ab-medioresultants of members of X are members of X. If E is a set of equations, we say that X is equinormal in E if it is equinormal in all members of E. We denote X^{eq} the equinormal closure of X, namely the smallest set of equations containing X and equinormal in itself.*

In the sequel we will denote by Eq the set of all the equations (in type D or B). At this point, like in [1], we have a theorem.

Theorem 18. *Let A be a set of D-equations and let $p =_T q$ be an equation. If $PA \cap Eq$ is equinormal in $p =_T q$ and PA and RA are prenormal in $p =_T q$, then PA and RA are normal in $p =_T q$.*

Proof: given the hypotheses of the theorem, it is enough to show that if x is in PA and u is a pq-resultant of x, then u is in PA, and if y is in RA and v is a pq-resultant of y, then v is in RA. Now, in case u is actually a pq-preresultant of x, then u is in PA by the prenormality of PA in $p =_T q$; similarly if y is in RA and v is a pq-preresultant of y, then v is in RA. So we can assume that u is a pq-medioresultant of x, and that v is a pq-medioresultant of y. We proceed by induction.

Assume first that x is an equation. Then x is in $PA \cap Eq$; by the equinormality of $PA \cap Eq$ in $p =_T q$, and since u is a pq-medioresultant of x, u is in $PA \cap Eq$, hence u is in PA.

Likewise assume that y is an equation. Then y is in $RA \cap Eq$, so y is a D-equation and y is not in A, hence y is not in $PA \cap Eq$. Since v is a pq-medioresultant of y, by the equinormality of $PA \cap Eq$, v is not in $PA \cap Eq$, and v is still a D-equation, hence v is not in A, hence v is in RA.

Now assume that x and y are added to PA and RA by some of the rules 3 to 8 of def. 11. Assume for instance that x is in PA by the rule 3. Then x is $M_{nij}d_1 \dots d_n$ for some $d_1, \dots, d_n$ such that $d_i d_j$ is in PA. Since u is a pq-medioresultant of x, u is $M_{nij}d'_1 \dots d'_n$, where each d'_k is a pq-resultant of d_k, and only one d'_k is different from d_k. So $d'_i d'_j$ is a pq-resultant of $d_i d_j$. By inductive hypothesis applied to $d_i d_j$, we have that $d'_i d'_j$ is in PA, hence $M_{nij}d'_1 \dots d'_n$ is in PA by rule 3, namely u is in PA. The other cases are analogous. ∎

Corollary 19. *If $PA \cap Eq$ is equinormal in itself and if PA and RA are prenormal in $PA \cap Eq$, then PA and RA are normal in $PA \cap Eq$.*

6 Conormality and autonormality

In this section we consider the key definitions of our construction, namely that of conormality and autonormality. Our definition of conormality is different from [1], though it is similar in the spirit.

Definition 20. *(conormality and autonormality) Let A be a set of D-equations and let e be an equation (in type B or D). A is said to be conormal in e if: for every set X of D-equations including A and such that e is in PX and $PX \cap Eq$ is equinormal in $A \cup \{e\}$, we have that PX and RX are prenormal in e.*

If E is a set of equations, A is said to be conormal in E if A is conormal in every member of E.

A is said to be autonormal if A is conormal in itself.

We are interested mostly in autonormal sets. The existence of autonormal sets is guaranteed by the following theorem (whose proof is trivial):

Theorem 21. *The empty set $\emptyset$ is autonormal.*

Now we obtain a theorem saying that B-equations are quite irrelevant for conormality, in the following sense:

Theorem 22. *Let A be a set of D-equations and let $a =_B b$ be a B-equation in PA. Then:*

1. *$PA \cap Eq$ is equinormal in $a =_B b$;*
2. *PA and RA are normal in $a =_B b$;*
3. *A is conormal in $a =_B b$.*

Proof:

1. Let $c =_T d$ be in $PA \cap Eq$ and let $c' =_T d'$ be an ab-medioresultant of $c =_T d$. Then it must be $T = B$, and some term between c and d must be equal to a or b, and $c' =_B d'$ is obtained from $c =_B d$ by replacing the term above with b or a respectively. Suppose for instance that a is equal to c (the other cases are analogous). By definition of PA, since $a =_B b$ is in PA, we have a is in PA if and only if b is in PA; and since $c =_T d$ is in PA we have c in PA if and only if d is in PA. But c is equal to a by hypothesis; so b is in PA if and only if d is in PA; and since b is c' and d is d', we conclude that c' is in PA if and only if d' is in PA, hence $c' =_B d'$ is in PA.
2. Assume that $a =_B b$ is in $PA \cap Eq$. Then a is in PA iff b is in PA, and a is in RA iff b is in RA. Hence by Lemma 15 PX and RX are prenormal in $a =_B b$, and since $PA \cap Eq$ is equinormal in $a =_B b$ by the previous point, we conclude that PA and RA are normal in $a =_B b$ by theorem 18.
3. Let X be a set of D-equations including A and assume that $a =_B b$ is in PX. Then, by Lemma 15, a is in PX iff b is in PX, and a is in RX iff b is in RX. Hence PX and RX are prenormal in $a =_B b$.

∎

We now give an important definition.

Definition 23. *Given a set A of D-equations, let us denote $Dcon(A)$ the set of all the D-equations in which A is conormal.*

We have a way to "enlarge" autonormal sets, similar to [1]:

Theorem 24. *Let A be an autonormal set of D-equations, and let $A' = (Dcon(A))^{eq}$. Then A' is autonormal.*

Proof: Let $a =_D b$ be an equation in A' and let X be any set of D-equations including A', such that $a =_D b$ is in PX and $PX \cap Eq$ is equinormal in A'. We want to show that PX and RX are prenormal in $a =_D b$. Now, since $PX \cap Eq$ is equinormal in A' and A' includes $Dcon(A)$, $PX \cap Eq$ is equinormal in $Dcon(A)$. Moreover PX and RX are prenormal in $Dcon(A)$ because A is conormal in $Dcon(A)$; hence PX and RX are normal in $Dcon(A)$ by theorem 18; hence PX and RX are normal in A' because A' is the smallest set of D-equations containing $Dcon(A)$ and closed under the medioresultant relation. Then PX and RX are prenormal in A', hence PX and RX are prenormal in $a =_D b$. ∎

We can repeat the reasoning above for increasing ordinal sequences of autonormal sets:

Theorem 25. *Let $(A_\beta)_{\beta<\alpha}$ be an increasing ordinal sequence of autonormal sets of D-equations and let $A' = (\bigcup_{\beta<\alpha} Dcon(A_\beta))^{eq}$. Then A' is autonormal.*

Proof: Let $a =_D b$ be an equation in A' and let X be any set of D-equations including A', such that $PX \cap Eq$ is equinormal in A'. We want to show that PX and RX are prenormal in $a =_D b$. Now since $PX \cap Eq$ is equinormal in A' and A' includes $Dcon(A_\beta)$, $PX \cap Eq$ is equinormal in $Dcon(A_\beta)$. Moreover PX and RX are prenormal in $Dcon(A_\beta)$ by the conormality of A_β in $Dcon(A_\beta)$; hence PX and RX are normal in every $Dcon(A_\beta)$ by theorem 18; hence PX and RX are normal in A' because, once again, A' is the smallest set of D-equations containing $\bigcup_{\beta<\alpha} Dcon(A_\beta)$ and closed under the medioresultant relation. Then PX and RX are prenormal in A', hence PX and RX are prenormal in $a =_D b$. ∎

We conclude this section with a theorem and a corollary which will be useful for proving the extensionality of our model of SF_3:

Theorem 26. *Let A be a class of D-equations and let $a =_D b$ a D-equation. If PA (resp. RA) is prenormal in $ac =_B bc$ for every c of type D, then PA (resp. RA) is prenormal in $a =_D b$.*

Proof: since PA is prenormal in $ac =_B bc$ for any c of type D, we have for every c that ac is in PA iff bc is in PA; but this means exactly that PA is prenormal in $a =_D b$. The same holds for RA. ∎

Corollary 27. *Let A be a set of D-equations and let $a =_D b$ a D-equation. If A is conormal in $ac =_B bc$ for any c of type D, then A is conormal in $a =_D b$.*

Proof: Let X be a set of D-equations including A, containing $a =_D b$ and such that $PX \cap Eq$ is equinormal in $A \cup \{a =_D b\}$. By lemma 15, $PX \cap Eq$ contains $ac =_B ac$ for every term c of type D. Hence, by equinormality of $PX \cap Eq$ in $a =_D b$, we have that $PX \cap Eq$ contains $ac =_B bc$ for every term c of type D. Moreover $PX \cap Eq$ is equinormal in $ac =_B bc$ by theorem 22. Hence we can use the conormality of A in $ac =_B bc$ for every c, and we can infer that PX and RX are prenormal in $ac =_B bc$ for every c. So by the previous theorem PX and RX are prenormal in $a =_D b$. Hence A is conormal in $a =_D b$. ∎

7 The system $C\Delta_3$

Now we are ready to construct a set Δ_3 of D-equations which will be the basis for the construction of a model of SF_3.

Definition 28. *(Δ_3) Let X_α the ordinal sequence of sets defined by:*

- $X_0 = \emptyset$*;*
- $X_{\alpha+1} = (Dcon(X_\alpha))^{eq}$*;*
- $X_\lambda = (\bigcup_{\alpha<\lambda} X_\alpha)^{eq}$ *for λ limit ordinal.*

We let $\Delta_3 = \bigcup_\alpha X_\alpha$.

We note that the sequence X_α is an increasing sequence of sets of D-equations, and since all the D-equations form a set, the sequence must stabilize at some ordinal δ, namely $X_\delta = X_{\delta+1} = \ldots = \Delta_3$.

Now a key theorem follows:

Theorem 29. *1. Δ_3 is autonormal;*

2. if Δ_3 is conormal in a D-equation $a =_D b$, then $a =_D b$ belongs to Δ_3;

3. Δ_3 is equinormal in itself.

Proof:

1. by the previous theorems, every X_α is autonormal; but $\Delta_3 = X_\delta$, hence Δ_3 is autonormal.
2. If Δ_3 is conormal in $a =_D b$, and $\Delta_3 = X_\delta$, then $a =_D b$ is in $Dcon(X_\delta)$ which is included in $(Dcon(X_\delta))^e q = X_{\delta+1} = \Delta_3$, hence $a =_D b$ is in Δ_3.
3. We have $\Delta_3 = X_\delta = X_{\delta+1} = (Dcon(X_\delta))^e q$, hence Δ_3 is equinormal in itself.

∎

8 The model of SF_3

We are ready to define a model $\mathcal{M}$ of SF_3:

Definition 30. *We define $\mathcal{M}$ to be the model with universe M, membership $\in_\mathcal{M}$, bar-membership $\overline{\in}_\mathcal{M}$ and equality $=_\mathcal{M}$, where:*

- *M is the set of the terms of $C\Delta_3$ of type D;*
- *$x \in_\mathcal{M} y$ iff yx is in $P\Delta_3$;*
- *$x \overline{\in}_\mathcal{M} y$ iff yx is in $R\Delta_3$;*
- *$x =_\mathcal{M} y$ iff $x =_D y$ is in Δ_3.*

Theorem 31. *$\mathcal{M}$ is a nontrivial model of SF_3 in the logic without Leibniz rules. Equality of $\mathcal{M}$ is a nontrivial equivalence relation.*

Proof: first, we check that $=_{\mathcal{M}}$ is an equivalence relation on M.

Now reflexivity follows because Δ_3 is conormal in $d =_D d$ for every d of type D; symmetry follows because the definition of ab-resultant is symmetric in a and b; transitivity follows because if $a =_D b$ and $b =_D c$ are in Δ_3, then $a = c$ is an ab-medioresultant of $b =_D c$, hence by the equinormality of Δ_3 in itself we have $a =_D c$ in Δ_3.

Now we check the nonlogical axioms of SF_3.

Mutual exclusion holds because $P\Delta_3$ and $R\Delta_3$ are disjoint.

Extensionality holds. In fact let x, y be two terms of type D with the same members and the same barmembers: this means that for any z of type D, xz is in $P\Delta_3$ iff yz is in $P\Delta_3$, and xz is in $R\Delta_3$ iff yz is in $R\Delta_3$. Then $xz =_B yz$ is in $P\Delta_3$ for any z, hence Δ_3 is conormal in $xz =_B yz$ for any z; hence by corollary 27 Δ_3 is conormal in $x =_D y$; hence by theorem 29 $x =_D y$ is in Δ_3, namely x and y are equal in $\mathcal{M}$.

Finally comprehension holds. In fact let $\phi(x_1, \ldots, x_n, c_1, \ldots, c_m)$ be a positive formula with n free variables and m constants, and let $a_1, \ldots, a_m$ be terms of type D; by induction we can construct a term $T_\phi(a_1, \ldots, a_m)$ of type T_n such that for any $d_1, \ldots, d_n$ of type D:

- $T_\phi(a_1, \ldots, a_m)d_1 \ldots d_n$ is in $P\Delta_3$ iff $\mathcal{M} \models \phi(d_1, \ldots, d_n, a_1, \ldots, a_m)$;
- $T_\phi(a_1, \ldots, a_m)d_1 \ldots d_n$ is in $R\Delta_3$ iff $\mathcal{M} \models \overline{\phi}(d_1, \ldots, d_n, a_1, \ldots, a_m)$.

In fact we can suppose that ϕ is prenex (a quantifier-free formula preceded by zero or more quantifiers) and irreflexive (namely no subformula $x \in x$ or $x \overline{\in} x$ or $x = x$ or $\neg(x = x)$ occurs). This last restriction is irrelevant as we can simulate, for instance, $x \in x$ with $\exists y(x = y \wedge x \in y)$.

1. if ϕ is basic positive and contains at least one variable, then T_ϕ can be constructed using M_{kij}, I_{kij}, $\neg_k$ and $P_{k\sigma}$. For instance, the filter of a set a, namely the set $\{x | a \in x\}$, can be constructed as $P_{2\tau}M_{212}a$, where τ is the transposition which exchanges 1 and 2.

2. if ϕ is basic positive and contains two constants, then in particular ϕ is a sentence, hence either ϕ is true in the model and $\overline{\phi}$ is false, or ϕ is false in the model and $\overline{\phi}$ is true, or both ϕ and $\overline{\phi}$ are false. In the first case, ϕ can be encoded by any identically true predicate of type T_n, for instance $true_n = E_n I_{n+1,1,2}$. In the second case, ϕ can be encoded by by any identically false predicate of type T_n, for instance $false_n = \neg_n true_n$. Finally in the third case, ϕ can be encoded by any identically undefined predicate of type T_n, for instance $bottom_n$.

3. If ϕ is quantifier-free, then ϕ is a boolean combination of basic positive formulas, hence T_ϕ can be constructed using $\neg_k$, $\vee_k$ and T_ψ for ψ basic.

4. If ϕ has quantifiers and is a sentence, then we can proceed as in point b.

5. If ϕ has quantifiers and is not a sentence, then ϕ is a sequence of quantifiers followed by a quantifier-free formula, hence T_ϕ can be constructed using E_k, $\neg_k$, $P_{k\sigma}$ and T_ψ for ψ quantifier-free.

As an example of application of the definitions, we prove that equality in $\mathcal{M}$ is non trivial. In fact take any term a of type D. Then $= a$ is the singleton of a, i.e. the set containing exactly the elements of the model which are equal to a, and $\neg_2(= a)$ is the complement of the singleton of a. We show that $= a$ is different from $\neg_2(= a)$ in $\mathcal{M}$. In fact suppose for an absurdity that they are equal in $\mathcal{M}$. Let η denote the equation $(= a) =_D (\neg_2(= a))$; then η is in Δ_3, hence Δ_3 is conormal in it. Then take X to be the set of all the D-equations; of course X includes Δ_3 and X is equinormal in itself, hence in $\Delta_3 \cup \{\eta\}$; so we can apply to X the conormality of Δ_3 in η and we find that PX is prenormal in η. Hence, for any term d of type D, $(= a)d$ lies in PX if and only if $(\neg_2(= a))d$ lies in PX. Now by definition of X, $(= a)d$ lies in PX for every d, hence $(\neg_2(= a))d$ lies in PX (by the above) and in RX (by the rules about the negation). But this is absurd since $PX \cap RX = \emptyset$. ■

9 Conclusion

First we remark that the above arguments show the existence of a nontrivial model of SF_3 in a logic without Leibniz rules, but this is not completely satisfactory because one usually wants the Leibniz rules. However, I do conjecture that the model $\mathcal{M}$ of the previous section does verify the Leibniz rules; this is because, intuitively, Δ_3 is a "small" set of somewhat "necessary" equations, which are selected by quantifying universally on a large set of structures (i.e. all those calculi PX, RX which are involved in the property of the autonormality of Δ_3).

Whatever is the answer to the conjecture above, I think that the study of $\mathcal{M}$ or other "Fitch-like" models of SF_3 or parts of it could be an interesting task. It is possible that some other Fitch-like construction (more clever than mine) delivers directly the unconditional consistency of SF_3, but this seems to require further investigations.

Summing up, this paper is not intended as a conclusion of something, but rather the beginning of some (hopefully not too hard) work.

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